Some Fixed point Theorems in Generalization Metric space

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Abstract
In this paper we establish some fixed point results for mappings satisfying sufficient contractive conditions on a complete G-metric space.

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1. Introduction
In 1992, Bapure Dhage in his Ph.D. thesis introduced the concept of a new class of generalized metric space called D-metric spaces[2-3]. In 2005 Mustafa and Sims[6] showed that most of the results concerning Dhage’s D-metric spaces are invalid. Therefore, they introduced an improved version of the generalized metric space structure, which are called G-metric spaces as generalization of metric space $\langle X,d \rangle$. To develop and to introduce a new fixed point theory for a variety of mappings in this new setting, also to extend known metric space theorems to a more general setting.

For more details on G-metric spaces, one can refer to the papers [6]-[9]. Now, we give preliminaries and basic definitions which are used throughout the paper.

In 2004, Mustafa and Sims [7] introduced the concept of G-metric spaces as follows:

Definition 1.1[7] Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying the following properties:

$$(G1) \quad G(x,y,x) = 0 \quad \text{if} \quad x = y = z;$$

$$(G2) \quad 0 < G(x,y,z) \quad \text{for all} \quad x, y, z \in X \quad \text{with} \quad x \neq y;$$

$$(G3) \quad G(x,y,z) \leq G(x,x,y) + G(x,y,z); \quad \text{for all} \quad x, y, z \in X \quad \text{with} \quad x \neq y;$$

$$(G4) \quad G(x,y,z) = G(x, z, y) = G(y, z, x) = \ldots \quad \text{(symmetry in all three variables);}$$

$$(G5) \quad G(x,y,z) \leq G(x,a,a) + G(a,y,z), \quad \text{for all} \quad x,y,z,a \in X, \quad \text{(rectangle inequality).}$$

Then the function $G$ is called a generalized metric, or, more specifically a G-metric on $X$, and the pair $(X,G)$ is called a G-metric space.

Definition 1.2. [7] Let $(X,G)$ be a G-metric space, and let $(x_n)$ be a sequence of points of $X$. A point $x \in X$ is said to be the limit of the sequence $(x_n)$ if

$$\lim_{n,m \to \infty} G(x_n,x_m) = 0,$$

and one says that the sequence $(x_n)$ is G-convergent to $x$. Thus, if $x_n \to x$ in a G-metric space $(X,G)$, then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x) < \varepsilon$, for all $n \geq N$. (we mean by $N$ the Natural numbers).

Proposition 1.3.[7] Let $(X,G)$ be G-metric space. Then the following are equivalent.

1. $(x_n)$ is G-convergent to $x$.
2. $G(x_n,x) \to 0$, as $n \to \infty$.
3. $G(x_n,x) \to 0$, as $n \to \infty$.
4. $G(x_n,x_n) \to 0$, as $m \to \infty$.

Definition 1.4.[7] Let $(X,G)$ be a G-metric space, a sequence $(x_n)$ is called G-Cauchy if given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_{n+1}) < \varepsilon$, for all $n,m \in \mathbb{N}$. Then $G(x_n,x_m) \to 0$ as $m,n \to \infty$.

Proposition 1.5.[7] In a G-metric space, $(X,G)$, the following are equivalent.

1. The sequence $(x_n)$ is G-Cauchy.
2. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_{n+m}) < \varepsilon$, for all $n,m \geq N$.

Definition 1.6. [7] A G-metric space $(X,G)$ is said to be G-complete ( or complete G-metric) if every G-Cauchy sequence in $(X,G)$ is G-convergent in $(X,G)$.

Proposition 1.7. [7] A G-metric space $(X,G)$ is G-complete if and only if $(X,d^G)$ is a complete metric space.

Theorem 1.8. [7] Let $(X,d)$ be a complete metric space, and $R$ be a function mapping $X$ into it self, satisfy the following condition.

$$d(R(x),R(y)) \leq ad(x,R(x)) + bd(y,R(y)) + cd(x,y), \quad \forall x,y \in X,$$

where $a,b,c$ are nonnegative numbers satisfying $a + b + c < 1$. Then, $R$ has a unique fixed point (i.e., there exists $u \in X$; $R(u) = u$).
3. Main Results

In this section, we will present several fixed point results on a complete G-metric space.

