Double Weighted Distribution & Double Weighted Exponential Distribution

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Abstract

Since the widely using of the weighted distribution in many fields of real life such various areas including medicine, ecology, reliability, and so on, then we try to shed light and record our contribution in this field thru the research. Derivation Double Weighted distribution and Double Weighted Exponential distribution with some of statistical properties is discussed in this paper.

Keywords: Weighted distribution, Double Weighted distribution, Double Weighted Exponential distribution.

1. Introduction

Weighted distribution provides an approach to dealing with model specification and data interpretation problems. Fisher (1934) and Rao (1965) introduced and unified the concept of weighted distribution. Fisher (1934) studied on how methods of ascertainment can influence the form of distribution of recorded observations and then Rao (1965) introduced and formulated it in general terms in connection with modeling statistical data where the usual practice of using standard distributions for the purpose was not found to be appropriate. In Rao' s paper, he identified various situations that can be modeled by weighted distributions. These situations refer to instances where the recorded observations cannot be considered as a random sample from the original distributions. This may occur due to non-observability of some events or damage caused to the original observation resulting in a reduced value, or adoption of a sampling procedure which gives unequal chances to the units in the original.

Weighted distributions occur frequently in research related to reliability bio-medicine, ecology and branching processes can be seen in Patil and Rao (1978), Gupta and Kirmani(1990), Gupta and Keating(1985), Oluyede (1999) and in references there in. Within the context of cell kinetics and the early detection of disease, Zelen (1974) introduced weighted distributions to represent twhat he broadly perceived as length-biased sampling (introduced earlier in Cox, D.R. (1962)). For additional and important results on weighted distributions, see Rao (1997), Patil and Ord(1997), Zelen and Feinleib (1969), see El-Shaarawi and Walter (2002) for application examples for weighted distribution, and there are many researches for weighted distribution as, Priyadarshani (2011) introduced a new class of weighted generalized gamma distribution and related distribution, theoretical properties of the generalized gamma model, Jing (2010) introduced the weighted inverse Weibull distribution and beta-inverse Weibull distribution, theoretical properties of them, Castillo and Perez-Casany (1998) introduced new exponential families, that come from the concept of weighted distribution, that include and generalize the poisson distribution, Shaban and Boudrissa (2000) have shown that the length-biased version of the Weibull distribution known as Weibull Length-biased (WLB) distributin is unimodal throughout examining its shape, with other properties, Das and Roy (2011) discussed the length-biased Weighted Generalized Rayleigh distribution with its properties, also they are develop the length-biased from of the weighted Weibull distribution see Das and Roy (2011). On Some Length-Biased Weighted Weibull Distribution, Patil and Ord (1976), introduced the concept of size-biased sampling and weighted distributions by identifying some of the situations where the underlying models retain their form. For more important results of weighted distribution you can see also (Oluyede and George (2000), Ghitany and Al-Mutairi (2008), Ahmed ,Reshi and Mir (2013), Broderick X. S., Oluyede and Pararai (2012), Oluyede and Terbeche M (2007)).

Suppose X is a non-negative random variable with its pdf $f(x)$, then the pdf of the weighted random variable X_w is given by:

$$f_w(x) = \frac{w(x)f(x)}{\mu_w}, \quad x > 0 \qquad (1)$$

Where $w(x)$ be a non-negative weight function and $\mu_w = E(w(X)) < \infty$.

When we use weighted distributions as a tool in the selection of suitable models for observed data is the choice of the weight function that fits the data. Depending upon the choice of weight function (x), we have different weighted models. For example, when $w(x) = x$, the resulting distribution is called length-biased. In this case, the pdf of a length-biased (rv) X_L is defined as

$$f_L(x) = \frac{xf(x)}{\mu} \tag{2}$$

Where $\mu = E(X) < \infty$ .More generally, when $w(x) = x^c$; $c > 0$, then the resulting distribution is called size-biased. This type of sampling is a generalization of length-biased sampling and majority of the literature is centered on this weight function. Denoting $\mu_c = E(x^c) < \infty$, distribution of the size-biased (rv)$X_s$of order c is specified by the pdf

$$f_s(x) = \frac{x^c f(x)}{\mu_c} \tag{3}$$

Clearly, when $c = 1$, (1) reduces to the pdf of a length-biased (rv).
In this paper we present the Double Weighted distribution (DWD), taking one type of weighted function, $w(x) = x$, and using the exponential distribution as original distribution, then we derive the pdf, cdf, and some other useful distributional properties.

2. Double Weighted Distribution

Definition(2.1): The Double Weighted distribution (DWD) is given by:

$$f_w(x; c) = \frac{w(x)\, f(x) F(cx)}{W_D} \;, \qquad x \geq 0, c > 0 \tag{4}$$

Where $W_D = \int_0^\infty w(x)\, f(x) F(cx) dx$

And the first weight is $w(x)$ and the second is (cx) , $F(cx)$ depend on the original distribution $f(x)$.

