Positivity Preserving Schemes for Black-Scholes Equation

Mohammad Mehdizadeh Khalsaraei (Corresponding author)
Faculty of Mathematical Science, University of Maragheh, Maragheh, Iran
E-mail: Muhammad.mehdizadeh@gmail.com

Reza Shokri Jahandizi
Faculty of Mathematical Science, University of Maragheh, Maragheh, Iran
E-mail: reza.shokri.j@gmail.com

Abstract
Mathematical finance is a field of applied mathematics, concerned with financial markets. In the market of financial derivatives the most important problem is the so called option valuation problem, i.e. to compute a fair value for the option. The solution of the Black-Scholes equation determines the option price, respectively according to the used initial conditions. In this paper, first we show that the positivity is not ensured with classical finite difference schemes when applied to the Black-Scholes equation for very small time steps. Next, by reforming the discretization of the reaction term of equation, a family of efficient explicit schemes are derived that is free of spurious oscillations around discontinuities and preserving positivity.
Keywords: Positivity, Nonstandard discretization, Black-Scholes equation

1. Introduction

In this work, we are interested in the *option valuation problem* satisfies the Black-Scholes partial differential equation presented in [5] as:

$$-\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}-rV=0, \tag{1}$$

where $V(S,t)$ is the price of the option and endowed with initial and boundary conditions:

$$V(S,0)=\max(S-K,0)1_{[L,U]}(S),$$

$$V(S,t)\to 0 \text{ as } S\to 0 \text{ or } S\to\infty,$$

with updating of the initial condition at the monitoring dates $t_i,\ i=1,\cdots,F$:

$$V(S,t_i)=V(S,t_i^-)1_{[L,U]}(S),\qquad 0=t_0<t_1<\cdots<t_F=T,$$

where $1_{[L,U]}(S)$ is the indicator function, i.e.,

$$1_{[L,U]}(S)=\begin{cases}1 & if \quad S\in[L,U]\\ 0 & if \quad S\notin[L,U]\end{cases}, \tag{2}$$

here, the parameter $r>0$ is the interest rate and the reference volatility is $\sigma>0$.

To obtain the finite difference approximation for equation (1), let the computational domain $[0,S_{\max}]\times[0,T]$ is discretized by a uniform mesh with steps ΔS, Δt in order to obtain grid points $(j\Delta S, n\Delta t)$, $j=1,\cdots,M$ and $n=0,1,\cdots,N$ so that $S_M=S_{\max}=M\Delta S$ and $T=N\Delta t$. By using the forward difference for $\frac{\partial V}{\partial t}$ and centered difference for discretization of $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ and approximations V_j^n of V at the grid points, we have the following explicit finite difference method [6]:

$$-\frac{V_j^{n+1}-V_j^n}{\Delta t}+rS_j\frac{V_{j+1}^n-V_{j-1}^n}{2\Delta S}+\frac{1}{2}(\sigma S_j)^2\frac{V_{j-1}^n-2V_j^n+V_{j+1}^n}{\Delta S^2}-rV_j^n=0. \tag{3}$$

This method has lower accuracy and often generates numerical drawbacks such as spurious oscillations and negative values in the solution when applied to (1). Whenever the financial parameters of the Black-Scholes model σ and r satisfy the relationship $\sigma^2 \ll r$, see Figure 1. The parameters used in this simulation is taken from [5].