**Theorem 2.1.** Let \((X, G)\) be a complete G-metric space, and let \(R : X \rightarrow X\) be a mapping satisfies the following condition

\[
\begin{align*}
\alpha G(Rx, Ry, Rz) + \beta [G(x, Rx, Rx) + G(y, Ry, Ry) + G(z, Rz, Rz)] & \leq c G(x, y, z) \\
& \quad \text{for all } x, y, z \in X, \text{ where the constants } a, b, c \text{ satisfy } a, b, c > 0; \\
0 & < c < a + b; \quad a \neq 0.
\end{align*}
\]

**Proof.** Take an arbitrary and define a sequence \(x_{n+1} = x_n, \quad n = 0, 1, 2, \ldots\)

Substituting \(x = x_n, y = x_{n+1}, z = x_{n+2}\), then we have

\[
\begin{align*}
\alpha G(Rx_n, Rx_{n+1}, Rx_{n+2}) + \beta [G(x_n, Rx_n, Rx_n) + G(x_{n+1}, Rx_{n+1}, Rx_{n+1}) + G(x_{n+2}, Rx_{n+2}, Rx_{n+2})] & \leq c G(x_n, x_{n+1}, x_{n+2}) \\
\alpha G(x_{n+1}, x_{n+2}, x_{n+3}) + \beta [G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3})] & \leq c G(x_n, x_{n+1}, x_{n+2}) \\
\alpha G(x_{n+1}, x_{n+2}, x_{n+3}) + \beta G(x_n, x_{n+2}, x_{n+3}) & \leq c G(x_n, x_{n+1}, x_{n+2}) \\
G(x_{n+1}, x_{n+2}, x_{n+3}) & \leq \frac{c - b}{a} G(x_n, x_{n+1}, x_{n+2})
\end{align*}
\]

Since \(0 < c < a + b = 0 < c - b < a \Rightarrow 0 < \frac{c - b}{a} < 1\)

We assume that \(\frac{c - b}{a} = k\) then

\[
G(x_{n+1}, x_{n+2}, x_{n+3}) \leq k G(x_n, x_{n+1}, x_{n+2})
\]

Similarly we can show that

\[
G(x_n, x_{n+1}, x_{n+2}) \leq k G(x_{n-1}, x_n, x_{n+1})
\]

Processing \(n\) times

\[
G(x_{n+1}, x_{n+2}, x_{n+3}) \leq k^{n+1} G(x_0, x_1, x_2)
\]

Next we show that \(\{x_n\}\) is Cauchy sequence. Without loss of generality assume that \(n > m\), then

\[
\begin{align*}
G(x_n, x_m, x_{m+1}) & \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{m+2}) + G(x_{n+2}, x_{n+3}, x_{m+3}) + \ldots + G(x_{n-1}, x_m, x_{m+1}) \\
& \leq k^n G(x_0, x_1, x_2) + k^{n-1} G(x_0, x_1, x_2) + \ldots + k G(x_0, x_1, x_2) \\
& \leq \frac{k^n (1 + k + k^2 + \ldots + k^{n-m})}{1 - k^{-1}} G(x_0, x_1, x_2) \\
& \leq \lim_{n \to \infty} \frac{k^n}{1 - k^{-1}} G(x_0, x_1, x_2)
\end{align*}
\]

Hence, limit \(n \to \infty\)

\[
\lim_{n \to \infty} G(x_n, x_m, x_{m+1}) = 0
\]

i.e. \(\{x_n\}\) is Cauchy sequence.

Since \((X, G)\) is complete, so there exists \(w \in X\) such that \(x_n \to w\), which implies

\[
\lim_{n \to \infty} G(x_n, x_m, w) = 0.
\]

Next we will show that \(w\) is fixed point of \(R\). we take \(x = x_n\) and \(y = z = w\ in (2.1(i))\) then

\[
\begin{align*}
\alpha G(Rx_n, Rw, Rw) + \beta [G(x_n, Rx_n, Rx_n) + G(w, Rw, Rw)] & \leq c G(x_n, w, w) \\
\alpha G(x_{n+1}, Rw, Rw) + \beta [G(x_n, x_{n+1}, x_{n+1}) + 2 G(w, Rw, Rw)] & \leq c G(x_n, w, w)
\end{align*}
\]

As \(n \to \infty\) we have

\[
(a + 2b) G(w, Rw, Rw) \leq 0
\]

Which is contradiction, so \(Rw = w\) i.e. \(w\) is fixed point of \(R\). Uniqueness:

Let \(p\) and \(q\) are two more fixed points of \(R\), different from \(w\), i.e. \(w \neq p \neq q\).

