Characterized Proximity L-spaces, Characterized Compact L-spaces and Characterized Uniform L-spaces

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Abstract
In this research work, three new spaces are proposed and investigated. The spaces are named characterized proximity L-space, characterized compact L-space and characterized uniform L-space. The properties of such spaces are deeply studied. Some sort of relationship were introduced among such spaces and other published spaces presented by the author. The published spaces are named characterized \( FT_s \)-spaces and characterized \( FR_k \)-spaces ([2,3,4]), for \( s \in \{0, 1, 2, \frac{1}{2}, 3, 4\} \) and \( k \in \{0, 1, 2, 3\} \). New properties were already added to the characterized \( FT_s \)-spaces and to the characterized \( FR_k \)-spaces. Moreover, the characterized proximity L-space, characterized compact L-space and characterized uniform L-space are classified according to the characterized \( FT_s \)-spaces and the characterized \( FR_k \)-spaces for \( s \in \{0, 1, 2, 4\} \) and \( k \in \{2, 3\} \).

Keywords: L-filter, topological L-space, operations, isotone and idempotent, characterized L-space, \( \phi_{1,2} \)-neighborhood filters, uniform L-structure, L-proximity, characterized proximity L-space, characterized compact L-space, characterized uniform L-space, characterized \( FT_s \)-space, \( F_{\phi_{1,2}} \)-T space, characterized \( FR_k \)-space and \( F_{\phi_{1,2}} \)-R space for \( s \in \{0, 1, 2, 4\} \) and \( k \in \{2, 3\} \).

1. Introduction
The notion of fuzzy filter which is named here L-filter has been introduced by Eklund et al. [13]. By means of this notion a point-based approach to L-topology related to usual points has been developed. The more general concept for L-filter introduced by Gähler in [15] and L-filters are classified by types. Because of the specific type of L-filter however the approach of Eklund is related only to L-topologies which are stratified, that is, all constant L-sets are open. The more specific L-filters considered in the former papers are called now homogeneous.

On the ordinary topological space \((X, T)\), the operation has been defined by Kasahara ([22]) as a mapping \( \phi \) from \( T \) into \( 2^X \) such that \( A \subseteq A^\phi \), for all \( A \in T \). Abd El-Monsef it al. ([7]) extend Kasahara operation to the power set \( P(X) \) of a set \( X \). Kandil et al.[(21)] extended Kasahara's and Abd El-Monsef's operations by introducing an operation on the class of all L-sets endowed with an L-topology \( \tau \) as a mapping \( \phi : L^X \to L^X \) such that \( \text{int} \mu \leq \mu^\phi \) for all \( \mu \in L^X \), where \( \mu^\phi \) denotes the value of \( \phi \) at \( \mu \).

The notions of the L-filters and the operations on the class of all L-sets on \( X \) endowed with an L-topology \( \tau \) are applied in [1,5,6] to introduce a more general theory including all the weaker and stronger forms of the L-topology. By means of these notions the notion of \( \phi_{1,2} \)-interior of L-set, \( \phi_{1,2} \)-L-convergence and \( \phi_{1,2} \)-L-neighborhood filters are defined and applied to introduced many special classes of separation axioms. The notion of \( \phi_{1,2} \)-interior operator for L-sets is defined as a mapping \( \phi_{1,2} \cdot \text{int} : L^X \to L^X \) which fulfill (I1) to (I5) in [1]. There is a one-to-one correspondence between the class of all \( \phi_{1,2} \)-open L-subsets of \( X \) and these operators, that is, the class \( \phi_{1,2} \cdot OF(X) \) of all \( \phi_{1,2} \)-open L-subsets of \( X \) can be characterized by these operators. Then the triple \((X, \phi_{1,2} \cdot \text{int})\) as well as the triple \((X, \phi_{1,2} \cdot OF(X))\) will be called the characterized
L-space of $\phi_{1,2}$-open L-subsets. The characterized L-spaces are characterized by many of characterizing notions in [1,5], for example by: $\phi_{1,2}$L-neighborhood filters, $\phi_{1,2}$L-interior of the L-filters and by the set of $\phi_{1,2}$-inner points of the L-filters. Moreover, the notions of closeness and compactness in characterized L-spaces are introduced and studied in [6]. The notions of characterized $FT_{s}$-spaces, $F\phi_{1,2}$-$T_{s}$ spaces, characterized $FR_{k}$-spaces and $F\phi_{1,2}$-$R_{k}$ spaces are introduced and studied in [2,3,4] for all $s \in \{0, 1, 2, 3, 4\}$ and $k \in \{0, 1, 2, 3\}$. This paper is devoted to introduce and study three new spaces named characterized proximity L-space, characterized compact L-space and characterized uniform L-space. Many relations between these spaces and our spaces characterized $FT_{s}$-spaces, $F\phi_{1,2}$-$T_{s}$ spaces, characterized $FR_{k}$-spaces and $F\phi_{1,2}$-$R_{k}$ spaces are investigated for $s \in \{0, 1, 2, 4\}$ and $k \in \{2, 3\}$.

In section 2, some definitions and notions related to L-sets, L-topologies, L-filters, L-proximity, operations on L-sets, $\phi_{1,2}$L-neighborhood filters, characterized L-spaces, characterized $FT_{s}$-spaces, $F\phi_{1,2}$-$T_{s}$ spaces, characterized $FR_{k}$-spaces and $F\phi_{1,2}$-$R_{k}$ spaces are given for $s \in \{0, 1, 2, 4\}$ and $k \in \{2, 3\}$. Section 3, is devoted to introduce and study the relation between characterized proximity L-spaces and our classes of characterized $FT_{s}$-spaces and characterized $FR_{k}$-spaces. It will be shown that in the characterized L-space $(X, \phi_{1,2}, \text{int})$, the L-proximity $\delta$ will be identified with the finer relation on the $\phi_{1,2}$L-neighborhood filters. Also, we will show that any L-proximity is separated if and only if the associated characterized proximity L-space is characterized $FT_{0}$ and to each L-proximity is associated a characterized $FR_{2}$-space in our sense. Generally, it will be shown that the associated characterized proximity L-space $(X, \phi_{1,2}, \text{int})$ is characterized $FR_{2}$-space if the related topological L-space $(X, \tau)$ is $F\phi_{1,2}$-$R_{2}$ space. Moreover, for each characterized $FR_{3}$-space the binary relation on $L^{X}$ defined by means the $\phi_{1,2}$-closure operator $\phi_{1,2}$-$\text{cl}$ of $\tau$ in Eq. (3.6), is L-proximity on $X$ and conversely, to each L-proximity $X$, which has a $\phi_{1,2}$-closure operator fulfills the binary relation given in (3.6), is associated characterized $FR_{3}$-space $(X, \phi_{1,2}, \text{int})$.

There is a good notion of $\phi_{1,2}$-compactness of the L-filters and of topological L-spaces introduced and studied by Abd-Allah et all. in [6]. This notion fulfills main properties, for example, it fulfills the Tychonoff Theorem. In section 4, we used this notion to introduce and study the notion of characterized compact L-spaces. It will be shown that every $\phi_{1,2}$-closed subset of a characterized compact L-space is $\phi_{1,2}$-compact and each $\phi_{1,2}$-compact subset of a characterized $FT_{2}$-space is $\phi_{1,2}$-closed. Also, it will be shown that each characterized compact $FT_{2}$-space is characterized $FT_{4}$-space. Moreover, if $(X, \psi_{1,2}, \text{int})$ is characterized compact L-space finer than the characterized $FT_{2}$-space $(X, \phi_{1,2}, \text{int})$, then $(X, \psi_{1,2}, \text{int})$ is isomorphic to $(X, \phi_{1,2}, \text{int})$. The notion of fuzzy uniform structure which is named here uniform SLS-structure had been introduced and studied by Gähler et all. in [17]. We used in the least section this notion to introduced the notion of characterized uniform L-spaces and the notion of associated characterized L-spaces. Finally, we show that the uniform L-space $(X, U)$ is separated if and only if the associated characterized stratified L-space.
\( (X, \varphi_{1,2}, \text{int}(U)) \) is characterized FT\(_0\) space and specially, \((X, \mathcal{U})\) is separated if and only if the associated stratified topological L-space \((X, \tau_\mathcal{U})\) is \(F\varphi_{1,2}-\text{T}_0\) space.

2. Preliminaries

We begin by recalling some facts on the L-filters. Let L be a completely distributive complete lattice with different least and last elements 0 and 1, respectively. Let \(L_0 = L \setminus \{0\}\). Sometimes we will assume more specially that L is complete chain, that is, L is a complete lattice whose partial ordering is a linear one. For a set \(X\), let \(L^X\) be the set of all L-subsets of \(X\), that is, of all mappings \(f : X \rightarrow L\). Assume that an order-reversing involution \(\alpha \mapsto \alpha'\) of L is fixed. For each L-set \(\mu \in L^X\), let \(\mu'\) denote the complement of \(\mu\) and it is defined by: 
\[
\mu'(x) = \mu(x)' \quad \text{for all } x \in X.
\]
Let \(\alpha\) be the constant L-subset of \(X\) with value \(\alpha \in L\). For all \(x \in X\) and for all \(\alpha \in L_0\), the L-subset \(x_\alpha\) of \(X\) whose value \(\alpha\) at \(x\) and 0 otherwise is called an L-point in \(X\).

The fuzzy filter on \(X\) ([15]) which is named here L-filter is a mapping \(\mathcal{M} : L^X \rightarrow L\) such that the following conditions are fulfilled:

\begin{align*}
\text{(F1)} & \quad \mathcal{M}(\varnothing) \leq \alpha \quad \text{for all } \alpha \in L \quad \text{and} \quad \mathcal{M}(\overline{\varnothing}) = 1.
\end{align*}

\begin{align*}
\text{(F2)} & \quad \mathcal{M}(\mu \lor \rho) = \mathcal{M}(\mu) \lor \mathcal{M}(\rho) \quad \text{for all } \mu, \rho \in L^X.
\end{align*}

The L-filter \(\mathcal{M}\) is called homogeneous ([13]) if \(\mathcal{M}(\varnothing) = \alpha\) for all \(\alpha \in L\). For each \(x \in X\), the mapping \(\hat{x} : L^X \rightarrow L\) defined by \(\hat{x}(\mu) = \mu(x)\) for all \(\mu \in L^X\) is a homogeneous L-filter on \(X\). For each \(\mu \in L^X\), the mapping \(\mathcal{M} : L^X \rightarrow L\) defined by \(\mathcal{M}(\eta) = \bigwedge_{0 \leq \eta(x)} \eta(x)\) for all \(\eta \in L^X\) is also homogeneous L-filter on \(X\), called homogenous L-filter at the L-subset \(\mu \in L^X\). Let \(\mathcal{F}_L X\) and \(\mathcal{F}_L X\) will be the denote the sets of all L-filters and of all homogeneous L-filters on a set \(X\), respectively. If \(\mathcal{M}\) and \(\mathcal{N}\) are L-filters on a set \(X\), \(\mathcal{M}\) is said to be finer than \(\mathcal{N}\), denoted by \(\mathcal{M} \preceq \mathcal{N}\), provided \(\mathcal{M}(\mu) \geq \mathcal{N}(\mu)\) holds for all \(\mu \in L^X\). Noting that if L is a complete chain then \(\mathcal{M}\) is not finer than \(\mathcal{N}\), denoted by \(\mathcal{M} \not\preceq \mathcal{N}\), provided there exists \(\mu \in L^X\) such that \(\mathcal{M}(\mu) < \mathcal{N}(\mu)\) holds.