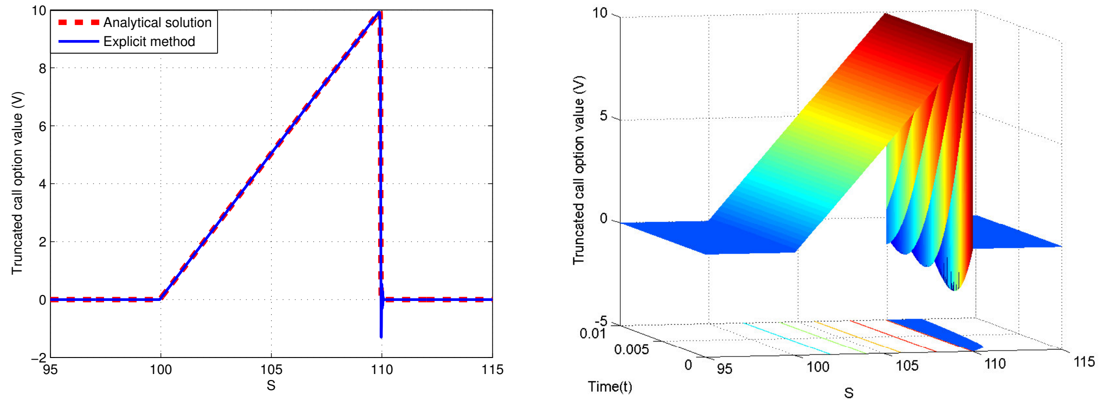

Figure 1. Truncated call option value for explicit method with $\Delta S = 0.01$, $\Delta t = 10^{-5}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{\max} = 120$.

2. Construction of new scheme

Following the suggested strategy in [1, 2, 3, 4, 5], by reforming the discretization of the reaction term to

$V(S,t) = aV_{j-1}^n + V_j^n + V_{j+1}^n - (a+b)V_j^{n+1}$,

we get a family of explicit nonstandard finite difference method as follows:

$$\left(\frac{1}{\Delta t} - r(a+b)\right)V_j^{n+1} = \left(-\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra\right)V_{j-1}^n + \left(\frac{1}{\Delta t} - \left(\frac{\sigma S_j}{\Delta S}\right)^2 - r\right)V_j^n + \left(\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - rb\right)V_{j+1}^n, \tag{4}$$

and the matrix form of the (4) is

$$\left(\frac{1}{\Delta t} - r(a+b)\right)V^{n+1} = AV^n, \tag{5}$$

where A is the following tridiagonal matrix

$$A = \left\{-\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra\,,\ \frac{1}{\Delta t} - \left(\frac{\sigma S_j}{\Delta S}\right)^2 - r\,,\ \frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - rb\right\}. \tag{6}$$

The constant a is chosen according to the following theorem:

Theorem 1. Sufficient for scheme (5) to be positive is

$$a \le -\frac{r}{8\sigma^2} \quad , \quad b \le -\frac{r}{8\sigma^2} \quad , \quad \Delta t < \frac{1}{(\sigma M)^2 + r}. \tag{7}$$

Proof. From (5) it is enough to show that

$$-\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra \ge 0, \tag{8}$$

$$\frac{rS_j}{2\Delta S}+\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2-rb\geq 0, \tag{9}$$

$$\frac{1}{\Delta t}-(\frac{\sigma S_j}{\Delta S})^2-r\geq 0, \tag{10}$$

$$\frac{1}{\Delta t}-r\left(a+b\right)\geq 0, \tag{11}$$

from (8) we can write

$$ra\leq\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2-\frac{rS_j}{2\Delta S},$$

$$\Leftrightarrow a\leq\frac{1}{r}\left[\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2-\frac{rS_j}{2\Delta S}\right],$$

$$\Leftrightarrow a\leq\frac{\sigma^2}{2r}\left[(\frac{S_j}{\Delta S})^2-\frac{r}{\sigma^2}(\frac{S_j}{\Delta S})+\frac{r^2}{4\sigma^4}-\frac{r^2}{4\sigma^4}\right], \tag{12}$$

$$\Leftrightarrow a\leq\frac{\sigma^2}{2r}\left[(\frac{S_j}{\Delta S}-\frac{r}{2\sigma^2})^2-\frac{r^2}{4\sigma^4}\right],$$

$$\Leftrightarrow a\leq\frac{\sigma^2}{2r}(\frac{S_j}{\Delta S}-\frac{r}{2\sigma^2})^2-\frac{r}{8\sigma^2},$$

now, the last inequality in (10) shows sufficiency of $a\leq-\frac{r}{8\sigma^2}$ for (7), and similarly, from (9) we derive that $b\leq-\frac{r}{8\sigma^2}$. In the other hand (10) and (11) leads to $\Delta t<\frac{1}{(\sigma M)^2+r}$, and the proof is complete.