We take \(x = w, y = p, z = q\) in (2.1(i)) then

\[
\begin{align*}
\alpha G(Rw, Rp, Rq) + \beta [G(w, Rw, Rw) + G(p, Rp, Rp) + G(q, Rq, Rq)] & \leq c G(w, p, q) \\
\Rightarrow G(w, p, q) & \leq \frac{1}{c} G(w, p, q)
\end{align*}
\]

Which is contradiction, so \(w = p = q\), i.e. \(w\) is unique fixed point of \(R\).

This complete the proof of theorem.

**Theorem 2.2.** Let \((X, G)\) be a complete G-metric space, and let \(R : X \rightarrow X\) be a mapping satisfies the following condition
\[
\min \{ G(x, y, z), G(x, z, y), G(z, x, y) \} \leq \alpha G(x, y, z) \quad \ldots \quad \ldots \quad \ldots \quad (2.2(i))
\]
for all \(x, y, z \in X\), where \(0 \leq \alpha < 1\).

**Proof.** Take an arbitrary and define a sequence \(x_{n+1} = x_n, \quad n = 0, 1, 2, \ldots\)
Substituting \(x = x_n, \quad y = x_{n+1}, \quad z = x_{n+1}\) in (2.2(i)) then we have
\[
\min \{ G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+1}, x_{n+1}) \} \leq \alpha G(x_n, x_{n+1}, x_{n+1})
\]
\[
\min \{ G(x_{n+1}, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+1}, x_{n+1}) \} \leq \alpha G(x_{n+1}, x_{n+1}, x_{n+1})
\]
\[
\min \{ G(x_{n+1}, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+1}, x_{n+1}) \} \leq \alpha G(x_{n+1}, x_{n+1}, x_{n+1})\]
If we take
\[
\min \{ G(x_{n+1}, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+1}, x_{n+1}) \} = G(x_{n+1}, x_{n+1}, x_{n+1})
\]
Then from (2.2(ii))
\[
G(x_{n+1}, x_{n+1}, x_{n+1}) \leq \alpha G(x_n, x_{n+1}, x_{n+1})
\]
And if we take
\[
\min \{ G(x_{n+1}, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+1}, x_{n+1}) \} = G(x_n, x_{n+1}, x_{n+1})
\]
Then
\[
G(x_n, x_{n+1}, x_{n+1}) \leq \alpha G(x_n, x_{n+1}, x_{n+1})
\]
Which is contradiction, so that
\[
G(x_n, x_{n+1}, x_{n+1}) \leq \alpha G(x_n, x_{n+1}, x_{n+1})
\]
Similarly we can show that
\[
G(x_n, x_{n+1}, x_{n+1}) \leq \alpha G(x_n, x_{n+1}, x_{n+1})
\]
Next we show that \(\{x_n\}\) is Cauchy sequence. Without loss of generality assume that \(n > m\).
Then
\[
G(x_n, x_m) \leq G(x_n, x_{n+1}) + G(x_{n+1}, x_{n+1}) + \ldots + G(x_m, x_{n+1}) + k^m G(x_0, x_0, x_0)
\]
\[
\leq k^m G(x_0, x_0, x_0) + k^m + k^m + \ldots + k^m + k^m = \frac{k^m}{1-k^m} G(x_0, x_0, x_0)
\]
\[
\lim_{n,m \to \infty} G(x_n, x_m) = 0
\]
i.e. \(\{x_n\}\) is Cauchy sequence.
Since \((X, G)\) is complete \(G\)-metric space which gives \(w \in X\) such that \(x_n \to w\) as \(n \to \infty\).
Next we will show that \(w\) is fixed point of \(R\) for this we take \(x = x_n\) and \(y = z = w\) in (2.2(i)) then
\[
\min \{ G(x_n, w, w), G(x_n, w, w), G(x_n, w, w) \} \leq \alpha G(x_n, w, w)
\]
\[
\min \{ G(x_n, w, w), G(x_n, w, w), G(x_n, w, w) \} \leq \alpha G(x_n, w, w)
\]
As \(n \to \infty\) we have
\[
\min \{ G(Rw, Rw, Rw), G(w, Rw, Rw), G(w, Rw, Rw) \} \leq \alpha G(w, w, w)
\]
Which is contradiction, so \(Rw = w\) i.e. \(w\) is fixed point of \(R\).

**Uniqueness:**
Let \(p\) and \(q\) are two more fixed points of \(R\), different from \(w\), i.e. \(w \neq p \neq q\).
We take \(x = w, y = p, z = q\) in (2.2(i)) then
\[
\min \{ G(Rw, Rp, Rq), G(wp, Rw, Rw), G(p, Rp, Rp), G(q, Rq, Rq) \} \leq \alpha G(w, p, q)
\]
\[
\min \{ G(Rw, Rp, Rq), G(wp, Rw, Rw), G(p, Rp, Rp), G(q, Rq, Rq) \} \leq \alpha G(w, p, q)
\]
Which is contradiction, so \(w = p = q\), i.e. \(w\) is unique fixed point of \(R\).