**Proposition 2.1** [11] For all \(\mu, \rho \in L^X\), we have
\[
\mu \leq \rho \quad \text{if and only if} \quad \hat{\mu} \leq \hat{\rho}
\]
For each non-empty set \(A\) of the L-filters on \(X\) the supremum \(\bigvee_{\mathcal{M} \in A} \mathcal{M}\) exists ([15]) and given by:
\[
\bigvee_{\mathcal{M} \in A} \mathcal{M}(\mu) = \bigwedge_{\mathcal{M} \in A} \mathcal{M}(\mu)
\]
for all \(\mu \in L^X\). Whereas the infimum \(\bigwedge_{\mathcal{M} \in A} \mathcal{M}\) of \(A\) does not exists in general as an L-filter. If the infimum \(\bigwedge_{\mathcal{M} \in A} \mathcal{M}\) exists, then we have:
\[
\bigwedge_{\mathcal{M} \in A} \mathcal{M}(\mu) = \bigvee_{\mu \leq \mu_n} \mathcal{M}(\mu_n) \quad \text{for all } \mu \in L^X,
\]
where \(n\) is a positive integer, \(\mu_1, \ldots, \mu_n\) is a collection such that \(\mu_1 \land \ldots \land \mu_n \leq \mu\) and \(\mathcal{M}_1, \ldots, \mathcal{M}_n\) are L-filters from \(A\). Let \(X\) be a set and \(\mu \in L^X\), then the homogeneous L-filter \(\hat{\mu}\) at \(\mu \in L^X\) is the L-filter on \(X\) given by:
\[
\hat{\mu}(x) = \bigvee_{0 \leq \mu(x)} \hat{x}
\]

**L-filter bases.** A family \(\{\mathcal{B}_\alpha\}_{\alpha \in L_0}\) of non-empty subsets of \(L^X\) is called a valued L-filter base ([15]) if the following conditions are fulfilled:
(V1) \( \mu \in B_\alpha \) implies \( \alpha \leq \sup \mu \).

(V2) For all \( \alpha, \beta \in L_0 \) with \( \alpha \land \beta \in L_0 \) and all \( \mu \in B_\alpha \) and \( \rho \in B_\beta \) there are \( \gamma \geq \alpha \land \beta \) and \( \eta \geq \mu \land \sigma \) such that \( \eta \in B_\gamma \).

**Proposition 2.2** [15] Each valued base \((B_\alpha)_{\alpha \in L_0}\) defines the L-filter \( M \) on \( X \) by \( M(\mu) = \bigvee_{\rho \in \mathbb{R}_L, \rho \leq \mu} \alpha^\mu \) for all \( \mu \in L^X \). Conversely, each L-filter \( M \) can be generated by a valued base, e.g. by \((\alpha \text{-pr} \ M)_{\alpha \in L_0}\) with \( \alpha \text{-pr} \ M = \{ \mu \in L^X : \alpha \leq M(\mu) \} \).

\((\alpha \text{-pr} \ M)_{\alpha \in L_0}\) is a family of prefilters on \( X \) and is called the large valued base of \( M \). Recall that a prefilter on \( X \) ([25]) is a non-empty proper subset \( F \) of \( L^X \) such that:

(1) \( \mu, \rho \in F \) implies \( \mu \land \rho \in F \) and (2) from \( \mu \in F \) and \( \mu \leq \rho \) it follows \( \rho \in F \).

**Proposition 2.3** [15] Let \( \mathcal{A} \) be a set of L-filters on a set \( X \). Then the following are equivalent:

(1) The infimum \( \bigwedge \mathcal{M} \) of \( \mathcal{A} \) with respect to the finer relation for L-filters exists.

(2) \( \mathcal{M}(\mu_1) \land \ldots \land \mathcal{M}(\mu_n) \leq \sup(\mu_1 \land \ldots \land \mu_n) \) for all finite subset \( \{\mathcal{M}_1, \ldots, \mathcal{M}_n\} \) of \( \mathcal{A} \) and \( \mu_1, \ldots, \mu_n \in L^X \).

(3) \( \alpha \leq \sup(\mu_1 \land \ldots \land \mu_n) \) holds for all non-empty finite subset \( \{\mu_1, \ldots, \mu_n\} \) of \( \bigcup_{\alpha \in L_0} \alpha \text{-pr} \ M \) and \( \alpha \in L_0 \)

**L-topology.** By an L-topology on a set \( X \) ([12,20]), we mean a subset of \( L^X \) which is closed with respect to all suprema and all finite infima and contains the constant L-sets \( 0 \) and \( 1 \). A set \( X \) equipped with an L-topology \( \tau \) on \( X \) is called topological L-space. For each topological L-space \((X, \tau)\), the elements of \( \tau \) are called open L-subsets of this space. If \( \tau_1 \) and \( \tau_2 \) are L-topologies on a set \( X \), \( \tau_2 \) is said to be finer than \( \tau_1 \) and \( \tau_1 \) is said to be coarser than \( \tau_2 \) provided \( \tau_1 \subseteq \tau_2 \) holds. The topological L-space \((X, \tau)\) and also \( \tau \) are said to be stratified provided \( \bar{\tau} \in \tau \) holds for all \( \alpha \in L \), that is, all constant L-sets are open ([24]).

**Proximity L-space.** A binary relation \( \delta \) on \( L^X \) is called L-proximity on \( X \) ([23]), provided it fulfill the following conditions:

(P1) \( \mu \overset{\delta}{=} \rho \) implies \( \rho \overset{\delta}{=} \mu \) for all \( \mu, \rho \in L^X \), where \( \delta \) is the negation of \( \delta \).

(P2) \( (\mu \lor \rho) \overset{\delta}{=} \eta \) if and only if \( \mu \overset{\delta}{=} \eta \) and \( \rho \overset{\delta}{=} \eta \) for all \( \mu, \rho, \eta \in L^X \).

(P3) \( \mu = 0 \) or \( \rho = 0 \) implies \( \mu \overset{\delta}{=} \rho \) for all \( \mu, \rho \in L^X \).

(P4) \( \mu \overset{\delta}{=} \rho \) implies \( \mu \leq \rho' \) for all \( \mu, \rho \in L^X \).

(P5) If \( \mu \overset{\delta}{=} \rho \), then there is an \( \eta \in L^X \) such that \( \mu \overset{\delta}{=} \eta \) and \( \eta \overset{\delta}{=} \rho \).

A set \( X \) equipped with an L-proximity \( \delta \) on \( X \) is called a proximity L-space and will be denoted by \((X, \delta)\). Every L-proximity \( \delta \) on a set \( X \) is associated an L-topology on \( X \) denoted by \( \tau_\delta \). The L-proximity \( \delta \) on a set \( X \) is said to be separated if and only if for all \( x, y \in X \) such that \( x \neq y \) we have \( x_\alpha \overset{\delta}{=} y_\beta \) for all \( \alpha, \beta \in L_0 \).

**Operation on L-sets.** In the sequel, let a topological L-space \((X, \tau)\) be fixed. By the operation ([21]) on a set \( X \) we mean a mapping \( \varphi : L^X \rightarrow L^X \) such that \( \text{int} \mu \leq \mu^\varphi \) holds, for all \( \mu \in L^X \), where, \( \mu^\varphi \) denotes the value of \( \varphi \) at \( \mu \). The class of all operations on \( X \) will be denoted by \( O_{(L^X, \tau)} \). By identity operation on \( O_{(L^X, \tau)} \), we mean the operation \( I_{L^X} : L^X \rightarrow L^X \) such that \( I_{L^X}(\mu) = \mu \), for all \( \mu \in L^X \). Also by constant
operation on \( O_{(L^X,\tau)} \) we mean the operation \( c_{L^X}: L^X \rightarrow L^X \) such that \( c_{L^X}(\mu) = \overline{\mu} \), for all \( \mu \in L^X \). If \( \leq \) is a partially ordered relation on \( O_{(L^X,\tau)} \) defined as follows: \( \varphi_1 \leq \varphi_2 \Leftrightarrow \mu^\varphi_1 \leq \mu^\varphi_2 \) for all \( \mu \in L^X \), then obviously, \( O_{(L^X,\tau)} \) is a completely distributive lattice. As an application on this partially ordered relation, the operation \( \varphi: L^X \rightarrow L^X \) will be called:

(i) Isotone if \( \mu \leq \rho \) implies \( \mu^\varphi \leq \rho^\varphi \), for all \( \mu, \rho \in L^X \).

(ii) Weakly finite intersection preserving (wfip, for short) with respect to \( \mathcal{A} \subseteq L^X \) if \( \rho \wedge \mu^\varphi \leq (\rho \wedge \mu)^\varphi \) holds, for all \( \rho \in \mathcal{A} \) and \( \mu \in L^X \).

(iii) Idempotent if \( \mu^\varphi = (\mu^\varphi)^\varphi \), for all \( \mu \in L^X \).

The operations \( \varphi, \psi \in O_{(L^X,\tau)} \) are said to be dual if \( \mu^\varphi = co(co \mu)^\psi \) or equivalently \( \mu^\varphi = co((co \mu)^\psi \) for all \( \mu \in L^X \), where \( co \mu \) denotes the complementation of \( \mu \). The dual operation of \( \varphi: L^X \rightarrow L^X \) will be denoted by \( \hat{\varphi}: L^X \rightarrow L^X \). In the classical case of \( L = \{0,1\} \), by the operation on a set \( X \) we mean the mapping \( \varphi: P(X) \rightarrow P(X) \) such that \( \text{int}A \subseteq A^\varphi \) for all \( A \in \text{power set} P(X) \). The identity operation on the class of all ordinary operations \( O_{(P(X) \cap L)} \) on \( X \) will be denoted by \( i_{P(X)} \), where \( i_{P(X)}(A) = A \) for all \( A \in P(X) \).