The proposed positive scheme is convergent due to following theorem:

Theorem 2. Under conditions (7), the new scheme is stable and convergent with local truncation error $O\left(\Delta t,\Delta S^2\right)$ for $a=b$ and $O\left(\Delta t,\Delta S\right)$ otherwise.

Proof. Under conditions (7) we have $\frac{1}{\Delta t}-r\left(a+b\right)\geq 0$. In the other hand we have $\|A\|_\infty=\frac{1}{\Delta t}-r\left(a+b\right)-r$, and for spectral radius, ρ of the iteration matrix we derive

$$\rho\left(\frac{1}{\frac{1}{\Delta t}-r\left(a+b\right)}A\right)\leq\left\|\frac{1}{\frac{1}{\Delta t}-r\left(a+b\right)}A\right\|_\infty\leq\left|\frac{1}{\frac{1}{\Delta t}-r\left(a+b\right)}\right|\|A\|_\infty\leq\frac{\frac{1}{\Delta t}-r\left(a+b\right)-r}{\frac{1}{\Delta t}-r\left(a+b\right)}<1,$$

therefor the scheme is stable and convergent with local truncation error:

$$T_j^n=-\frac{V(S_j,t_{n+1})-V(S_j,t_n)}{\Delta t}+rS_j\frac{V(S_{j+1},t_n)-V(S_{j-1},t_n)}{2\Delta S}$$

$$+\frac{1}{2}(\sigma S_j)^2\frac{V(S_{j-1},t_n)-2V(S_j,t_n)+V(S_{j+1},t_n)}{\Delta S^2}$$

$$-r\left(aV(S_{j-1},t_n)+V(S_j,t_n)+bV(S_{j+1},t_n)-(a+b)V(S_j,t_{n+1})\right), \tag{14}$$

by Taylor's expansion, we have

$$V\left(S_j, t_{n+1}\right) = V\left(S_j, t_n\right) + \Delta t\left(\frac{\partial V}{\partial t}\right)_j^n + \frac{1}{2}\Delta t^2\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + \frac{1}{6}\Delta t^3\left(\frac{\partial^3 V}{\partial t^3}\right)_j^n + \cdots,$$

$$V\left(S_{j+1}, t_n\right) = V\left(S_j, t_n\right) + \Delta S\left(\frac{\partial V}{\partial S}\right)_j^n + \frac{1}{2}\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + \frac{1}{6}\Delta S^3\left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \cdots,$$

$$V\left(S_{j+1}, t_n\right) = V\left(S_j, t_n\right) - \Delta S\left(\frac{\partial V}{\partial S}\right)_j^n + \frac{1}{2}\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n - \frac{1}{6}\Delta S^3\left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \cdots,$$

substitution into the expression for T_j^n then gives

$$\begin{aligned} T_j^n = &\left(-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV\right)_j^n - \frac{1}{2}\Delta t\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + r(a-b)\Delta S\left(\frac{\partial V}{\partial S}\right)_j^n \\ &-\frac{r}{2}(a+b)\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + r(a+b)\Delta t\left(\frac{\partial V}{\partial t}\right)_j^n + \frac{r}{2}(a+b)\Delta t^2\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + \cdots. \end{aligned} \tag{15}$$

But V is the solution of the Black-Scholes equation so

$$\left(-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV\right)_j^n = 0.$$

Therefore, if $a = b$ then $T_j^n = O\left(\Delta t, \Delta S^2\right)$, otherwise $T_j^n = O\left(\Delta t, \Delta S\right)$.