3. Double Weighted Exponential Distribution

Consider the first weight function $w(x) = x$ and the pdf of Exponential distribution given by :-

$$f(x) = \lambda e^{-\lambda x},\ x \geq 0,\ \lambda > 0 \quad \text{and} \quad F(cx; \lambda) = 1 - e^{-c\lambda x},\ c > 0$$

And $W_D = \int_0^\infty w(x)\, f(x) F(cx) dx = \int_0^\infty x\lambda e^{-\lambda x}\left(1 - e^{-c\lambda x}\right) dx = \frac{(c+1)^2 - 1}{(c+1)^2 \lambda}$

Then the pdf of Double Weighted Exponential distribution (DWED) is:

$$g_w(x; \lambda, c) = \frac{(c+1)^2 \lambda^2}{(c+1)^2 - 1} x e^{-\lambda x}\left(1 - e^{-c\lambda x}\right), x \geq 0,\ \lambda, c > 0 \tag{5}$$

The cdf of DWED is:

$$G_w(x; \lambda, c) = \frac{(c+1)^2 \lambda^2}{(c+1)^2 - 1} \int_0^x x e^{-\lambda t}\left(1 - e^{-c\lambda t}\right) dt$$

$$= \frac{(c+1)^2 \lambda^2}{(c+1)^2 - 1}\left[\frac{e^{-\lambda t}}{\lambda}\left(\frac{e^{-c\lambda t}}{(c+1)}\left(x + \frac{1}{\lambda(c+1)}\right) - \frac{\lambda x + 1}{\lambda}\right)\right] + 1 \tag{6}$$

Using $\int_0^x t e^{-\lambda t} dt = \frac{1}{\lambda^2} - \frac{1}{\lambda^2} e^{-\lambda t} - \frac{1}{\lambda} x e^{-\lambda t}$ and

$$\int_0^x t e^{-(c+1)\lambda t} dt = \frac{1}{(c+1)^2 \lambda^2} - \frac{1}{(c+1)\lambda} x e^{-(c+1)\lambda t} - \frac{1}{(c+1)^2 \lambda^2} e^{-(c+1)\lambda t}$$

4. The Shape

The shape of the density function given in (5) can be clarified by studying this function defined over the positive real line $[0, \infty)$ and the behavior of its derivative as follows:

4.1 Limit and Mode of the function

Note that the limits of the Density function given in (5) is as follow:-

$$\lim_{x \to 0} g_w(x; \lambda, c) = \lim_{x \to 0} \frac{(c+1)^2\lambda^2}{(c+1)^2-1} x e^{-\lambda x}\left(1 - e^{-c\lambda x}\right) = 0 \qquad (7)$$

$$\lim_{x \to \infty} g_{w_1}(x; \lambda, c) = \frac{(c+1)^2\lambda^2}{(c+1)^2-1} \lim_{x \to \infty} x e^{-\lambda x}\left(1 - e^{-c\lambda x}\right) = 0 \qquad (8)$$

Since $\lim_{x \to \infty} e^{-\lambda x} = 0$ and $\lim_{x \to \infty}\left(1 - e^{-c\lambda x}\right) = 1$

From these limits, we conclude the pdf of DWED when $w(x) = x$ has one mode say x_0 as follow:

Note that $\text{Log}\, g_w(x; \lambda, c) = \text{Log}\left(\frac{(c+1)^2\lambda^2}{(c+1)^2-1}\right) + \text{Log}x - \lambda x + \text{Log}\left(1 - e^{-c\lambda x}\right) \qquad (9)$

Differentiating equation (9) with respect to x, we obtain

$$\frac{\partial}{\partial x}\text{Log}\, g_w(x; \lambda, c) = \frac{1}{x} - \lambda + \frac{c\lambda e^{-c\lambda x}}{1 - e^{-c\lambda x}}$$

The mode of the Double Weighted Exponential distribution is obtained by solving the following nonlinear equation with respect to x

$$\frac{1}{x} - \lambda + \frac{c\lambda e^{-c\lambda x}}{1 - e^{-c\lambda x}} = 0 \qquad (10)$$

The mode of Double Weighted Exponential distribution is given by **Table 1.**

The reliability function, hazard function and reverse hazard function of DWED are respectively given by:

$$R_{g_w}(x; \lambda, c) = 1 - G_w(x; \lambda, c) = \frac{(c+1)^2\lambda^2}{(c+1)^2-1}\left[\frac{e^{-\lambda t}}{\lambda}\left(\frac{\lambda x+1}{\lambda} - \frac{e^{-c\lambda t}}{(c+1)}\left(x + \frac{1}{\lambda(c+1)}\right)\right)\right] \qquad (11)$$

$$h_{g_w}(x; \lambda, c) = \frac{g_w(x;\lambda,c)}{R(x;\lambda,c)} = \frac{\lambda^2(c+1)^2 x\left(1-e^{-c\lambda x}\right)}{(\lambda x+1)(c+1)^2 - e^{-c\lambda x}[\lambda(c+1)x+1]} \qquad (12)$$

$$\varphi_{g_w}(x; \lambda, c) = \frac{g_w(x;\lambda,c)}{G_w(x;\lambda,c)} = \frac{(c+1)^2\lambda^2\, x\, e^{-\lambda x}\left(1-e^{-c\lambda x}\right)}{\lambda e^{-c\lambda x}\left((c+1)x e^{-\lambda x}+1\right) - (c+1)^2[\lambda(\lambda x+1)+1] - 1} \qquad (13)$$