The \( \varphi - \text{open} \) \( L \)-sets. Let a topological \( L \)-space \( (X,\tau) \) be fixed and \( \varphi \in O_{(L^X,\tau)} \). The \( L \)-set \( \mu: X \rightarrow L \) is called \( \varphi - \text{open} \) \( L \)-set if \( \mu \leq \mu^\varphi \) holds. We will denote the class of all \( \varphi - \text{open} \) \( L \)-sets on \( X \) by \( \varphi \text{OF}(X) \).

The \( L \)-set \( \mu \) is called \( \varphi - \text{closed} \) if its complement \( co \mu \) is \( \varphi - \text{open} \). The two operations \( \varphi, \psi \in O_{(L^X,\tau)} \) are equivalent and written \( \varphi \sim \psi \) if \( \varphi \text{OF}(X) = \psi \text{OF}(X) \).

The \( \varphi_{1,2} - \text{interiors} \) of \( L \)-sets. Let a topological \( L \)-space \( (X,\tau) \) be fixed and \( \varphi_1, \varphi_2 \in O_{(L^X,\tau)} \). Then the \( \varphi_{1,2} - \text{interior} \) of the \( L \)-set \( \mu: X \rightarrow L \) is the mapping \( \varphi_{1,2}\text{int} \mu: X \rightarrow L \) defined by:

\[
\varphi_{1,2}\text{int} \mu = \bigvee_{\rho \in \varphi \text{OF}(X), \rho^\varphi \leq \mu} \rho \tag{2.2}
\]

\( \varphi_{1,2}\text{int} \mu \) is the greatest \( \varphi_1 - \text{open} \) \( L \)-set \( \rho \) such that \( \rho^\varphi \) less than or equal to \( \mu \) (11). The \( L \)-set \( \mu \) is said to be \( \varphi_{1,2} - \text{open} \) if \( \mu \leq \varphi_{1,2}\text{int} \mu \). The class of all \( \varphi_{1,2} - \text{open} \) \( L \)-sets on \( X \) will be denoted by \( \varphi_{1,2}\text{OF}(X) \).

The complement \( co \mu \) of a \( \varphi_{1,2} - \text{open} \) \( L \)-subset \( \mu \) will be called \( \varphi_{1,2} - \text{closed} \), the class of all \( \varphi_{1,2} - \text{closed} \) \( L \)-subsets of \( X \) will be denoted by \( \varphi_{1,2}\text{CF}(X) \). In the classical case of \( L = \{0,1\} \), the topological \( L \)-space \( (X,\tau) \) is up to an identification by the ordinary topological space \( (X,\tau) \) and \( \varphi_{1,2}\text{int} \mu \) is the classical one. Hence, in this case the ordinary subset \( A \) of \( X \) is \( \varphi_{1,2} - \text{open} \) if \( A \subseteq \varphi_{1,2}\text{int} A \). The complement of a \( \varphi_{1,2} - \text{open} \) subset \( A \) of \( X \) will be called \( \varphi_{1,2} - \text{closed} \). The class of all \( \varphi_{1,2} - \text{open} \) and the class of all \( \varphi_{1,2} - \text{closed} \) subsets of \( X \) will be denoted by \( \varphi_{1,2}O(X) \) and \( \varphi_{1,2}C(X) \), respectively. Clearly, \( F \) is \( \varphi_{1,2} - \text{closed} \) if and only if \( \varphi_{1,2}\text{cl}_\tau F = F \).

**Proposition 2.4** [1] If \( (X,\tau) \) be a topological \( L \)-space and \( \varphi_1, \varphi_2 \in O_{(L^X,\tau)} \). Then, the mapping \( \varphi_{1,2}\text{int} \mu: X \rightarrow L \) fulfills the following axioms:

(i) If \( \varphi_2 \geq 1_{L^X} \), then \( \varphi_{1,2}\text{int} \mu \leq \mu \) holds.

(ii) \( \varphi_{1,2}\text{int} \mu \) is isotone, i.e, if \( \mu \leq \rho \) then \( \varphi_{1,2}\text{int} \mu \leq \varphi_{1,2}\text{int} \rho \) holds for all \( \mu, \rho \in L^X \).
(iii) $\phi_{1,2}.\text{int} \overline{1} = \overline{1}$.

(iv) If $\phi_2 \geq 1_{L^X}$ is isotone and $\phi_1$ is whip with respect to $\phi_1OF(X)$, then $\phi_{1,2}.\text{int}(\mu \wedge \rho) = \phi_{1,2}.\text{int} \mu \wedge \phi_{1,2}.\text{int} \rho$ for all $\mu, \rho \in L^X$.

(v) If $\phi_2$ is isotone and idempotent operation, then $\phi_{1,2}.\text{int} \mu = \phi_{1,2}.\text{int}(\phi_{1,2}.\text{int} \mu)$ holds.

(vi) $\phi_{1,2}.\text{int}(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} \phi_{1,2}.\text{int} \mu_i$ for all $\mu_i \in \phi_{1,2}.OF(X)$.

**Proposition 2.5** [1] Let $(X, \tau)$ be a topological L-space and $\phi_1, \phi_2 \in O_{(L^X, \tau)}$. Then the following are fulfilled:

(i) If $\phi_2 \geq 1_{L^X}$, then the class $\phi_{1,2}.OF(X)$ of all $\phi_{1,2}$-open L-sets on $X$ forms an extended L-topology on $X$, denoted by $\tau_{\phi_1,2}^{\phi_2}$ ([18]).

(ii) If $\phi_2 \geq 1_{L^X}$, then the class $\phi_{1,2}.OF(X)$ of all $\phi_{1,2}$-open L-sets on $X$ forms a supra L-topology on $X$, denoted by $\tau_{\phi_1,2}^{\phi_2}$ ([18]).

(iii) If $\phi_2 \geq 1_{L^X}$ is isotone and $\phi_1$ is whip with respect to $\phi_1OF(X)$, then $\phi_{1,2}.OF(X)$ is a pre L-topology on $X$, denoted by $\tau_{\phi_1,2}^{\phi_2}$ ([18]).

(iv) If $\phi_2 \geq 1_{L^X}$ is isotone and idempotent operation and $\phi_1$ is whip with respect to $\phi_1OF(X)$, then $\phi_{1,2}.OF(X)$ forms an L-topology on $X$, denoted by $\tau_{\phi_1,2}^{\phi_2}$ ([12, 20]).

From Propositions 2.4 and 2.5, if the topological L-space $(X, \tau)$ is fixed and $\phi_1, \phi_2 \in O_{(L^X, \tau)}$, Then

$$\phi_{1,2}.OF(X) = \{ \mu \in L^X \mid \mu \leq \phi_{1,2}.\text{int} \mu \}$$

(2.3)

and the following conditions are fulfilled:

(11) If $\phi_2 \geq 1_{L^X}$, then $\phi_{1,2}.\text{int} \mu \leq \mu$ holds for all $\mu \in L^X$.

(12) If $\mu \leq \rho$ then $\phi_{1,2}.\text{int} \mu \leq \phi_{1,2}.\text{int} \rho$ holds for all $\mu, \rho \in L^X$.

(13) $\phi_{1,2}.\text{int} \overline{1} = \overline{1}$.

(14) If $\phi_2 \geq 1_{L^X}$ is isotone and $\phi_1$ is whip with respect to $\phi_1OF(X)$, then $\phi_{1,2}.\text{int}(\mu \wedge \rho) = \phi_{1,2}.\text{int} \mu \wedge \phi_{1,2}.\text{int} \rho$ for all $\mu, \rho \in L^X$.

(15) If $\phi_2 \geq 1_{L^X}$ is isotone and idempotent operation, then $\phi_{1,2}.\text{int}(\phi_{1,2}.\text{int} \mu) = \phi_{1,2}.\text{int} \mu$ for all $\mu \in L^X$.

Independently on the L-topologies, the notion of $\phi_{1,2}$-interior operator for L- sets can be defined as a mapping $\phi_{1,2}.\text{int} : L^X \rightarrow L^X$ which fulfill (11) to (15). It is well-known that (2.2) and (2.3) give a one-to-one correspondence between the class of all $\phi_{1,2}$-open L- sets and these operators, that is, $\phi_{1,2}.OF(X)$ can be characterized by $\phi_{1,2}$-interior operators. In this case $(X, \phi_{1,2}.\text{int})$ as well as $(X, \phi_{1,2}.OF(X))$ will be called characterized L-space ([11]) of $\phi_{1,2}$-open L-subsets of $X$. If $(X, \phi_{1,2}.\text{int})$ and $(X, \psi_{1,2}.\text{int})$ are two characterized L-spaces, then $(X, \phi_{1,2}.\text{int})$ is said to be finer than $(X, \psi_{1,2}.\text{int})$ and denoted by $\phi_{1,2}.\text{int} \leq \psi_{1,2}.\text{int}$ provided $\phi_{1,2}.\text{int} \mu \geq \psi_{1,2}.\text{int} \mu$ holds for all $\mu \in L^X$. The characterized L-space $(X, \phi_{1,2}.\text{int})$ of all $\phi_{1,2}$-open L-sets is said to be stratified if and only if $\phi_{1,2}.\text{int} \overline{\alpha} = \overline{\alpha}$ for all $\alpha \in L$. As shown in [1], the characterized L-space $(X, \phi_{1,2}.\text{int})$ is stratified if the related L-topology is stratified. Moreover, the characterized L-space $(X, \phi_{1,2}.\text{int})$ is said to have the weak infimum property ([18]) provided
for all $\mu \in L^X$ and $\alpha \in L$. The characterized L-space $(X, \mathcal{Q}_{1,2} \text{.int})$ is said to be strongly stratified ([18]) provided $\mathcal{Q}_{1,2} \text{.int}$ is stratified and have the weak infimum property.

If $\mathcal{Q}_1 = \text{int}$ and $\mathcal{Q}_2 = 1_X$ , then the class $\mathcal{O}_X (X)$ of all $\mathcal{Q}_{1,2}$-open L-set of $X$ coincide with $\tau$ which is defined in [12,20] and hence the characterized L-space $(X, \mathcal{Q}_{1,2} \text{.int})$ coincide with the topological L-space $(X, \tau)$. Another special choices for the operations $\mathcal{Q}_1$ and $\mathcal{Q}_2$ obtained in Table(1).

