Common Fixed Point Theorems of Integral Type in Menger Pm Spaces

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Abstract: In this paper, we propose integral type common fixed point theorems in Menger spaces satisfying common property (E.A). Our results generalize several previously known results in Menger as well as metric spaces.

Keywords: Menger space; Common property (E.A); weakly compatible pair of mappings and t-norm.

1. Introduction:
In 1942 Menger [21] initiated the study of probabilistic metric space (often abbreviated as PM space) and by now the theory of probabilistic metric spaces has already made a considerable progress in several directions (see [32]). The idea of Menger was to use distribution functions (instead of nonnegative real numbers) as values of a probabilistic metric. This PM space can cover even those situations where in one can not exactly ascertain a distance between two points, but can only know the possibility of a possible value for the distance (between a pair of points). In 1986, Jungck [13] introduced the notion of compatible mappings and utilized the same to improve commutativity conditions in common fixed point theorems. This concept has been frequently employed to prove existence theorems on common fixed points. However, the study of common fixed points of non-compatible mappings was initiated by Pant [32]. Recently, Aamri and Moutawakil [1] and Liu et al. [39] respectively defined the property (E.A) and the common property (E.A) and proved interesting common fixed point theorems in metric spaces. Most recently, Kubiaczyk and Sharma [15] adopted the property (E.A) in PM spaces and used it to prove results on common fixed points. Recently, Imdad et al. [26] adopted the common property (E.A) in PM spaces and proved some coincidence and common fixed point results in Menger spaces.

The theory of fixed points in PM spaces is a part of probabilistic analysis and continues to be an active area of mathematical research. Thus far, several authors studied fixed point and common fixed point theorems in PM spaces which include [5, 7, 8, 10, 16, 17, 18, 24, 26, 28, 30, 34, 31, 36, and 37] besides many more. In 2002, Branciari [3] obtained a fixed point result for a mapping satisfying an integral analogue of Banach contraction principle. The authors of the papers [2, 4, 6, 26, 11, and 29] proved a host of fixed point theorems involving relatively more general integral type contractive conditions. In an interesting note, Suzuki [35] showed that Meir-Keeler contractions of integral type are still Meir-Keeler contractions. The aim of this paper is to prove integral type fixed point theorems in Menger PM spaces satisfying common property (E.A).

2 Preliminaries:

Definition 2.1 [7] A mapping $F: \mathbb{R} \to [0, 1]$ is called distribution function if it is non-decreasing, left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

Let $L$ be the set of all distribution functions whereas $H$ be the set of specific distribution functions (also known as Heaviside function) defined by

$$
H(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
1, & \text{if } x > 0
\end{cases}
$$

Definition 2.2 [21] Let $X$ be a nonempty set. An ordered pair $(X, F)$ is called a PM space if $F$ is a mapping from $X \times X$ into $L$ satisfying the following conditions:

(i) $F_{p,q}(x) = H(x)$ if and only if $p = q$,
(ii) $F_{p,q}(x) = F_{q,p}(x)$,
(iii) $F_{p,q}(x) = 1$ and $F_{r,q}(y) = 1$, then $F_{p,r}(x + y) = 1$, for all $p, q, r \in X$ and $x, y \geq 0$.

Every metric space $(X, d)$ can always be realized as a PM space by considering $F : X \times X \to L$ defined by $F_{p,q}(x) = H(x - d(p, q))$ for all $p, q \in X$. So PM spaces offer a wider framework (than that of the metric spaces) and are general enough to cover even wider statistical situations.

Definition 2.3. [7] A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if

(i) $\Delta (a, 1) = a, \Delta (0, 0) = 0$,
(ii) $\Delta (a, b) = \Delta (b, a)$,
(iii) $\Delta (c, d) \geq \Delta (a, b)$ for $c \geq a, d \geq b$. 

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Definition 2.10. The following are the four basic t-norms:

(i) The minimum t-norm: \( T_a(x, y) = \min\{x, y\} \).
(ii) The product t-norm: \( T_p(x, y) = x \cdot y \).
(iii) The Lukasiewicz t-norm: \( T_l(x, y) = \max\{x + y - 1, 0\} \).
(iv) The weakest t-norm, the drastic product:

\[
T_d(x, y) = \begin{cases} \min\{x, y\}, & \text{if } \min\{x, y\} = 1 \\ 0, & \text{otherwise} \end{cases}
\]

Example 2.4. Clearly, a weakly commuting pair is compatible but every compatible pair need not be weakly commuting.