5. Moment Of DWED

The k^{th} moment of DWED is given by:

$$E_{gw}(x^k) = \frac{\Gamma(k+2)[(c+1)^{k+2}-1]}{\lambda^k(c+1)^k[(c+1)^2-1]} = \frac{\Gamma(k+2)\varepsilon_{k+2}}{\lambda^k(c+1)^k\varepsilon_2}, \quad k = 1,2,\ldots \qquad (14)$$

Where $\varepsilon_s = [(c+1)^s - 1]$, $s = 1,2,3,\ldots$

Proof:

Let $y = (c+1)\lambda x \Longrightarrow x = \frac{y}{(c+1)\lambda}$, $dx = \frac{1}{(c+1)\lambda}dy$

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Then $\int_0^\infty x^{k+1}e^{-(c+1)\lambda x}dx = \int_0^\infty \frac{y^{k+1}}{\lambda^{k+1}(c+1)^{k+1}} \times \frac{1}{(c+1)\lambda} \times \mathrm{e}^{-\mathrm{y}}\mathrm{dy}$

$$= \frac{1}{\lambda^{k+2}(c+1)^{k+2}}\int_0^\infty y^{k+1}\,\mathrm{e}^{-\mathrm{y}}\mathrm{dy} = \frac{\Gamma(\mathrm{k}+2)}{\lambda^{k+2}(c+1)^{k+2}} \quad \text{And} \quad \int_0^\infty x^{k+1}e^{-\lambda x}dx = \frac{\Gamma(\mathrm{k}+2)}{\lambda^{k+2}}$$

Then $E_{gw}(x^k) = \frac{(c+1)^2\lambda^2}{(c+1)^2-1}\int_0^\infty x^{k+1}e^{-\lambda x}\left(1-e^{-c\lambda x}\right)dx$

$$= \frac{(c+1)^2\lambda^2}{(c+1)^2-1}\left[\int_0^\infty x^{k+1}e^{-\lambda x}dx - \int_0^\infty x^{k+1}e^{-(c+1)\lambda x}dx\right]$$

$$= \frac{(c+1)^2\lambda^2}{(c+1)^2-1}\left[\frac{\Gamma(\mathrm{k}+2)}{\lambda^{k+2}} - \frac{\Gamma(\mathrm{k}+2)}{\lambda^{k+2}(c+1)^{k+2}}\right] = \frac{(\alpha+1)^{k+1}\Gamma(\mathrm{k}+2)-\Gamma(\mathrm{k}+2)}{\lambda^k(c+1)^k[(c+1)^2-1]}$$

$$= \frac{\Gamma(\mathrm{k}+2)[(\mathrm{c}+1)^{\mathrm{k}+2}-1]}{\lambda^k(c+1)^k[(\mathrm{c}+1)^2-1]} = \frac{\Gamma(\mathrm{k}+2)\varepsilon_{k+2}}{\lambda^k(c+1)^k\varepsilon_2} \qquad \blacksquare$$

Then from equation (14) we can find mean, variance, coefficient of variation, skewness and kurtosis as follow:

Mean:

$$\mu_{gw}(x) = \frac{2[(\mathrm{c}+1)^3-1]}{\lambda(\mathrm{c}+1)[(\mathrm{c}+1)^2-1]} = \frac{2\varepsilon_3}{\lambda(c+1)\varepsilon_2}\ , \quad x>0, \lambda, c>0 \tag{15}$$

Variance:

$$\sigma^2{}_{gw}(x) = \frac{6[(c+1)^2-1][(c+1)^4-1]-4[(c+1)^3-1]^2}{\lambda^2(c+1)^2[(c+1)^2-1]^2} = \frac{6\varepsilon_2\varepsilon_4-4\varepsilon^2{}_3}{\lambda^2(c+1)^2\varepsilon^2{}_2} \tag{16}$$

Coefficient of Variation:

$$CV_{gw} = \frac{\left[3\left((c+1)^4-1\right)\left((c+1)^2-1\right)-2\left((c+1)^3-1\right)^2\right]^{\frac{1}{2}}}{2\left((c+1)^3-1\right)} = \frac{[3\varepsilon_4\varepsilon_2-2\varepsilon^2{}_3]^{\frac{1}{2}}}{2\varepsilon_3} \tag{17}$$

Coefficient of skewness:

$$CS_{gw} = \frac{4[\varepsilon^2{}_2\varepsilon_5-9\varepsilon_2\varepsilon_3\varepsilon_4+4\varepsilon^3{}_3]}{[3\varepsilon_2\varepsilon_4-2\varepsilon^2{}_3]^{\frac{3}{2}}} \tag{18}$$