The $\mathcal{Q}_{1,2}$-neighborhood filters. An important notion in the characterized L-space $(X, \mathcal{Q}_{1,2} \text{.int})$ is that of a $\mathcal{Q}_{1,2}$-neighborhood filter at the point and at the ordinary subset in this space. Let $(X, \tau)$ be a topological L-space and $\mathcal{Q}_1, \mathcal{Q}_2 \in O_{(L^X, \tau)}$. As follows by (I1) to (I5) for each $x \in X$, the mapping $\mathcal{N}_{\mathcal{Q}_{1,2}} (x) : L^X \rightarrow L$ which is defined by:

$$\mathcal{N}_{\mathcal{Q}_{1,2}} (x) (\mu) = (\mathcal{Q}_{1,2} \text{.int} \mu)(x) \quad (2.4)$$

for all $\mu \in L^X$ is L-filter, called $\mathcal{Q}_{1,2}$-neighborhood filter at $x$ ([11]). If $\mathcal{Q} \neq F \subseteq P(X)$, then the $\mathcal{Q}_{1,2}$-neighborhood filter at $F$ will be denoted by $\mathcal{N}_{\mathcal{Q}_{1,2}} (F)$ and it will be defined by:

$$\mathcal{N}_{\mathcal{Q}_{1,2}} (F) = \bigvee_{x \in F} \mathcal{N}_{\mathcal{Q}_{1,2}} (x).$$

Since $\mathcal{N}_{\mathcal{Q}_{1,2}} (x)$ is L-filter for all $x \in X$, then $\mathcal{N}_{\mathcal{Q}_{1,2}} (F)$ is also L-filter on $X$. Moreover, because of $[\mathcal{X}_F] = \bigvee_{x \in F} x$, then we have $\mathcal{N}_{\mathcal{Q}_{1,2}} (F) \geq [\mathcal{X}_F]$ holds.

If the related $\mathcal{Q}_{1,2}$-interior operator fulfill the axioms (I1) and (I2) only, then the mapping $\mathcal{N}_{\mathcal{Q}_{1,2}} (x) : L^X \rightarrow L$, which is defined by (2.4) is an L-stack ([18]), called $\mathcal{Q}_{1,2}$-neighborhood stack at $X$. Moreover, if the $\mathcal{Q}_{1,2}$-interior operator fulfill the axioms (I1), (I2) and (I4) such that in (I4) instead of $\rho \in L^X$ we take $\bar{\alpha}$, then the mapping $\mathcal{N}_{\mathcal{Q}_{1,2}} (x) : L^X \rightarrow L$, is an L-stack with the cutting property, called here $\mathcal{Q}_{1,2}$-neighborhood stack with the cutting property at $X$. Obviously, the $\mathcal{Q}_{1,2}$-neighborhood filters fulfill the following axioms:

(N1) $x \leq \mathcal{N}_{\mathcal{Q}_{1,2}} (x)$ holds for all $x \in X$.

(N2) $\mathcal{N}_{\mathcal{Q}_{1,2}} (x) (\mu) \leq \mathcal{N}_{\mathcal{Q}_{1,2}} (x) (\rho)$ holds for all $\mu, \rho \in L^X$ and $\mu \leq \rho$.

(N3) $\mathcal{N}_{\mathcal{Q}_{1,2}} (x) (y \mapsto \mathcal{N}_{\mathcal{Q}_{1,2}} (y) (\mu)) = \mathcal{N}_{\mathcal{Q}_{1,2}} (x) (\mu)$, for all $x \in X$ and $\mu \in L^X$.

Clearly, $y \mapsto \mathcal{N}_{\mathcal{Q}_{1,2}} (y) (\mu)$ is the L-set $\mathcal{Q}_{1,2} \text{.int} \mu$.

The characterized L-space $(X, \mathcal{Q}_{1,2} \text{.int})$ of all $\mathcal{Q}_{1,2}$-open L-subsets of a set $X$ is characterized as a filter pre L-topology ([11]), that is, as a mapping $\mathcal{N}_{\mathcal{Q}_{1,2}} (x) : X \rightarrow \mathcal{F}_L X$ such that the axioms (N1) to (N3) are fulfilled.

The $\mathcal{Q}_{1,2}$-convergence. Let a topological L-spaces $(X, \tau)$ be fixed and $\mathcal{Q}_1, \mathcal{Q}_2 \in O_{(L^X, \tau)}$. If $X$ is a point in the characterized L-space $(X, \mathcal{Q}_{1,2} \text{.int})$, $F \subseteq X$ and $\mathcal{M}$ is L-filter on $X$. Then $\mathcal{M}$ is said to be $\mathcal{Q}_{1,2}$-convergence ([11]) to $x$ and written $\mathcal{M} \xrightarrow{\mathcal{Q}_{1,2} \text{.int}} x$, provided $\mathcal{M}$ is finer than the $\mathcal{Q}_{1,2}$-neighborhood filter $\mathcal{N}_{\mathcal{Q}_{1,2}} (x)$. Moreover, $\mathcal{M}$ is said to be $\mathcal{Q}_{1,2}$-convergence to $F$ and written $\mathcal{M} \xrightarrow{\mathcal{Q}_{1,2} \text{.int}} F$, provided $\mathcal{M}$
is finer than the $\phi_{1,2}$-neighborhood filter $\mathcal{N}_{\phi_{1,2}}(x)$ for all $x \in F$, that is, $\mathcal{M}$ is finer than the $\phi_{1,2}$-neighborhood filter $\mathcal{N}_{\phi_{1,2}}(F)$.

Internal $\phi_{1,2}$-closure of L-sets and $\phi_{1,2}$-closure operators. Let a topological L-space $(X, \tau)$ be fixed and $\phi_1, \phi_2 \in O_{(L^X, \tau)}$. The internal $\phi_{1,2}$-closure of the L-set $\mu : X \rightarrow L$ is the mapping $\phi_{1,2}\text{cl}\mu : X \rightarrow L$ defined by:

$$\phi_{1,2}\text{cl}\mu(x) = \bigvee_{\mathcal{M} \leq \mathcal{N}_{\phi_{1,2}}(x)} \mathcal{M}(\mu)$$

(2.5)

for all $x \in X$. In (2.5) the L-filter $\mathcal{M}$ may have additional properties, e.g., we may assume that is homogeneous or even that is ultra. Obviously, $\phi_{1,2}\text{cl}\mu \geq \mu$ holds for all $\mu \in L^X$. The mapping

$$\phi_{1,2}\text{cl} : \mathcal{F}_L X \rightarrow \mathcal{F}_L X$$

which assigns $\phi_{1,2}\text{cl} \mathcal{M}$ to each L-filter $\mathcal{M}$ on $X$, that is,

$$(\phi_{1,2}\text{cl} \mathcal{M})(\mu) = \bigvee_{\phi_{1,2}\text{cl} \mu \leq \mathcal{P} \mu} \mathcal{M}(\rho)$$

(2.6)

is called $\phi_{1,2}$-closure operator ([6]) of the characterized L-space $(X, \phi_{1,2}, \text{int})$ with respect to the related L-topology $\tau$. Obviously, the $\phi_{1,2}$-closure operator $\phi_{1,2}\text{cl}$ is isotone hull operator, that is, for all $\mathcal{M}, \mathcal{N} \in \mathcal{F}_L X$ we have

$$\mathcal{M} \leq \mathcal{N} \quad \text{implies} \quad \phi_{1,2}\text{cl} \mathcal{M} \leq \phi_{1,2}\text{cl} \mathcal{N}$$

and that $\mathcal{M} \leq \phi_{1,2}\text{cl} \mathcal{M}$.

Characterized $\mathcal{F}_T$ and $\mathcal{F}\phi_{1,2} - T_s$ spaces. The notions of characterized $\mathcal{F}_T$ and $\mathcal{F}\phi_{1,2} - T_s$ spaces are investigated and studied in [2,3] for all $s \in \{0, 1, 2, 2^1, 3, 4\}$. These spaces depend only on the usual points and the operation defined on the class of all L-subsets of $X$ endowed with a topological L-space $(X, \tau)$.

Let a topological L-space $(X, \tau)$ be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then the characterized L-space $(X, \varphi_{1,2}, \text{int})$ is said to be:

1. $\mathcal{F}_{T_0}$-space (resp. $\mathcal{F}_{T_1}$-space) if for all $x, y \in X$ such that $x \neq y$ there exists $\mu \in L^X$ and $\alpha \in L_0$ such that $\mu(x) < \alpha \leq (\varphi_{1,2}\text{int}\mu)(y)$ holds or (resp. and) there exist $\rho \in L^X$ and $\beta \in L_0$ such that $\rho(y) < \beta \leq (\varphi_{1,2}\text{int}\mu)(x)$ holds. The related topological L-space $(X, \tau)$ is said to be $\mathcal{F}\varphi_{1,2} - T_0$ (resp. $\mathcal{F}\varphi_{1,2} - T_1$) if for all $x, y \in X$ such that $x \neq y$ we have $x \not\in \mathcal{N}_{\varphi_{1,2}}(y)$ or (resp. and) $y \not\in \mathcal{N}_{\varphi_{1,2}}(x)$.

2. $\mathcal{F}_{T_2}$-space if for all $x, y \in X$ such that $x \neq y$, the infimum $\mathcal{N}_{\varphi_{1,2}}(x) \wedge \mathcal{N}_{\varphi_{1,2}}(y)$ does not exists. The related topological L-space $(X, \tau)$ is said to be $\mathcal{F}\varphi_{1,2} - T_2$ if $\mathcal{M} \xrightarrow{\varphi_{1,2}\text{int}} x, y$ implies $x = y$ for all $\mathcal{M} \in \mathcal{F}_L X$ and for all $x, y \in X$.

Characterized $\mathcal{F}R_k$ and $\mathcal{F}\phi_{1,2} - R_k$ spaces. The notions of characterized $\mathcal{F}R_k$ and $\mathcal{F}\phi_{1,2} - R_k$ spaces are introduced and studied in [3] for all $k \in \{0, 1\}$. Moreover, the notion of $\phi_{1,2}$ L-neighborhood filter at a point and at the ordinary subset of the characterized L-space $(X, \phi_{1,2}, \text{int})$ is applied in [4], to introduced and studied the notions of $\mathcal{F}R_k$-spaces for $k \in \{2, 3\}$. However, the notions of $\mathcal{F}\phi_{1,2} - R_k$ spaces are also given by means of the $\phi_{1,2}$ L-convergence at a point and at the ordinary set in the space.
Let a topological L-space \((X, \tau)\) be fixed and \(\varphi_1, \varphi_2 \in O_{(L^X, \tau)}\). Then the characterized L-space \((X, \varphi_{1,2,\text{int}})\) is said to be:

1. \(\text{FR}_{2}\)-space (resp. \(\text{FR}_{1}\)-space), if for all \(x \in X\), \(F \in \varphi_{1,2,C}(X)\) such that \(x \notin F\) (resp. \(F_1, F_2 \in \varphi_{1,2,C}(X)\) such that \(F_1 \cap F_2 = \varphi\)), the infimum \(\mathcal{N}_{\varphi_{1,2}}(x) \wedge \mathcal{N}_{\varphi_{1,2}}(F)\) (resp. \(\mathcal{N}_{\varphi_{1,2}}(F_1) \wedge \mathcal{N}_{\varphi_{1,2}}(F_2)\)) does not exist. The related topological L-space \((X, \tau)\) is said to be \(F\varphi_{1,2,\text{R}}\)-space (resp. \(F\varphi_{1,2,\text{R}_3}\)-space) if for all \(x \in X\), (resp. \(F \in \varphi_{1,2,C}(X)\)) and \(M \in \mathcal{F}_L X\) such that \(M \xrightarrow{\varphi_{1,2,\text{int}}} x\) (resp. \(F \varphi_{1,2,\text{cl}} M \xrightarrow{\varphi_{1,2,\text{int}}} x\)) we have \(\varphi_{1,2,\text{cl}} M \xrightarrow{\varphi_{1,2,\text{int}}} F \varphi_{1,2,\text{int}}\).