This complete the proof of theorem .

**Theorem 2.3.** Let \((X, G)\) be a complete \(G\)-metric space, and let \(R : X \to X\) be a mapping satisfies the following condition
\[
\min \{ G(x, y, z), G(x, z, y), G(z, x, y) \} \leq \alpha G(x, y, z) \quad \ldots \quad \ldots \quad \ldots \quad (2.3(i))
\]
for all \(x, y, z \in X\), where \(0 \leq \alpha < 1\).

**Proof.** Take an arbitrary and define a sequence \(x_{n+1} = x_n, \quad n = 0, 1, 2, \ldots\)
Substituting \(x = x_n, \quad y = x_{n+1}, \quad z = x_{n+1}\) in (2.3(i)) then we have
\[
\min \{ G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}) \} \leq \alpha G(x_n, x_{n+1}, x_{n+1})
\]
\[
\min \{ G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}) \} \leq \alpha G(x_n, x_{n+1}, x_{n+1})
\]
Case I: If we take
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
Then from (2.3(ii))
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
Case II:
if we take
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
Then
\[
G(x_{n+1}, x_{n+2}, x_{n+3}) \leq \min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
Case III:
if we take
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
Then
\[
G(x_{n+1}, x_{n+2}, x_{n+3}) \leq \min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
Case IV:
if we take
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
Then
\[
G(x_{n+1}, x_{n+2}, x_{n+3}) \leq \min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
\[
\min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
Similarly we can show that
\[
G(x_{n+1}, x_{n+2}, x_{n+3}) \leq \min \left\{ \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right), \left( x_{n+1}, x_{n+2}, x_{n+3} \right) \right\}
\]
Next we show that \(G_{n+1}\) is Cauchy sequence. Without loss of generality assume that \(n > m\).

Then
\[
G(x_{n+1}, x_{n+1}) \leq G(x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+1})
\]
\[
\leq G(x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+1}) + \ldots + G(x_{n+1}, x_{n+1})
\]
\[
\leq k^m G(x_0, x_1) + k^{m-1} G(x_0, x_1) + \ldots + k G(x_0, x_1) + G(x_0, x_1)
\]
\[
\leq k^m \frac{1}{k \times \cdots \times k} G(x_0, x_1)
\]
Hence, limit \( m, n \to \infty \)
\[
\lim_{m, n \to \infty} G(x_n, x_n, x_n) = 0
\]
i.e. \( \{x_n\} \) is Cauchy sequence.

Since \((X, G)\) is complete \( G \) – metric space which gives \( w \in X \) such that \( \{x_n\} \to w \),
as \( n \to \infty \).

Next we will show that \( w \) is fixed point of \( R \). For this we take \( x = x_n \) and
\[
y = z = w \text{ in (2.3(i)) then}
\]
\[
\min \{G(Rx_n, Rw, Rw), G(x_n, Rx_n, Rx_n), G(w, Rw, Rw), G(w, Rw, Rw)\} \leq \alpha G(x_n, w, w)
\]
\[
\min \{G(Rx_n, Rw, Rw), G(x_n, Rx_n, Rx_n), G(w, Rw, Rw)\} \leq \alpha G(x_n, w, w)
\]
As \( n \to \infty \) we have
\[
\min \{G(Rw, Rw, Rw), G(w, Rw, Rw), G(w, Rw, Rw)\} \leq \alpha G(w, w, w)
\]

Which is contradiction, so \( Rw = w \) i.e. \( w \) is fixed point of \( R \).

**Uniqueness:**
Let \( p \) and \( q \) are two more fixed points of \( R \), different from \( w \), i.e. \( w \neq p \neq q \).

We take \( x = w, y = p, z = q \) in (2.3(i)) then
\[
\min \{G(Rw, Rp, Rq), G(w, Rw, Rw), G(p, Rp, Rp), G(q, Rq, Rq)\} \leq \alpha G(w, p, q)
\]
\[
\min \{G(Rw, Rw, Rw), G(w, Rw, Rw), G(w, Rw, Rw)\} \leq \alpha G(w, w, w)
\]
Which is contradiction, so \( w = p = q \), i.e. \( w \) is unique fixed point of \( R \).

This complete the proof of theorem.

**References**

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