Coefficient of kurtosis:

$$CK_{gw} = \frac{6[5\varepsilon^3{}_2\varepsilon_6-8\varepsilon^2{}_2\varepsilon_3\varepsilon_5+6\varepsilon_2\varepsilon^2{}_3\varepsilon_4-2\varepsilon^4{}_3]}{\varepsilon_4-2\varepsilon^4{}_3} \tag{19}$$

Table 2 shows the mode, mean, standard deviation (STD), coefficient of variation CV_{gw}, coefficient of skewness CS_{gw} and coefficient of kurtosis CK_{gw} with some values of the parameters λ and c. **Moment Generating Function Of DWED**

The moment generating function of DWED is given by:

$$M_{gw}(t) = \frac{(c+1)^2\lambda^2}{(c+1)^2-1}\int_0^\infty x\,e^{tx}e^{-\lambda x}\left(1-e^{-c\lambda x}\right)dx$$

$$= \frac{(c+1)^2\lambda^2}{(c+1)^2-1}\left[\int_0^\infty x\,e^{-(\lambda-t)x}dx - \int_0^\infty x\,e^{-(\lambda(1+c)-t)x}dx\right]$$

Let $(\lambda(1+c)-t)x = y \implies x = \frac{y}{\lambda(1+c)-t}\ , dx = \frac{1}{\lambda(1+c)-t}dy$

So $\int_0^\infty x\, e^{-(\lambda(1+c)-t)x} dx = \frac{1}{[\lambda(1+c)-t]^2} \int_0^\infty y e^{-y} dy = \frac{1}{[\lambda(1+c)-t]^2}$

And $\int_0^\infty x\, e^{-(\lambda-t)x} dx = \frac{1}{(\lambda-t)^2}$, then

$$M_{gw}(t) = \frac{(c+1)^2\lambda^2}{(c+1)^2-1}\left[\frac{1}{(\lambda-t)^2} - \frac{1}{[\lambda(1+c)-t]^2}\right] \quad (20)$$

6. Fisher Information Of DWIWD

The Fisher information (FI) that x contains about the parameters $\vartheta = (\lambda, c)$ are obtained as follows:

Note that

$$\log g_w(x;\lambda,c) = 2\log(c+1) + 2\log\lambda - \log[(c+1)^2-1] + \log x - \lambda x + \log\left[1-e^{-\lambda c x}\right] \quad (21)$$

Differentiating (21) with respect to c,we get

$$\frac{\partial}{\partial c}\log g_w(x;\lambda,c) = \frac{2}{c+1} - \frac{2(c+1)}{(c+1)^2-1} + \frac{\lambda x e^{-\lambda c x}}{1-e^{-\lambda c x}} \quad (22)$$

$$\frac{\partial^2}{\partial c^2}\log g_w(x;\lambda,c) = \frac{-2}{(c+1)^2} - \frac{2[(c+1)^2-1]-4(c+1)^2}{[(c+1)^2-1]^2} - \frac{\lambda^2 x^2 e^{-\lambda c x}}{\left[1-e^{-\lambda c x}\right]^2} \quad (23)$$

$$\therefore\ -E\left[\frac{\partial^2}{\partial c^2}\log g_w(x;\lambda,c)\right] = \frac{-2}{(c+1)^2} - \frac{2[(c+1)^2-1]-4(c+1)^2}{[(c+1)^2-1]^2} - \frac{\lambda^4(c+1)^2}{(c+1)^2-1}\int_0^\infty \frac{x^3 e^{-(c+1)\lambda x}}{1-e^{-\lambda c x}} dx$$

Let $1-e^{-\lambda c x} = y \implies e^{-\lambda c x} = 1-y \implies x = \frac{-\log(1-y)}{\lambda c}$, $dx = \frac{1}{(1-y)\lambda c} dy$

So $\int_0^\infty \frac{x^3 e^{-(c+1)\lambda x}}{1-e^{-\lambda c x}} dx = \frac{-1}{\lambda^4 c^4}\int_0^1 [\log(1-y)]^3 (1-y)^{\frac{(c+1)}{c}-1}\frac{1}{y} dy$, we have

$$(1-y)^{\frac{(c+1)}{c}-1} = \sum_{j=0}^\infty \frac{(-1)^j \Gamma\left(\frac{(c+1)}{c}\right) y^j}{\Gamma\left(\frac{(c+1)}{c}-j\right) j!} \quad (24)$$

Then $\int_0^\infty \frac{x^3 e^{-(c+1)\lambda x}}{1-e^{-\lambda c x}} dx = \frac{-1}{\lambda^4 c^4}\int_0^1 [\log(1-y)]^3 \sum_{j=0}^\infty \frac{(-1)^j \Gamma\left(\frac{(c+1)}{c}\right) y^j}{\Gamma\left(\frac{(c+1)}{c}-j\right) j!}\frac{1}{y} dy$

$$= \frac{-1}{\lambda^4 c^4}\sum_{j=0}^\infty \frac{(-1)^j \Gamma\left(\frac{(c+1)}{c}\right)}{\Gamma\left(\frac{(c+1)}{c}-j\right) j!}\int_0^1 [\log(1-y)]^3 y^{j-1} dy$$