2. \(\text{FT}_{s}\)-space if and only if it is \(\text{FR}_{s}\) and \(\text{FT}_{1}\). The related topological L-space \((X, \tau)\) is said to be \(F\varphi_{1,2,\text{R}_k}\)-space (resp. \(F\varphi_{1,2,\text{T}_s}\)-space) if and only if it is \(F\varphi_{1,2,\text{R}_k}\) and \(F\varphi_{1,2,\text{T}_s}\) for \(k \in \{2, 3\}\) and \(s \in \{3, 4\}\).

3. Characterized Proximity L-spaces

In this section we are going to introduce and study the notion of characterized proximity L-spaces. We make at first the relation between the farness on L-sets and the finer relation on L-filters. So, we define a \(\varphi_{1,2}\) LS-neighborhood filter \(\mathcal{N}_{\varphi_{1,2}}(\mu)\) at the L-subset \(\mu \in L^X\) and we show some results for this notion. The notion of homogeneous L-filter \(\mu\) which is defined in (2.1) and the notion of \(\varphi_{1,2}\) L-neighborhood filter \(\mathcal{N}_{\varphi_{1,2}}(x)\) at the L-subset \(\mu \in L^X\) are applied to study the relation between the L-proximity \(\delta\) defined in [23] and our L-separation axioms in [2,3,4]. Moreover, the relation between characterized proximity L-spaces and the characterized FT\(_s\)-spaces, characterized \(\text{FR}_{2}\)-spaces and characterized \(\text{FR}_{3}\)-spaces are introduce

**Proposition 3.1** Let a topological L-space \((X, \tau)\) be fixed and \(\varphi_1, \varphi_2 \in O_{(L^X, \tau)}\) such that \(\varphi_2 \geq 1_{L^X}\) is isotone and idempotent and \(\varphi_1\) is wfsp with respect to \(\varphi_1 \text{OF}(X)\). Then the supremum of the \(\varphi_{1,2}\) L-neighborhood filters \(\mathcal{N}_{\varphi_{1,2}}(x)\) at \(x \in X\) which is given by:

\[
\mathcal{N}_{\varphi_{1,2}}(x) = \bigvee_{\varphi_{1,2}} \mathcal{N}_{\varphi_{1,2}}(x)
\]

for all \(\mu \in L^X\) is L-filter on \(X\) called a \(\varphi_{1,2}\) L-neighborhood filter at \(\mu \in L^X\).

**Proof.** Fix an \(\alpha \in L_0\), then because of (2.4) and the condition \(\varphi_2 \geq 1_{L^X}\), we have

\[
\mathcal{N}_{\varphi_{1,2}}(\alpha) = \bigwedge_{\varphi_{1,2}} \mathcal{N}_{\varphi_{1,2}}(y)(\alpha) = \bigwedge_{\varphi_{1,2}} (\varphi_{1,2,\text{int}} \alpha)(y) = \bigwedge_{\varphi_{1,2}} \alpha(y) = \alpha
\]

and

\[
\mathcal{N}_{\varphi_{1,2}}(\bar{\alpha}) = \bigwedge_{\varphi_{1,2}} \mathcal{N}_{\varphi_{1,2}}(y)(\bar{\alpha}) = \bigwedge_{\varphi_{1,2}} (\varphi_{1,2,\text{int}} \bar{\alpha})(y) = \bigwedge_{\varphi_{1,2}} \bar{\alpha}(y) = 1.
\]

Thus, condition (F1) is fulfilled. To prove condition (F2), let \(\rho, \eta \in L^X\), then because of Proposition 2.4 and (2.4) we have

\[
\mathcal{N}_{\varphi_{1,2}}(\mu)(\rho \wedge \eta) = \bigwedge_{\varphi_{1,2,\text{int}}} (\varphi_{1,2,\text{int}} \rho)(y) \wedge \bigwedge_{\varphi_{1,2,\text{int}}} (\varphi_{1,2,\text{int}} \eta)(y) = \mathcal{N}_{\varphi_{1,2}}(\mu)(\rho) \wedge \mathcal{N}_{\varphi_{1,2}}(\mu)(\eta).
\]
Hence, $N_{\Theta_2}(\mu)$ is $L$-filter on $X$. Since $(\phi_{1,2}, \text{int})(x) \leq \rho(x)$ holds for all $x \in X$ and $\rho \in L^X$, then $N_{\Theta_2}(\mu)(\rho) \leq \mu(\rho)$ holds for all $\rho \in L^X$. Thus, $\mu \leq N_{\Theta_2}(\mu)$ and therefore $N_{\Theta_2}(\mu)$ fulfills condition (N1). For condition (N2), let $\rho, \eta \in L^X$ such that $\rho \leq \eta$. Because of Proposition 2.4, we have $\phi_{1,2}, \text{int}\rho \leq \phi_{1,2}, \text{int}\eta$ which implies $\bigwedge_{0<\mu(y)} (\phi_{1,2}, \text{int}\rho)(y) \leq \bigwedge_{0<\mu(y)} (\phi_{1,2}, \text{int}\eta)(y)$ holds for all $y \in X$. Hence $N_{\Theta_2}(\mu)(\rho) \leq N_{\Theta_2}(\mu)(\eta)$ and therefore (N2) is fulfilled. Since for any $y \in X$ we have $\bigwedge_{0<\mu(y)} y \mapsto \bigwedge_{0<\mu(y)} (\phi_{1,2}, \text{int}\rho)(y)$ represents the mapping $\phi_{1,2}, \text{int}\rho$. Then from Proposition 2.4 we have $N_{\Theta_2}(\mu)(\rho) = \bigwedge_{0<\mu(x)} (\phi_{1,2}, \text{int}\rho(x)) = \bigwedge_{0<\mu(x)} (\phi_{1,2}, \text{int}\rho)(x)$, and then $N_{\Theta_2}(\mu)(\bigwedge_{0<\mu(y)} y \mapsto \bigwedge_{0<\mu(y)} (\phi_{1,2}, \text{int}\rho)(y)) = N_{\Theta_2}(\mu)(\rho)$ for all $y \in X$ and $\rho \in L^X$. Thus, condition (N3) is also fulfilled and therefore $N_{\Theta_2}(\mu)$ fulfilled the conditions (N1) to (N3) of the $\phi_{1,2}$-neighborhood filters. □

Not that in [11] the supremum of the empty set of the $L$-filters is the finest $L$-filter. This means $N_{\Theta_2}(\emptyset) \leq \mu'$ for all $\mu \in L^X$. Because of (2.4) the equations (2.1) and (2.2) can be written as in the following:

$$\mu'(\rho) = \bigwedge_{0<\mu(x)} \rho(x)$$  \hspace{1cm} (3.2)

$$N_{\Theta_2}(\mu)(\rho) = \bigwedge_{0<\mu(x)} N_{\Theta_2}(\mu)(\rho) = \bigwedge_{0<\mu(x)} (\phi_{1,2}, \text{int}\rho)(x)$$  \hspace{1cm} (3.3)

for all $\rho \in L^X$. Here a useful remark is given

Remark 3.1. The homogeneous $L$-filter $x$ at the ordinary point $x$ is nothing that a homogeneous $L$-filter $x_\alpha$ at the $L$-point $x_\alpha$, that is, $x_\alpha = x$ for all $x \in X$ and $\alpha \in L_0$. Moreover, the $\phi_{1,2}$ $L$-neighborhood filter $N_{\Theta_2}(x)$ at $x \in X$ is itself a $\phi_{1,2}$-neighborhood filter $N_{\Theta_2}(x_\alpha)$ at $x_\alpha$.

The $\phi_{1,2}$ L-neighborhood filter $N_{\Theta_2}(\mu)$ at the L-subset $\mu \in L^X$ and the homogeneous L-filter $\mu'$ fulfill the following properties.

Lemma 3.1. Let $(X, \tau)$ be a topological $L$-space and $\phi_1, \phi_2 \in O_{(L^X, \tau)}$. Then for all $\mu, \rho \in L^X$ the following properties are fulfilled:

1. $\mu \leq \rho$ implies $N_{\Theta_2}(\rho') \leq \mu'$ and $N_{\Theta_2}(\mu) \leq \rho$ implies $N_{\Theta_2}(\rho') \leq \mu'$.

2. $\mu \leq \rho$ implies $N_{\Theta_2}(\mu) \leq N_{\Theta_2}(\rho)$.

3. $N_{\Theta_2}(\mu \vee \rho) = N_{\Theta_2}(\mu) \vee N_{\Theta_2}(\rho)$.

4. $N_{\Theta_2}(\mu) \leq \rho$ implies $\mu \leq \rho$.

5. $N_{\Theta_2}(\mu) \leq \rho$ implies there is an $\eta \in L^X$ such that $N_{\Theta_2}(\mu) \leq \eta$ and $N_{\Theta_2}(\eta) \leq \rho$.