Lemma 2.7. Let \( (X, F) \) be a Menger space. If there exists some \( k \in (0, 1) \) such that for all \( n \geq 1 \),

\[
\lim_{n \to \infty} F_{\Delta}(x, y) = F(x, y)
\]

then \( F_{\Delta} \) is continuous at the point \( t \in X \) as \( n \to \infty \).

In respect of above mentioned t-norms, we have the following ordering:

\[
T_d < T_p < T_l < T_d
\]

Definition 2.5. Let \( (X, F, \Delta) \) be a Menger PM space \( (X, F) \) is a PM space and \( \Delta \) is a t-norm satisfying the following condition:

\[
F_{\Delta}(x) ≥ \Delta(F_{\Delta}(x), F_{\Delta}(y))
\]

Definition 2.6. [28] A sequence \( \{p_n\} \) in a Menger PM space \( (X, F, \Delta) \) is said to be convergent to a point \( p \in X \) if for every \( \varepsilon > 0 \) and \( \lambda > 0 \), there is an integer \( M(\varepsilon, \lambda) \) such that \( F_{\Delta}(p_n, p) > 1 - \lambda \) for all \( n ≥ M(\varepsilon, \lambda) \).

Lemma 2.7. [7, 9] Let \( (X, F, \Delta) \) be a Menger space with a continuous t-norm \( \Delta \) and \( \{x_n\}, \{y_n\} \subset X \) such that \( x_n \) converges to \( x \) and \( y_n \) converges to \( y \). If \( F_{\Delta}(x, y) \) is continuous at the point \( t \in X \), then

\[
\lim_{n \to \infty} F_{\Delta}(x_n, y_n) = F_{\Delta}(x, y)
\]

Definition 2.8. Let \( (A, S) \) be a pair of maps from a Menger PM space \( (X, F, \Delta) \) into itself. Then the pair of maps \( (A, S) \) is said to be weakly commuting if \( F_{\Delta, S}(x, y) ≥ F_{\Delta, S}(x, z) \) for each \( x, y, z \in X \) and \( t > 0 \).

Definition 2.9. [34] A pair \( (A, S) \) of self mappings of a Menger PM space \( (X, F, \Delta) \) is said to be compatible if \( F_{\Delta, S}(p_n, q_n) ≥ F_{\Delta, S}(p, q) \) for all \( n = 1 \) and \( t > 0 \).

Definition 2.10. [19] A pair \( (A, S) \) of self mappings of a Menger PM space \( (X, F, \Delta) \) is said to be non-compatible if and only if there exists at least one sequence \( \{x_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} A_{x_n} = \lim_{n \to \infty} S_{x_n} = t \in X
\]

Implies that \( \lim_{n \to \infty} F_{\Delta, S}(x_n, t) \) (for some \( t > 0 \)) is either less than 1 or non-existent.

Definition 2.11. [15] A pair \( (A, S) \) of self mappings of a Menger PM space \( (X, F, \Delta) \) is said to satisfy the property \( (E.A) \) if there exists a sequence \( \{x_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} A_{x_n} = \lim_{n \to \infty} S_{x_n} = t \in X
\]

Clearly, a pair of compatible mappings as well as noncompatible Mappings satisfies the property \( (E.A) \).

Inspired by Liu et al. [39], Imdad et al. [26] defined the following:

Definition 2.12. Two pairs \( (A, S) \) and \( (B, T) \) of self mappings of a Menger PM space \( (X, F, \Delta) \) are said to satisfy the common property \( (E.A) \) if there exist two sequences \( \{x_n\}, \{y_n\} \subset X \) and some \( t \in X \) such that

\[
\lim_{n \to \infty} A_{x_n} = \lim_{n \to \infty} S_{x_n} = t = \lim_{n \to \infty} B_{y_n} = \lim_{n \to \infty} T_{y_n}
\]

Definition 2.13. [12] A pair \( (A, S) \) of self mappings of a nonempty set \( X \) is said to be weakly compatible if the pair commutes on the set of their coincidence points i.e. \( A_p = S_p \) for some \( p \in X \) implies \( A S_p = S A_p \).

Definition 2.14. [24] Two finite families of self mappings \( \{A_i\} \) and \( \{B_i\} \) are said to be pairwise commuting if:

(i) \( A_i A_j = A_j A_i \), \( i, j \in \{1, 2...m\} \),
(ii) \( B_i B_j = B_j B_i \), \( i, j \in \{1, 2...n\} \),
(iii) \( A_i B_j = B_j A_i \), \( i \in \{1, 2...m\}, j \in \{1, 2...n\} \).