Let $1-y = u \implies y = 1-u$, $dy = -du$

Then $\int_0^\infty \frac{x^3 e^{-(c+1)\lambda x}}{1-e^{-\lambda c x}} dx = \frac{-1}{\lambda^4 c^4}\sum_{j=0}^\infty \frac{(-1)^j \Gamma\left(\frac{(c+1)}{c}\right)}{\Gamma\left(\frac{(c+1)}{c}-j\right) j!}\int_0^1 [\log(u)]^3 (1-u)^{j-1} du$

$$= \frac{-1}{\lambda^4 c^4}\sum_{j=0}^\infty \frac{(-1)^j \Gamma\left(\frac{(c+1)}{c}\right)}{\Gamma\left(\frac{(c+1)}{c}-j\right) j!}\sum_{i=0}^\infty \frac{(-1)^i \Gamma(j)}{\Gamma(j-i)\, i!}\int_0^1 [\log(u)]^3 u^i du$$

$$= \frac{6\,\Gamma\left(\frac{(c+1)}{c}\right)}{\lambda^4 c^4}\sum_{j=0}^\infty\sum_{i=0}^\infty \frac{(-1)^{j+i}\,\Gamma(j)}{\Gamma\left(\frac{(c+1)}{c}-j\right)\Gamma(j-i)\,(i+1)^4 i!\, j!}$$

$$-E\left[\frac{\partial^2}{\partial c^2}\log g_w(x;\lambda,c)\right] = \frac{2}{(c+1)^2} + \frac{2[(c+1)^2-1]-4(c+1)^2}{[(c+1)^2-1]^2} + \frac{6\,\Gamma\left(\frac{(c+1)}{c}\right)}{\lambda^4 c^4}\sum_{j=0}^\infty\sum_{i=0}^\infty \frac{(-1)^{j+i}\,\Gamma(j)}{\Gamma\left(\frac{(c+1)}{c}-j\right)\Gamma(j-i)\,(i+1)^4 i!\, j!}, \quad (25)$$

Now differentiating (21) with respect to λ, we get

$$\frac{\partial}{\partial\lambda}\log g_w\ (x;\lambda,c) = \frac{2}{\lambda} - x + \frac{cxe^{-\lambda cx}}{1-e^{-\lambda cx}} \tag{26}$$

$$\frac{\partial^2}{\partial\lambda^2}\log g_w\ (x;\lambda,c) = \frac{-2}{\lambda^2} - \frac{c^2x^2e^{-\lambda cx}}{\left(1-e^{-\lambda cx}\right)^2}, \quad \text{so}$$

$$-E\left[\frac{\partial^2}{\partial\lambda^2}\log g_w\ (x;\lambda,c)\right] = \frac{-2}{\lambda^2} - \frac{c^2\lambda^2(c+1)^2}{(c+1)^2-1}\int_0^\infty x^3e^{-(c+1)\lambda x}\frac{1}{1-e^{-\lambda cx}}dx$$

$$= \frac{-2}{\lambda^2} + \frac{(c+1)^2}{c^2\lambda^2[(c+1)^2-1]}\int_0^1[\log(1-y)]^3(1-y)^{\frac{(c+1)}{c}-1}\frac{1}{y}dy$$

$$= \frac{-2}{\lambda^2} + \frac{(c+1)^2}{c^2\lambda^2[(c+1)^2-1]}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma\left(\frac{(c+1)}{c}\right)}{\Gamma\left(\frac{(c+1)}{c}-j\right)j!}\int_0^1[\log(1-y)]^3y^{j-1}dy$$

$$= \frac{-2}{\lambda^2} + \frac{(c+1)^2}{c^2\lambda^2[(c+1)^2-1]}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma\left(\frac{(c+1)}{c}\right)}{\Gamma\left(\frac{(c+1)}{c}-j\right)j!}\int_0^1[\log(u)]^3(1-u)^{j-1}du$$

$$= \frac{-2}{\lambda^2} + \frac{(c+1)^2}{c^2\lambda^2[(c+1)^2-1]}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{j+i}\Gamma\left(\frac{(c+1)}{c}\right)\Gamma(j)}{\Gamma\left(\frac{(c+1)}{c}-j\right)\Gamma(j-i)\,i!\,j!}\frac{-6}{(i+1)^4}$$

$$= \frac{-2}{\lambda^2} - \frac{6(c+1)^2}{c^2\lambda^2[(c+1)^2-1]}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{j+i}\Gamma\left(\frac{(c+1)}{c}\right)\Gamma(j)}{\Gamma\left(\frac{(c+1)}{c}-j\right)\Gamma(j-i)(i+1)^4\,i!\,j!}$$

$$\therefore -E\left[\frac{\partial^2}{\partial\lambda^2}\log g_w\ (x;\lambda,c)\right] = \frac{2}{\lambda^2} + \frac{6(c+1)^2}{c^2\lambda^2[(c+1)^2-1]}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{j+i}\Gamma\left(\frac{(c+1)}{c}\right)\Gamma(j)}{\Gamma\left(\frac{(c+1)}{c}-j\right)\Gamma(j-i)(i+1)^4\,i!\,j!} \tag{27}$$