Proof. Let $\mu \leq \rho$. From (N1) we have $\mu \leq N_{\Theta_2}(\rho)$ and therefore for all $\eta \in L^X$ we have $\bigwedge_{0<\mu(x)} \eta(x) \geq \bigwedge_{0<\rho(y)} (\phi_{1,2}, \text{int}\eta)(y)$. Hence, $\bigwedge_{0<\mu(x)} \eta(x) \leq \bigwedge_{0<\rho(y)} (\phi_{1,2}, \text{int}\eta)(y)$. Thus, $N_{\Theta_2}(\rho')(\eta) \geq \mu'(\eta)$ and therefore, $N_{\Theta_2}(\rho') \leq \mu'$. Similarly, if $N_{\Theta_2}(\mu) \leq \rho'$, then from (N1) we have $\mu \leq \rho'$ which implies $N_{\Theta_2}(\rho') \leq \mu'$. Thus, (1) is fulfilled. Since $\mu \leq \rho$ implies $\mu(x) \leq \rho(x)$ for all $x \in X$, then $\bigwedge_{0<\mu(x)} (\phi_{1,2}, \text{int}\eta)(x) \geq \bigwedge_{0<\rho(x)} (\phi_{1,2}, \text{int}\eta)(x)$.
Hence, $N_{\Phi_{2,1}}(\mu)(\eta) \geq N_{\Phi_{2,1}}(\rho)(\eta)$ for all $\eta \in L^X$ and therefore $N_{\Phi_{2,1}}(\mu) \leq N_{\Phi_{2,1}}(\rho)$. Hence, (2) is fulfilled.

Since $\mu, \rho \leq \mu \lor \rho$, then from (2) we have $N_{\Phi_{2,1}}(\mu) \land N_{\Phi_{2,1}}(\rho) \leq N_{\Phi_{2,1}}(\mu \lor \rho)$. Now, let $\eta \in L^X$ then

$$(N_{\Phi_{2,1}}(\mu) \land N_{\Phi_{2,1}}(\rho))(\eta) = \bigvee_{k_1, k_2 \leq \eta} (N_{\Phi_{2,1}}(\mu)(k_1) \land N_{\Phi_{2,1}}(\rho)(k_2))$$

$$= \bigvee_{k_1, k_2 \leq \eta} (\bigwedge_{0 \leq \mu(x)} \phi_{1,2} \int k_1(x) \land \bigwedge_{0 \leq \rho(y)} \phi_{1,2} \int k_2(y))$$

$$\leq \bigvee_{k_1, k_2 \leq \eta} \bigwedge_{0 \leq \mu(x)} \phi_{1,2} \int (k_1 \land k_2)(z)$$

$$\leq \bigwedge_{0 \leq \mu(x)} \phi_{1,2} \int \eta(z) = N_{\Phi_{2,1}}(\mu \land \rho)(\eta)$$

Hence, $N_{\Phi_{2,1}}(\mu) \land N_{\Phi_{2,1}}(\rho) \geq N_{\Phi_{2,1}}(\mu \lor \rho)$ and therefore (3) is fulfilled. To prove (4), let $N_{\Phi_{2,1}}(\mu) \leq \rho$ holds. Because of (2.1), (3.1) and (N1) we have $\mu \leq N_{\Phi_{2,1}}(\mu)$ and then $\mu \leq \rho$. Hence, Proposition 2.1 implies $\mu \leq \rho$. Thus, (4) is fulfilled. Finally, let $N_{\Phi_{2,1}}(\mu) \leq \hat{\rho}$. Then $\bigwedge_{0 \leq \mu(x)} \phi_{1,2} \int \lambda(x) \geq \bigwedge_{0 \leq \rho(y)} \lambda(y)$ for all $\lambda \in L^X$. Hence there is $\eta \in L^X$ such that

$$\bigwedge_{0 \leq \mu(x)} \phi_{1,2} \int \lambda(x) \geq \bigwedge_{0 \leq \eta(z)} \lambda(z) \geq \bigwedge_{0 \leq \rho(y)} \lambda(y).$$

This means there is $\eta \in L^X$ such that $N_{\Phi_{2,1}}(\mu)(\lambda) \geq \eta(\lambda)$ and $N_{\Phi_{2,1}}(\mu)(\lambda) \geq \hat{\rho}(\lambda)$ are hold for all $\lambda \in L^X$. Thus, $N_{\Phi_{2,1}}(\mu) \leq \eta$ and $N_{\Phi_{2,1}}(\eta) \leq \hat{\rho}$ are also hold. Consequently, (5) is fulfilled. $\square$

In the characterized L-space $(X, \phi_{1,2}, \int)$, the L-proximity will be identified with the finer relation on the L-filters, specially with the finer relation on the $\phi_{1,2}$ L-neighborhood filters. This shown in the following proposition.

**Proposition 3.2** Let $(X, \tau)$ be a topological L-space and $\phi_1, \phi_2 \in O_{(L^X, \tau)}$. Then the binary relation $\delta$ on $L^X$ which is defined by:

$$\mu \delta \rho \text{ if and only if } N_{\Phi_{2,1}}(\rho) \leq \mu'$$

for all $\mu, \rho \in L^X$ is L-proximity on $X$.

**Proof.** Let $\mu, \rho \in L^X$ such that $\mu \delta \rho$, then $N_{\Phi_{2,1}}(\rho) \leq \mu'$. Because of (1) in Lemma 3.1, we have $N_{\Phi_{2,1}}(\mu) \leq \rho'$ and therefore $\rho \delta \mu$. Hence, condition (P1) is fulfilled.

Since $N_{\Phi_{2,1}}(\mu) \leq N_{\Phi_{2,1}}(\mu \lor \rho)$ and $N_{\Phi_{2,1}}(\rho) \leq N_{\Phi_{2,1}}(\mu \lor \rho)$ are hold for all $\mu, \rho \in L^X$, then $N_{\Phi_{2,1}}(\mu \lor \rho) \leq \eta'$ implies $N_{\Phi_{2,1}}(\mu) \leq \eta'$ and $N_{\Phi_{2,1}}(\rho) \leq \eta'$ are hold for all $\eta \in L^X$. This means $\eta \delta (\mu \lor \rho)$ implies $\eta \delta \mu$ and $\eta \delta \rho$. Conversely, let $\eta \delta \mu$ and $\eta \delta \rho$ for all $\mu, \rho, \eta \in L^X$, then $N_{\Phi_{2,1}}(\mu) \leq \eta'$ and $N_{\Phi_{2,1}}(\rho) \leq \eta'$ are hold. Hence, (3) in Lemma 3.1 implies $N_{\Phi_{2,1}}(\mu \lor \rho) \leq N_{\Phi_{2,1}}(\mu) \lor N_{\Phi_{2,1}}(\rho) \leq \eta'$ holds and therefore $\eta \delta (\mu \lor \rho)$. Consequently, (P2) is fulfilled. To prove (P3), since $N_{\Phi_{2,1}}(\rho) \leq \mu'$ holds for all $\mu \in L^X$. Then, $\mu \delta \rho$ for all $\mu \in L^X$. Hence, $\mu = \rho$ or $\rho = \rho$ implies $\mu \delta \rho$ for all $\mu, \rho \in L^X$. Thus, (P3) is fulfilled.

Let $\mu, \rho \in L^X$ such that $\mu \delta \rho$, then $N_{\Phi_{2,1}}(\rho) \leq \mu'$. Because of (1) and (4) in Lemma 3.1, we have $N_{\Phi_{2,1}}(\mu) \leq \delta$ and therefore $\mu \leq \rho'$, that is, (P4) is fulfilled. Finally, let $\mu, \rho \in L^X$ such that $\mu \delta \rho$; then
such that holds for all δ. Then, is L-proximity on \( \tau \) and \( \delta \) is separated L-proximity and let \( \Delta \) be a topological L-space, \( \phi_{\alpha,2} \in O_{(L^X, \tau)} \) and \( \delta \) is an L-proximity on \( X \). 

**Proposition 3.3** Let \((X, \tau)\) be a topological L-space, \( \phi_{1,2} \in O_{(L^X, \tau)} \) and \( \delta \) is an L-proximity on \( X \). Then the characterized proximity L-space \((X, \phi_{1,2} \cdot \text{int}_\delta)\) is characterized \( FT_0 \)-space if and only if \( \delta \) is separated.

**Proof.** Let \((X, \phi_{1,2} \cdot \text{int}_\delta)\) is characterized \( FT_0 \)-space and let \( x, y \in X \) such that \( x \neq y \). Then \( \phi_{1,2} \cdot \text{int}_\delta(x) \leq x' \), and therefore there is \( \mu \in L^X \) such that \( \phi_{1,2} \cdot \text{int}_\delta \mu(y) > \mu(x) \). Because of (3.4), we have \( \bigvee_{\rho} \phi_{1,2} \cdot \text{int}_\delta \mu(y) > \mu(\phi_{1,2} \cdot \text{int}_\delta \mu(x)) \) holds for all \( \rho \in L^X \)

with \( \mu \geq \rho \), that is, \( \mu(x) < \rho(y) \) holds for all \( \rho \in L^X \) with \( \phi_{1,2} \cdot \text{int}_\delta \mu(\rho) \leq \mu(\rho) \). Choice \( \mu = x' \) and \( \rho = y' \), then because of Remark 3.1, we get \( \phi_{1,2} \cdot \text{int}_\delta \mu(y) = y' \). Using Proposition 3.2 we get \( x' \delta y' \), and therefore \( x' \delta y' \) holds for all \( \alpha, \beta \in L_0 \). Thus, \( \delta \) is separated.

Conversely, let \( \delta \) is separated L-proximity and let \( x, y \in X \) such that \( x \neq y \). Then, \( x \delta y' \) and because of Proposition 3.2 and Remark 3.1, we have \( \phi_{1,2} \cdot \text{int}_\delta \mu(y) = y' \). Therefore, \( \phi_{1,2} \cdot \text{int}_\delta \mu(\rho) \leq \mu(\rho) \) holds for all \( \mu \in L^X \). Consider, \( \mu = x' \) we get \( \phi_{1,2} \cdot \text{int}_\delta x'(y) = 1 \) and \( x'(x) = 0 \). Hence, there exists \( \mu = x' \in L^X \) such that \( \phi_{1,2} \cdot \text{int}_\delta \mu(z) = 1 \geq \mu(x) \), that is, \( x' \leq \phi_{1,2} \cdot \text{int}_\delta \mu \) and therefore \((X, \phi_{1,2} \cdot \text{int}_\delta)\) is characterized \( FT_0 \)-space. \( \Box \)

In the following proposition, the \( \phi_{1,2} \cdot \text{closure} \) of the L-subsets in the characterized proximity L-space \((X, \phi_{1,2} \cdot \text{int}_\delta)\) are equivalent with the L-subsets by the L-proximity \( \delta \) on \( X \).