3. Main Result

The following lemma is useful for the proof of succeeding theorems.

Lemma 3.1. Let \( (X, F, \Delta) \) be a Menger space. If there exists some \( k \in (0, 1) \) such that for all \( p, q \in X \) and all \( x > 0 \),

\[
\int_0^x f_p(x) \phi(t) dt ≥ \int_0^x f_q(x) \phi(t) dt
\]

where \( \phi : [0, \infty) \to [0, \infty) \) is a summable non-negative Lebesgue integrable function such that

\[
\int_0^x \phi(t) dt > 0 \quad \text{for each} \quad t \in [0, 1], \quad \text{then} \quad p = q.
\]
Lemma 3.3. Let A, B, S and T be four self mappings of a Menger space (X, F, Δ) which satisfy the following conditions:
(i) the pair (A, S) (or (B, T)) satisfies the property (E.A),
(ii) B(yₙ) converges for every sequence {yₙ} in X whenever T(yₙ) converges,
(iii) for any p, q ∈ X and for all x > 0,
\[ \int_0^t \min[F_{Sp, Tq}(x), F_{Sp, Ap}(x), F_{Tq, Bq}(x), F_{Sp, Bq}(x), F_{Tq, Ap}(x)] \phi(t) dt \geq \int_0^t F_{Sp, Bq}(x) \phi(t) dt \]
where \( \phi : [0, \infty) \to [0, \infty) \) is a non-negative summable Lebesgue integrable function such that \( \int_0^1 \phi(t) dt > 0 \) for each \( t \in [0, 1) \), where \( 0 < k < 1 \) and
(iv) A(X) ⊂ T(X) (or B(X) ⊂ S(X)).
Then the pairs (A, S) and (B, T) share the common property (E.A).

**Proof.** Suppose that the pair (A, S) enjoys the property (E.A), then there exists a sequence \( \{x_n\} \) in X such that
\[ \lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = t, \]
for some \( t \in X \).
Since A(X) ⊂ T(X), for each \( x_n \) there exists \( y_n \in X \) such that \( A x_n = T y_n \) and hence
\[ \lim_{n \to \infty} T y_n = \lim_{n \to \infty} A x_n = t \]
Thus in all, we have \( A x_n \to t, S x_n \to t \) and \( T y_n \to t \). Now we assert that \( B y_n \to t \).
To accomplish this, using (3.3.1), with \( p = x_n, q = y_n \), one gets
\[ \int_0^t \min[F_{Sx_n, By_n}(x_k), F_{Sx_n, Ax_n}(x), F_{Tyn, By_n}(x), F_{Sx_n, By_n}(x), F_{Tyn, Ax_n}(x)] \phi(t) dt \]
Let \( \lim_{n \to \infty} B(y_n) \). Also, let \( x > 0 \) be such that \( F_{l, \cdot}(\cdot) \) is continuous in \( x \) and \( kx \). Then, on making \( n \to \infty \) in the above inequality, we obtain
\[ \int_0^{F_{l, \cdot}(kx)} \phi(t) dt \geq \int_0^{F_{l, \cdot}(x)} \phi(t) dt \]
or
\[ \int_0^{F_{l, \cdot}(kx)} \phi(t) dt \geq \int_0^{F_{l, \cdot}(x)} \phi(t) dt \]
This implies that \( l = t \) (in view of Lemma 3.1) which shows that the pairs (A, S) and (B, T) share the common property (E.A).

**Theorem 3.4.** Let A, B, S and T be self mappings of a Menger space (X, F, Δ) which satisfy the inequality (3.3.1) together with
(i) the pairs (A, S) and (B, T) share the common property (E.A),
(ii) S(X) and T(X) are closed subsets of X.
Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