Now differentiating (26) with respect to c, we get

$$\frac{\partial^2}{\partial\lambda\partial c}\log g_w\ (x;\lambda,c) = \frac{xe^{-\lambda cx}}{1-e^{-\lambda cx}} - \frac{c\lambda x^2e^{-\lambda cx}}{\left(1-e^{-\lambda cx}\right)^2} \tag{28}$$

$$-E\left[\frac{\partial^2}{\partial\lambda\partial c}\log g_w\ (x;\lambda,c)\right] = \frac{\lambda^2(c+1)^2}{(c+1)^2-1}\int_0^\infty x^2e^{-(c+1)\lambda x}dx - \frac{c\lambda^3(c+1)^2}{(c+1)^2-1}\int_0^\infty x^3e^{-(c+1)\lambda x}\frac{1}{1-e^{-(c+1)\lambda x}}dx$$

Let $(c+1)\lambda x = y \implies x = \frac{y}{(c+1)\lambda}$, $dx = \frac{1}{(c+1)\lambda}dy$, then

$$\frac{\lambda^2(c+1)^2}{(c+1)^2-1}\int_0^\infty x^2e^{-(c+1)\lambda x}dx = \frac{\lambda^2(c+1)^2}{(c+1)^2-1}\int_0^\infty\frac{y^2}{\lambda^2(c+1)^2}\frac{1}{(c+1)\lambda}e^{-y}dy = \frac{2}{\lambda(c+1)[(c+1)^2-1]},$$

$$\frac{c\lambda^3(c+1)^2}{(c+1)^2-1}\int_0^\infty x^3e^{-(c+1)\lambda x}\frac{1}{1-e^{-(c+1)\lambda x}}dx = \frac{-(c+1)^2}{\lambda c^3[(c+1)^2-1]}\int_0^1[\log(1-y)]^3(1-y)^{\frac{(c+1)}{c}-1}\frac{1}{y}dy$$

By the same way above we obtain

$$\frac{c\lambda^3(c+1)^2}{(c+1)^2-1}\int_0^\infty x^3e^{-(c+1)\lambda x}\frac{1}{1-e^{-(c+1)\lambda x}}dx = \frac{6(c+1)^2}{\lambda c^3[(c+1)^2-1]}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{j+i}\Gamma\left(\frac{(c+1)}{c}\right)\Gamma(j)}{\Gamma\left(\frac{(c+1)}{c}-j\right)\Gamma(j-i)(i+1)^4\,i!\,j!}$$

$$-E\left[\frac{\partial^2}{\partial\lambda\partial c}\log g_w\ (x;\lambda,c)\right] = \frac{6(c+1)^2}{\lambda c^3[(c+1)^2-1]}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{j+i}\Gamma\left(\frac{(c+1)}{c}\right)\Gamma(j)}{\Gamma\left(\frac{(c+1)}{c}-j\right)\Gamma(j-i)(i+1)^4\,i!\,j!} - \frac{2}{\lambda(c+1)[(c+1)^2-1]} \tag{29}$$

And $$-E\left[\frac{\partial^2}{\partial c\partial\lambda}\log g_w\ (x;\lambda,c)\right] = -E\left[\frac{\partial^2}{\partial\lambda\partial c}\log g_w\ (x;\lambda,c)\right] \tag{30}$$

Now, the Fisher Information Matrix (FIM) for DWED is given by:

$$I_{g_w}(c,\lambda) = \begin{pmatrix} -E\left[\frac{\partial^2}{\partial c^2}\log g_w\ (x;\lambda,c)\right] & -E\left[\frac{\partial^2}{\partial c\partial\lambda}\log g_w\ (x;\lambda,c)\right] \\ -E\left[\frac{\partial^2}{\partial\lambda\partial c}\log g_w\ (x;\lambda,c)\right] & -E\left[\frac{\partial^2}{\partial\lambda^2}\log g_w\ (x;\lambda,c)\right] \end{pmatrix} \quad (31)$$

7. Estimation Of Parameters In The DWED

Method of moment and maximum likelihood are used to estimate the parameters of DWED.