**Proposition 3.4** Let \((X, \tau)\) be a topological L-space, \( \phi_{1,2} \in O_{(L^X, \tau)} \) such that \( \phi_{1,2} \geq \Delta \) and \( \delta \) is an L-proximity on \( X \). Then, \( \mu \delta \rho \) if and only if \( \phi_{1,2} \cdot \text{cl}_\delta \mu(x) \geq \phi_{1,2} \cdot \text{cl}_\delta \mu(y) \) for all \( \mu, \rho \in L^X \)

**Proof.** Let \( \mu, \rho \in L^X \) such that \( \phi_{1,2} \cdot \text{cl}_\delta \mu(x) \geq \phi_{1,2} \cdot \text{cl}_\delta \mu(y) \), then Proposition 3.2 implies \( \phi_{1,2} \cdot \text{cl}_\delta \mu \geq \phi_{1,2} \cdot \text{cl}_\delta \mu \). Since \( \phi_{1,2} \geq \Delta \) and \( \phi_{1,2} \cdot \text{int}_\delta \mu \) is isotonous operator, then \( \mu \leq \phi_{1,2} \cdot \text{cl}_\delta \mu \)
and $N^\delta_{\varphi_{1,2}}(\rho) \leq N^\delta_{\varphi_{1,2}}(\varphi_{1,2,\text{cl}}\delta\rho)$ are hold for all $\mu, \rho \in L^X$. Hence, $N^\delta_{\varphi_{1,2}}(\rho) \leq \mu'$ and therefore $\mu \overline{\delta} \rho$.

Conversely, Let $\mu, \rho \in L^X$ such that $\mu \overline{\delta} \rho$. Because of Proposition 3.2 we have $N^\delta_{\varphi_{1,2}}(\rho) \leq \mu$. Since $\varphi_{1,2} \geq 1_{L^X}$ and $N^\delta_{\varphi_{1,2}}(\eta)$ is isotope operator, then $\mu' \leq \varphi_{1,2,\text{cl}}\delta\mu'$ holds for all $\mu' \in L^X$ and therefore $\rho \leq N^\delta_{\varphi_{1,2}}(\mu') \leq N^\delta_{\varphi_{1,2}}(\varphi_{1,2,\text{cl}}\delta\mu')$. From Lemma 3.1, we have $N^\delta_{\varphi_{1,2}}(\varphi_{1,2,\text{cl}}\delta\mu) \leq \rho'$ and then $\rho \overline{\delta} \varphi_{1,2,\text{cl}}\delta\mu$. Therefore, $\varphi_{1,2,\text{cl}}\delta\mu \leq N^\delta_{\varphi_{1,2}}(\rho')$. Using Lemma 3.1 we get $\varphi_{1,2,\text{cl}}\delta\mu \leq N^\delta_{\varphi_{1,2}}(\rho')$. Thus, $N^\delta_{\varphi_{1,2}}(\varphi_{1,2,\text{cl}}\delta\rho) \leq (\varphi_{1,2,\text{cl}}\delta\mu)'$ and therefore $\varphi_{1,2,\text{cl}}\delta\mu \overline{\delta} \varphi_{1,2,\text{cl}}\delta\rho$ for all $\mu, \rho \in L^X$. □

In the following theorem we give new description for the characterized $FR_2$-spaces and its related $F\varphi_{1,2}$-spaces.

**Theorem 3.1** Let $(X, \tau)$ be a topological L-space and $\varphi_1, \varphi_2 \in O_{(l^X, z)}$. Then the following statements are equivalent:

1. $(X, \tau)$ is $F\varphi_{1,2}$-R$_2$-space.
2. $(X, \varphi_{1,2}, \text{int})$ is characterized $FR_2$-space.
3. For all $x \in X$ and $\mu \in L^X$ with $N_{\varphi_{1,2}}(x) \leq \mu$, there exists $\rho \in L^X$ such that $N_{\varphi_{1,2}}(x) \leq \rho$ and $N_{\varphi_{1,2}}(\varphi_{1,2,\text{cl}}\delta\rho) \leq \mu$ are hold.

**Proof.** The equivalent between (1) and (2) is already proved in [4, Theorem 2.1]. Now, let $(X, \varphi_{1,2}, \text{int})$ is characterized $FR_2$-space and let $\rho \in L^X$ holds for all $x \in X$ and $\mu \in L^X$. Then because of part (5) in Lemma 3.1, there exists $\rho \in L^X$ such that $N_{\varphi_{1,2}}(x) \leq \rho$ and $N_{\varphi_{1,2}}(\rho) \leq \mu$ are hold. By using (2.1) and (3.1) we have also $\bigwedge_{0 \in \rho(z)} (\varphi_{1,2}, \text{int})\eta(\rho(z)) \geq \bigwedge_{0 \in \rho(y)} \eta(\rho(y))$ holds for all $\eta \in L^X$. Since $(X, \varphi_{1,2}, \text{int})$ is characterized $FR_2$-space, then we have $\varphi_{1,2,\text{cl}}\text{cl}(N_{\varphi_{1,2}}(x)) = N_{\varphi_{1,2}}(x)$ for all $x \in X$ and then $\varphi_{1,2,\text{cl}}\text{int}\eta(z) = \bigwedge_{0 \in \rho(z), \varphi_{1,2,\text{cl}}\text{cl} \varphi_{1,2,\text{cl}}\text{int} \sigma(z) \geq \bigwedge_{0 \in \rho(y)} \eta(\rho(y))$ holds for all $\eta \in L^X$. Thus, $\bigvee_{0 \in \rho(z), \varphi_{1,2,\text{cl}}\text{cl} \varphi_{1,2,\text{cl}}\text{int} \sigma(z) \geq \bigwedge_{0 \in \rho(y)} \eta(\rho(y))$ holds. Since $\varphi_{1,2,\text{cl}}\text{cl}$ is hull operator, then for all $\rho \in L^X$ we have $\bigvee_{0 \in \rho(z), \varphi_{1,2,\text{cl}}\text{cl} \varphi_{1,2,\text{cl}}\text{int} \sigma(z) = \bigwedge_{0 \in \rho(z), \varphi_{1,2,\text{cl}}\text{cl} \varphi_{1,2,\text{cl}}\text{int} \sigma(z) \geq \bigwedge_{0 \in \rho(y)} \eta(\rho(y))$ holds and therefore (2.4) and (2.6) imply $\varphi_{1,2,\text{cl}}\text{cl} \varphi_{1,2,\text{cl}}\text{int} \eta(z) = N_{\varphi_{1,2}}(z) \leq \hat{\varphi}_{1,2,\text{cl}}\delta\rho(z) \leq \hat{\varphi}_{1,2,\text{cl}}\delta\rho(z)$ for all $\eta \in L^X$. Then $\varphi_{1,2,\text{cl}}\text{cl} \varphi_{1,2,\text{cl}}\text{int} \eta(z) \geq \bigwedge_{0 \in \rho(y)} \eta(\rho(y))$ holds and therefore $N_{\varphi_{1,2}}(\varphi_{1,2,\text{cl}}\delta\rho)(\eta) \geq \hat{\varphi}_{1,2,\text{cl}}\delta\rho(\eta)$ is also holds. Thus, $N_{\varphi_{1,2}}(\varphi_{1,2,\text{cl}}\delta\rho) \leq \hat{\varphi}_{1,2,\text{cl}}\delta\rho$, that is, $N_{\varphi_{1,2}}(\varphi_{1,2,\text{cl}}\delta\rho) \leq \hat{\varphi}_{1,2,\text{cl}}\delta\rho$ are hold. Hence, (3) is fulfilled. Finally, let (3) is fulfilled and let $x \in X$ and $F \in \varphi_{1,2,\text{cl}}(X)$ such that $x \notin F$. Then, $x \in F'$ and therefore $x' \leq \hat{\varphi}_{1,2,\text{cl}}\delta\rho$. Because of Proposition 2.1 and Lemma 3.1 we have $N_{\varphi_{1,2}}(x) \leq \hat{\varphi}_{1,2,\text{cl}}\delta\rho$ holds and then there is $\rho \in L^X$ such that $N_{\varphi_{1,2}}(x) \leq \rho$ and $N_{\varphi_{1,2}}(\varphi_{1,2,\text{cl}}\delta\rho) \leq \rho$ are hold. By using part (1) from
Lemma 3.1. We get \( \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(F) \leq \rho^* \) and hence
\[
\mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(x)(\eta) \wedge \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(F)(\sigma) \geq \rho(\eta) \wedge \rho^*(\sigma) = \bigwedge_{\omega(\rho_{\varphi}(y))} \eta(y) \wedge \bigwedge_{\omega(\rho^*(\sigma))} \sigma(z) \] holds for all \( \eta, \sigma \in L^X \).

Consider \( \rho = x_1 \vee y_1 \) for all \( x \neq y \in F' \) we get
\[
\mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(x)(\eta) \wedge \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(F)(\sigma) \geq \bigwedge_{\omega(\rho_{\varphi}(y))} \eta(y) \wedge \bigwedge_{\omega(\rho^*(\sigma))} \sigma(z)
\]
holds for all \( \eta, \sigma \in L^X \). Since for choice \( \eta = x_1 \vee y_1 \) and \( \sigma = (x_1 \vee y_1)' \) we have \( \uparrow \eta \wedge \sigma = 0 \) and \( \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(\eta) \wedge \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(F)(\sigma) > 0 \) are fulfilled, then the infimum \( \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(x) \wedge \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(F) \) does not exists.

Hence, \((X, \varphi_{\rho} \int)\) is characterized \( FR_2 \)-space and therefore (3) and (2) are equivalent. □

In the following proposition we show that the associated characterized proximity \( L \)-space \((X, \varphi_{\rho,2} \cdot \text{int}_g)\) is characterized \( FR_2 \)-space if the related topological \( L \)-space \((X, \tau)\) is \( F \varphi_{1,2} \cdot R_2 \) space.

**Proposition 3.5** Let \((X, \tau)\) be a topological \( L \)-space, \( \varphi_1, \varphi_2 \in O(L^X, \tau) \) and \( \delta \) is an \( L \)-proximity on \( X \).

Then the associated characterized proximity \( L \)-space \((X, \varphi_{\rho,2} \cdot \text{int}_g)\) is characterized \( FR_2 \)-space if \((X, \tau)\) is \( F \varphi_{1,2} \cdot R_2 \) space.

**Proof.** Let \( x \in X \) and \( \mu \in L^X \) with \( \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(x) \leq \mu \). Because of Proposition 3.2, we have \( \mu' \delta x_1 \) and from (P5), there is \( \rho L^X \) such that \( \mu' \delta \rho \) and \( \rho' \delta x_1 \). Therefore Proposition 3.4 implies \( \varphi_{\rho,2} \cdot \text{cl}_g \mu \delta \varphi_{\rho,2} \cdot \text{cl}_g \rho \) and hence \( \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}} \leq (\varphi_{\rho,2} \cdot \text{cl}_g \mu)' \) and \( \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(x) \leq \rho \) are hold. Hence, \( \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}} \leq \mu \) implies there is \( \rho L^X \) such that \( \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}}(x) \leq \rho \) and \( \mathcal{N}^{\varphi_{\rho}}_{\varphi_{\rho}} \leq \mu \) are hold. Since \((X, \tau)\) is \( F \varphi_{1,2} \cdot R_2 \) space, then from Theorem 3.1 we have \((X, \varphi_{\rho,2} \cdot \text{int}_g)\) is characterized \( FR_2 \)-space. □

In the following theorem we give a new description for the characterized \( FR_3 \)-spaces and its related \( F \varphi_{1,2} \cdot R_3 \) spaces.