**Proof.** Since the pairs (A, S) and (B, T) share the common property (E.A), there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in X such that
\[ \lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T y_n = \lim_{n \to \infty} B y_n = t, \]
for some \( t \in X \).
Since S(X) is a closed subset of X, hence \( \lim_{n \to \infty} S x_n = t \in S(X) \). Therefore, there exists a point \( u \in X \) such that \( S u = t \). Now, we assert that \( A u = Su \). To prove this, on using (3.3.1) with \( p = u, q = y_n \), one gets
\[
\int_0^F_{A_u,B_y}(kx) \phi(t) dt \\
\geq \int_0^F_{S_u,T_y}(x)F_{S_u,A_u}(x)F_{T_y,B_y}(x)F_{T_y,A_u}(x) + F_{T_y,B_y}(x)F_{T_y,A_u}(x) \phi(t) dt
\]

which on making \( n \to \infty \), reduces to

\[
\int_0^F_{A_u}(kx) \phi(t) dt \geq \int_0^F_{S_u}(x)F_{S_u,A_u}(x) + F_{S_u}(x)F_{A_u}(x) \phi(t) dt
\]

or

\[
\int_0^F_{A_u}(kx) \phi(t) dt \geq \int_0^F_{A_u}(x) \phi(t) dt
\]

Now appealing to Lemma 3.1, we have \( A_u = t \) and hence \( A_u = Su \). Therefore, \( u \) is a coincidence point of the pair \( (A, S) \). Since \( T(X) \) is a closed subset of \( X \), therefore \( \lim_{n \to \infty} T_y = t \in T(X) \) and hence one can find a point \( w \in X \) such that \( Tw = t \). Now we show that \( Bw = Tw \). To accomplish this, on using (3.3.1) with \( p = x_n, q = w \), we have

\[
\int_0^F_{A_x,B_w}(kx) \phi(t) dt \\
\geq \int_0^F_{S_x,T_w}(x)F_{S_x,B_w}(x)F_{T_w,B_w}(x)F_{T_w,A_x}(x) + F_{T_w,B_w}(x)F_{T_w,A_x}(x) \phi(t) dt
\]

which on making \( n \to \infty \), reduces to

\[
\int_0^F_{L,B_w}(kx) \phi(t) dt \geq \int_0^F_{L,B_w}(x) \phi(t) dt
\]

or

\[
\int_0^F_{L,B_w}(kx) \phi(t) dt \geq \int_0^F_{L,B_w}(x) \phi(t) dt
\]

On employing Lemma 3.1, we have \( Bw = t \) and hence \( Tw = BW \). Therefore, \( w \) is a coincidence point of the pair \( (B, T) \).

Since the pair \( (A, S) \) is weakly compatible and \( A_u = Su \), therefore \( A_t = ASu = SAu = St \).

Again, on using (3.3.1) with \( p = t, q = w \), we have

\[
\int_0^F_{A_t,B_w}(kx) \phi(t) dt \\
\geq \int_0^F_{S_t,T_w}(x)F_{S_t,B_w}(x)F_{T_w,B_w}(x)F_{T_w,A_t}(x) + F_{T_w,B_w}(x)F_{T_w,A_t}(x) \phi(t) dt
\]

or

\[
\int_0^F_{A_t}(kx) \phi(t) dt \geq \int_0^F_{A_t}(x)F_{A_t}(x) + F_{A_t}(x)F_{A_t}(x) \phi(t) dt
\]

or

\[
\int_0^F_{A_t}(kx) \phi(t) dt \geq \int_0^F_{A_t}(x) \phi(t) dt
\]
Appealing to Lemma 3.1, we have $At = St = t$ which shows that $t$ is a common fixed point of the pair $(A, S)$. Also the pair $(B, T)$ is weakly compatible and $Bw = Tw$, hence $Bt = BTw = TBw = Tt$.

Next, we show that $t$ is a common fixed point of the pair $(B, T)$. In order to accomplish this, using (3.3.1) with $p = u$, $q = t$, we have

$$
\int_0^\frac{F_{Au,Bt}(kx)}{F_{Bu,Ct}(kx)} \phi(t) \, dt \\
\geq \int_0^\frac{\min(F_{Su,Tl}(x), F_{Su,Au}(x), F_{TL,Bt}(x), F_{Su,Bt}(x), F_{TL,Au}(x))}{F_{Su,Bt}(x)} \frac{F_{Su,Tl}(x)F_{TL,Bt}(x)F_{Su,Au}(x)}{F_{Su,Bt}(x)} \phi(t) \, dt \\
\text{or}

$$
\int_0^\frac{F_{TL,Bt}(kx)}{F_{BL,Bt}(kx)} \phi(t) \, dt \\
\geq \int_0^\frac{\min(F_{TL,Bt}(x), F_{TL,Tl}(x), F_{BL,Bt}(x), F_{BL,Tl}(x), F_{BL,Bt}(x), F_{BL,Tl}(x))}{F_{BL,Bt}(x)} \frac{F_{BL,Bt}(x)F_{BL,Bt}(x)F_{BL,Tl}(x)}{F_{BL,Bt}(x)} \phi(t) \, dt \\
\text{or}