8.1 Method Of Moment Estimators

The method of moments in general provides estimators which are consistent but not as efficient as the Maximum Likelihood ones. They are often used because they lead to very simple computations. If X follows DWED with parameters λ and c, then the k^{th} moment of X, say $E_{gw}(x^k)$ is given as

$$E_{gw}(x^k) = \frac{\Gamma(k+2)[(c+1)^{k+2}-1]}{\lambda^k(c+1)^k[(c+1)^2-1]}$$

For the case $k = 1$ and let $X_1, X_2, \ldots, X_n$ be an independent sample from the DWIWD we obtain the first sample moment as follow

$$\frac{2[(c+1)^3-1]}{\lambda(c+1)[(c+1)^2-1]} = \frac{1}{n}\sum_{j=1}^{n} X_j \quad (32)$$

The same way above we can find the estimate for parameter λ (or c) when c (or λ) is known as follow

$$\hat{\lambda} = \frac{2[(c+1)^3-1]}{\overline{X}(c+1)[(c+1)^2-1]} \quad , c \text{ known} \quad (33)$$

And

$$\hat{c} = \frac{\sqrt{(\lambda\overline{X}-2)(\lambda\overline{X}+6)}-3\lambda\overline{X}+6}{2\lambda\overline{X}-4} \quad , \lambda \text{ known} \quad (34)$$

We note that if we want to estimate any one of them where the other parameter is unknown, can be obtained by numerical methods.

8.2 **Maximum Likelihood Estimators**

This is the best known, most widely used, and most important of the methods of estimation. All that is done is to write down the likelihood function $L(\theta; x)$, and then find the value $\hat{\theta}$ of θ which maximizes $L(\theta; x)$. The log-likelihood function based on the random sample $x_1, x_2, \ldots, x_n$ is given by:

$$L(\lambda, c) = 2n\log(c+1) + 2\,n\log\lambda - n\log[(c+1)^2-1] + \sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\log(1-e^{-c\lambda x_i}) \quad (35)$$

which admits the partial derivatives

$$\frac{\partial L(\lambda,c)}{\partial\lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{cx_ie^{-c\lambda x_i}}{1-e^{-c\lambda x_i}} \quad (36)$$

And

$$\frac{\partial L(\lambda,c)}{\partial c} = \frac{2n}{c+1} - \frac{2n(c+1)}{(c+1)^2-1} + \sum_{i=1}^{n}\frac{\lambda x_ie^{-c\lambda x_i}}{1-e^{-c\lambda x_i}} \quad (37)$$

Equating these equations to zero, then we get

$$\frac{2n}{\lambda} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{cx_ie^{-c\lambda x_i}}{1-e^{-c\lambda x_i}} = 0 \quad (38)$$

$$\frac{2n}{c+1} - \frac{2n(c+1)}{(c+1)^2-1} + \sum_{i=1}^{n} \frac{\lambda x_i e^{-c\lambda x_i}}{1-e^{-c\lambda x_i}} = 0 \quad (39)$$

To find out the maximum likelihood estimators of λ, c we have to solve the above system of nonlinear equations (38)-(39) with respect to λ, c. As it seems, this system has no closed form solution in λ, c. Then we have to use a numerical technique method, such as Newton-Raphson method (see Adi (1966)), to obtain the solution. An example below show that.

Example(1):

Table 3 represents the total annual rainfall (mm) in the province of Babylon, and for the period of (1998-2009) see (Environmental Statistics in Iraq Report, (2009)). The data can be modeled by Double Weighted Exponential distribution and also we estimate the parameters λ and c using Newton Raphson method (see Adi (1966)), beginning with the initial estimates $\hat{\lambda} = 0.01$ and $\hat{c} = 0.7$. Then the estimate values are: $\hat{\lambda} = 0.0351$ and $\hat{c} = 0.0005$, for 6 iterated.

8. Conclusions

We can develop the weighted distribution into Derivation Double weighted distribution (DWD), like Double Weighted distribution, Double Weighted Exponential distribution. Therefore, We can discuss some of statistical properties on them.

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Table 1: mode of DWED.

c	λ	Mode (x_0)
2	1	1.2298
	2	0.6149
	5	0.2459
	7.6	0.1618
4	2	0.5308
6		0.5069
8.2		0.5011
9.7		0.5002

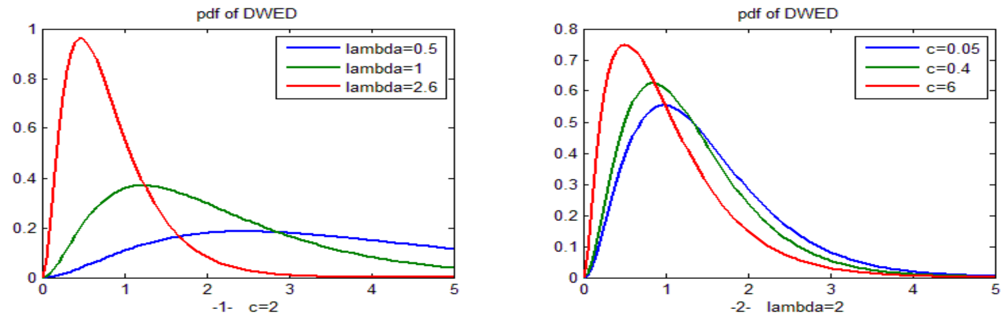

Figure 1: the pdf of DWED for specified values of λ and c.