3. Numerical Results

To illustrate the advantage of the designed new positive explicit scheme see Figure 2 that shows the new explicit scheme is positivity-preserving and spurious oscillations are avoided.

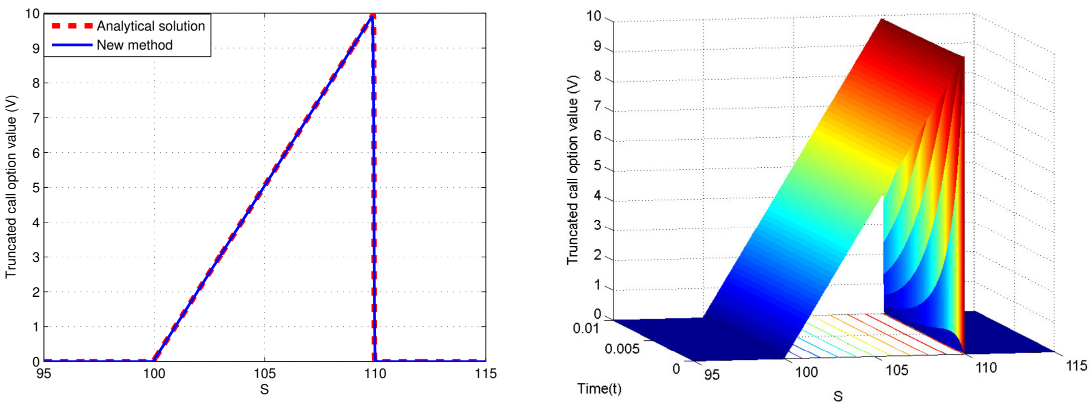

Figure 2. Truncated call option value for new explicit method with $\Delta S = 0.01$, $\Delta t = 10^{-5}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{\max} = 120$.

In the case of larger time step, we see the same behavior, see Figure 3. These numerical results are obtained with $\sigma^2 \ll r$.

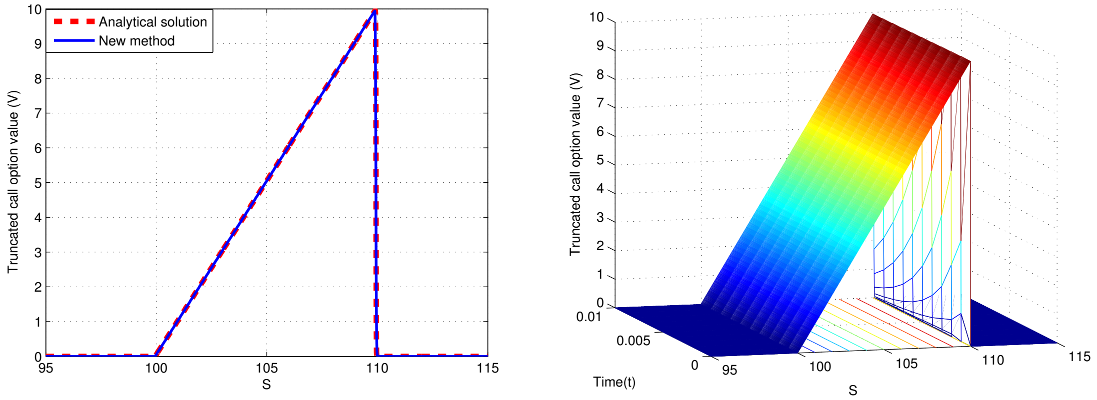

Figure 3. Truncated call option value for new explicit method with $\Delta S = 0.01$, $\Delta t = 10^{-3}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{\max} = 120$.

4. Conclusions and discussion

We constructed a family of explicit method based on a nonstandard discretization scheme to solve option valuation problem with double barrier knock-out call option. In particular, the proposed method uses a nonstandard discretization in reaction term and the spatial derivatives are approximated using standard finite difference scheme. It has been shown that the proposed new scheme preserve the positivity as well as stability and consistence. Future work will include extending the method to nonlinear Black-Scholes equation.

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