$$
\int_0^\frac{F_{TL,Bt}(kx)}{F_{BL,Bt}(kx)} \phi(t) \, dt \geq \int_0^\frac{F_{TL,Bt}(x)}{F_{BL,Bt}(x)} \phi(t) \, dt \\
\text{or}

$$
\text{or}

Using Lemma 3.1, we have $Bt = t$ which shows that $t$ is a common fixed point of the pair $(B, T)$. Hence $t$ is a common fixed point of both the pairs $(A, S)$ and $(B, T)$. Uniqueness of common fixed point is an easy consequence of the inequality (3.3.1). This completes the proof.

**Theorem 3.5.** Let $A, B, S$ and $T$ be self mappings of a Menger space $(X, F, \Delta)$ satisfying the inequality (3.3.1). Suppose that

(i) the pair $(A, S)$ (or $(B, T)$) has property (E.A),
(ii) $B(y_n)$ converges for every sequence $\{y_n\}$ in $X$ whenever $T(y_n)$ converges,
(iii) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$),
(iv) $S(X)$ (or $T(X)$) is a closed subset of $X$.

Then the pairs $(A, S)$ and $(B, T)$ have a point of coincidence each. Moreover, $A, B, S$ and $T$ have a unique common fixed point provided both the pairs $(A, S)$ and $(B, T)$ are weakly compatible.

**Proof.** In view of Lemma 3.3, the pairs $(A, S)$ and $(B, T)$ share the common property (E.A), i.e. there exists two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$
\lim_{n \to \infty} A x_n = \lim_{n \to \infty} A y_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T y_n = \lim_{n \to \infty} B y_n = t \text{ for some } t \in X.
$$

If $S(X)$ is a closed subset of $X$, then proceeding on the lines of Theorem 3.5, one can show that the pair $(A, S)$ has a coincidence point, say $u$, i.e. $Au = Su = t$.

Since $A(X) \subset T(X)$ and $Au \in A(X)$, there exists $w \in X$ such that $Au = Tw$. Now, we assert that $Bw = Tw$.

On using (3.3.1) with $p = x_n$, $q = w$, one gets

$$
\int_0^\frac{F_{Ax_n,Bw}(kx)}{F_{Bw,Cx_n}(kx)} \phi(t) \, dt \\
\geq \int_0^\frac{\min(F_{SSx_n,Tw}(x), F_{SSx_n,Ax_n}(x), F_{TW,Bw}(x), F_{SSx_n,Bw}(x), F_{TW,Ax_n}(x))}{F_{SSx_n,Bw}(x)} \frac{F_{SSx_n,Tw}(x)F_{TW,Bw}(x)F_{SSx_n,Bw}(x)F_{TW,Ax_n}(x)}{F_{SSx_n,Bw}(x)} \phi(t) \, dt \\
\text{or}

$$

which on making $n \to \infty$, reduces to
Owing to Lemma 3.1, we have $t = Bw$ and hence $T_w = Bw$ which shows that $w$ is a coincidence point of the pair $(B, T)$. Rest of the proof can be completed on the lines of the proof of Theorem 3.4. This completes the proof.

**Corollary 3.6.** Let $A$ and $S$ be self mappings on a Menger space $(X,F,\Delta)$. Suppose that
(i) the pair $(A, S)$ enjoys the property (E.A),
(ii) for all $p, q \in X$ and for all $x > 0$,
\[
\int_0^{F_{Ap,Aq}(kx)} \phi(t) dt \geq \int_0^{\min(F_{Sp,Sq}(x),F_{Sp,Ap}(x),F_{Sq,Aq}(x),F_{Sp,Aq}(x),F_{Sq,Ap}(x)) F_{Sp,Sq}(x) F_{Sq,Aq}(x) F_{Sp,Aq}(x) F_{Sp,Ap}(x) F_{Sq,Ap}(x)} \phi(t) dt,
\]

or
\[
\int_0^{F_{Ap}(kx)} \phi(t) dt \geq \int_0^{F_{Ap}(x)} \phi(t) dt
\]

(iii) $S(X)$ is a closed subset of $X$.

Then $A$ and $S$ have a coincidence point. Moreover, if the pair $(A, S)$ is weakly compatible, then $A$ and $S$ have a unique common fixed point.

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