Table 2 the mode, mean, standard deviation (STD), coefficient of variation CV_{gw}, coefficient of skewness CS_{gw} and coefficient of kurtosis CK_{gw} with some values of the parameters λ and c of DWED.

λ	c	Mode	Mean	STD	VAR	CV_{gw}	CS_{gw}	CK_{gw}
1	2	1.2298	2.1667	1.4044	1.9722	0.4583	3.9808	6.0268
	3	1.1207	2.1000	1.4018	1.9650	0.4720	4.0401	6.0802
	6.	1.0118	2.1208	1.4068	1.9791	0.4891	4.0450	6.0594
	2	1.0000	2.0110	1.4112	1.9914	0.4962	4.0218	6.0254
	12							
2	2	0.6149	1.0833	0.7022	0.4931	0.4583	3.9808	6.0268
	3	0.5603	1.0500	0.7009	0.4913	0.4720	4.0401	6.0802
	6.	0.5059	1.0169	0.7034	0.4948	0.4891	4.0450	6.0594
	2	0.5000	1.0055	0.7056	0.4979	0.4962	4.0218	6.0254
	12							
5	2	0.2459	0.4333	0.2809	0.0789	0.4583	3.9808	6.0268
	3	0.2241	0.4200	0.2804	0.0786	0.4720	4.0401	6.0802
	6.	0.2023	0.4068	0.2814	0.0792	0.4891	4.0450	6.0594
	2	0.2000	0.4022	0.2822	0.0797	0.4962	4.0218	6.0254
	12							
8.	2	0.1413	0.2490	0.1614	0.0261	0.4583	3.9808	6.0268
7	3	0.1288	0.2414	0.1611	0.0260	0.4720	4.0401	6.0802
	6.	0.1163	0.2338	0.1617	0.0261	0.4891	4.0450	6.0594
	2	0.1149	0.2311	0.1622	0.0263	0.4962	4.0218	6.0254
	12							

From **Table 2,** we note that the mode decreases at fixed λ and change c . And so the mean, except at $c = 6.2$. The STD and VAR take valuable oscillatory between decreasing at $c = 3$ with $\lambda = 1, 2, 5, 8.7$. we note that CV_{gw} , CS_{gw} and CK_{gw} do not depend on λ, but only on c. That is, the CV_{gw} increases at all values of c. While the CS_{gw} and CK_{gw} takes valuable oscillatory between increasing into decreasing.
The Figure follow shows the plots of CV_{gw} , CS_{gw} and CK_{gw} for DWED.

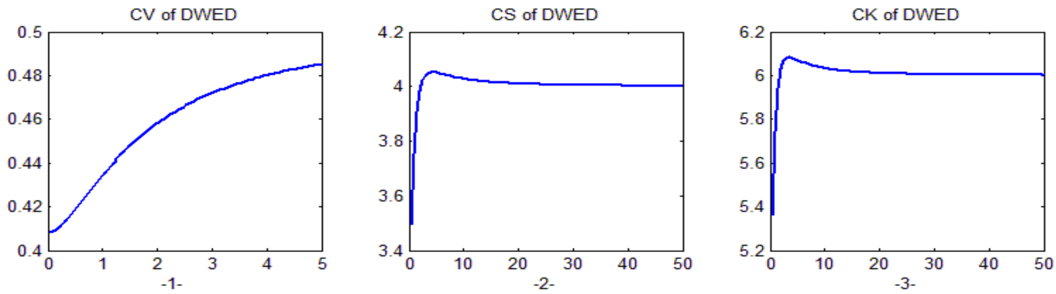

Figure 2 plot of CV_{gw} , CS_{gw} and CK_{gw}

From **Figure 2** (1, 2, 3), we note that CV_{gw} , CS_{gw} and CK_{gw} do not depend on the parameter λ. In -1-, the CV_{gw} behaves monotonically increases when c increasing. The relationship between c and CS_{gw} is shown in -2-, such that the CS_{gw} behaves monotonically increases when c increasing until it reaches the maximum value of CS_{gw} is 4.0525 for c =4.3 and then decreases to settle down in value $CS_{gw} = 4$ for the larger value of c. From our calculations it's clear that $CS_{gw} > 0$, then (Mean > Mode), see table 2 and the pdf of DWED is skewed to the right. The relationship between c and CK_{gw} is shown in -3-, such that the CK_{gw} behaves monotonically increases when c increasing until it reaches the maximum value of CK_{gw} is 6.0833 for c =3.5 and then decreases to settle down in value $CK_{gw} = 5$ for the larger value of c. Since in our calculations it's clear that $CK_{gw} > 3$ then the pdf of DWED shape is more peaked than the Normal pdf.

Table 3: Total annual rainfall (mm) in the province of Babylon, and for the period of (1998-2009)

1998	**1999**	**2000**	**2001**	**2003**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**
95.8	65.3	85.3	81.3	102.8	134.5	71.1	73.2	170.3	41.0	51.8	52.4