**Theorem 3.2** Let \((X, \tau)\) be a topological \( L \)-space and \( \varphi_1, \varphi_2 \in O(L^X, \tau) \). Then the following statements are equivalent:

1. \((X, \tau)\) is \( F \varphi_{1,2} \cdot R_3 \) space.
2. \((X, \varphi_1, \varphi_2) \cdot \text{int}\) is characterized \( FR_3 \)-space.
3. For all \( F \in \varphi_1, C(X) \) and \( \mu \in L^X \) with \( \mathcal{N}^{\varphi_2} \leq \mu \), there exists \( \rho L^X \) such that \( \mathcal{N}^{\varphi_2} \leq \rho \) and \( \mathcal{N}^{\varphi_2} \cdot \text{cl}_g \rho \) are hold.

**Proof.** The equivalent between (1) and (2) is already proved in [4, Theorem 3.1]. Now, let \((X, \varphi_1, \varphi_2) \cdot \text{int}\) is characterized \( FR_3 \)-space and let \( \mathcal{N}^{\varphi_2} \leq \mu \) holds for all \( F \in \varphi_1, C(X) \) and \( \mu \in L^X \). Then because of part (5) in Lemma 3.2, there exists \( \rho L^X \) such that \( \mathcal{N}^{\varphi_2} \leq \rho \) and \( \mathcal{N}^{\varphi_2} \cdot \mu \) are hold. By using (2.1) and (3.1) we also have \( \bigwedge_{\omega(\mu(x))} \varphi_{\rho,2} \cdot \text{int} \eta(z) \geq \bigwedge_{\omega(\mu(y))} \eta(y) \) holds for all \( \eta L^X \). Since \((X, \varphi_1, \varphi_2) \cdot \text{int}\) is characterized \( FR_3 \)-space, then \( \varphi_{\rho,2} \cdot \text{cl}(F) = \mathcal{N}^{\varphi_2} \cdot \mu \) and therefore \( \bigwedge_{x \in F} \varphi_{\rho,2} \cdot \text{int} \eta(x) = \bigwedge_{\varphi_{\rho,2} \cdot \text{int} \sigma(x) \in F} \varphi_{\rho,2} \cdot \text{int} \sigma(x) \) holds for all \( \eta L^X \). Since \( \mathcal{N}^{\varphi_2} \leq \rho \), then Proposition 2.1 and Lemma 3.1

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we have $\check{F} \leq \check{\rho}$ and therefore $F \subseteq S_0(\rho)$, that is, $\bigwedge_{x \in F} \varphi_{1,2} \cdot \text{int} \eta(x) \geq \bigwedge_{0 \leq \rho(x)} \varphi_{1,2} \cdot \text{int} \eta(x)$ holds for all $\eta \in L^X$. Hence,

$$\bigwedge_{x \in F} \varphi_{1,2} \cdot \text{int} \eta(x) \geq \bigwedge_{0 \leq \rho(x)} \bigvee_{x \in F} \varphi_{1,2} \cdot \text{int} \sigma(x) \geq \bigwedge_{0 \leq \rho(z)} \varphi_{1,2} \cdot \text{int} \eta(z) \geq \bigwedge_{0 \leq \mu(y)} \eta(y).$$

Since $\varphi_{1,2} \cdot \text{cl}$ is hull operator, then we have $\bigvee_{\varphi_{1,2} \cdot \text{cl} \eta(\sigma(z))} \bigwedge_{x \in F} \varphi_{1,2} \cdot \text{int} \sigma(x) \geq \bigwedge_{0 \leq \mu(y)} \eta(y)$ holds for all $\rho \in L^X$ and therefore from the distributivity of $L$ we get

$$\bigwedge_{0 \leq \rho(z)} \varphi_{1,2} \cdot \text{cl} \eta(x) = \bigwedge_{0 \leq \rho(z)} \bigwedge_{x \in F} \varphi_{1,2} \cdot \text{cl} \eta(x) \geq \bigwedge_{0 \leq \mu(y)} \eta(y).$$

Hence, $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2} \cdot \text{cl} \rho)(\eta) \geq \widetilde{\mu}(\eta)$ holds for all $\eta \in L^X$ and therefore $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2} \cdot \text{cl} \rho) \leq \widetilde{\mu}$, that is, $\mathcal{N}_{\varphi_{1,2}}(F) \leq \widetilde{\mu}$, implies that there exists $\rho \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(F) \leq \rho$ and $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2} \cdot \text{cl} \rho) \leq \widetilde{\mu}$ are hold. Hence, (3) is fulfilled. Finally, let (3) is fulfilled and let $F_1, F_2 \in \varphi_{1,2} \cdot C(X)$ such that $F_1 \cap F_2 = \varnothing$. Because of Proposition 2.1 and Lemma 3.1 we have $\mathcal{N}_{\varphi_{1,2}}(F_1) \leq \mathcal{N}_{\varphi_{1,2}}(F_2)$ holds and then there is $\rho \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(F_1) \leq \rho$ and $\mathcal{N}_{\varphi_{1,2}}(F_2) \leq (\varphi_{1,2} \cdot \text{cl}) \rho \leq \rho'$ are hold and hence $\mathcal{N}_{\varphi_{1,2}}(F_1)(\eta) \vee \mathcal{N}_{\varphi_{1,2}}(F_2)(\sigma) \geq \rho(\eta) \wedge \rho'(\sigma)$ holds for all $\eta, \sigma \in L^X$. Consider, $\rho = \mathcal{N}_{\varphi_{1,2}}(F_1) \vee x_1$ for all $x \in F_2 \backslash F_1$. Then for $\eta = \rho$ and $\sigma = \rho'$ we have $\sup(\eta \wedge \sigma) = 0$ and $\mathcal{N}_{\varphi_{1,2}}(F_1)(\eta) \vee \mathcal{N}_{\varphi_{1,2}}(F_2)(\sigma) \geq \sup(\eta \wedge \sigma) = 0$ are fulfilled, then the infimum $\mathcal{N}_{\varphi_{1,2}}(F_1) \wedge \mathcal{N}_{\varphi_{1,2}}(F_2)$ does not exists. Hence, $(X, \varphi_{1,2} \cdot \text{int})$ is characterized FR$_3$-space and therefore (3) and (2) are equivalent. □

In the following we are going to show an important relation between the associated characterized proximity L-space and the characterized FR$_3$-space.

**Proposition 3.5** Let $(X, \tau)$ be a topological L-space and $\varphi_1, \varphi_2 \in \mathcal{O}(L^X, \tau)$ such that $\varphi_2 \geq 1_{L^X}$ is isotone and $\varphi_1$ is wfp with respect to $\varphi_1 \cdot \text{OF}(X)$, where $L$ is a complete chain. If $(X, \tau)$ is F $\varphi_{1,2} \cdot \text{R}_3$ space, then the binary relation $\delta$ on $X$ which is defined by:

$$\mu \delta \rho \iff \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2} \cdot \text{cl}) \mu \leq (\varphi_{1,2} \cdot \text{cl}) \rho$$

(3.6)

for all $\mu, \rho \in L^X$ is L-proximity on $X$ and $(X, \delta)$ is proximity L-space. On other hand if $(X, \delta)$ is proximity L-space for $\delta$ is defined by (3.6), then the associated characterized proximity L-space $(X, \varphi_{1,2} \cdot \text{int})$ is characterized FR$_3$-space.

**Proof.** Let $(X, \tau)$ be F $\varphi_{1,2} \cdot \text{R}_3$-space and $\delta$ a binary relation on $X$ defined by (3.6). Then, $\mu \delta \rho$ implies $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2} \cdot \text{cl}) \mu \leq (\varphi_{1,2} \cdot \text{cl}) \rho$ and from Lemma 3.1 part (1) we get $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2} \cdot \text{cl} \rho) \leq (\varphi_{1,2} \cdot \text{cl}) \mu$ and then $\rho \delta \mu$. Hence, condition (P1) is fulfilled. For showing condition (P2), let $(\mu \vee \rho) \delta \eta$ for a fixed L-subsets $\mu, \rho, \eta \in L^X$. Then, $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2} \cdot \text{cl}(\mu \vee \rho)) \leq (\varphi_{1,2} \cdot \text{cl}) \eta$'. Since $L$ is complete chain, $\varphi_2 \geq 1_{L^X}$ is isotone and $\varphi_1$ is wfp with respect to $\varphi_1 \cdot \text{OF}(X)$, then $\varphi_{1,2} \cdot \text{cl}(\mu \vee \rho) = \varphi_{1,2} \cdot \text{cl} \mu \vee \varphi_{1,2} \cdot \text{cl} \rho$ and therefore
\( \mathcal{N}_{\varphi_{1,2}, \mu} (\varphi_{1,2}, cl \mu \land \varphi_{1,2}, cl \rho) \leq (\varphi_{1,2}, cl \eta)' \). Because of Lemma 3.1 part (3) we have 
\( \mathcal{N}_{\varphi_{1,2}, \mu} (\varphi_{1,2}, cl \mu) \leq (\varphi_{1,2}, cl \eta)' \) and 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \rho) \leq (\varphi_{1,2}, cl \eta)' \) are hold and therefore \( \mu \bar{\delta} \eta \) and \( \rho \bar{\delta} \eta \). Thus, \( (\mu \lor \rho) \bar{\delta} \eta \) implies \( \mu \bar{\delta} \eta \) and \( \rho \bar{\delta} \eta \). On the other hand let \( \mu \bar{\delta} \eta \) and \( \rho \bar{\delta} \eta \) are hold for all \( \mu, \rho, \eta \in L^X \). Then from Lemma 3.1 we have 
\( \mathcal{N}_{\varphi_{1,2}, \mu} (\varphi_{1,2}, cl \mu) \leq (\varphi_{1,2}, cl \eta)' \) and 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \rho) \leq (\varphi_{1,2}, cl \eta)' \) are hold and therefore 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl (\mu \lor \rho)) = \mathcal{N}_{\varphi_{1,2}, \mu} (\varphi_{1,2}, cl \mu) \land \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \rho) \leq (\varphi_{1,2}, cl \eta)' \), that is, \( \mu \bar{\delta} \eta \) and \( \rho \bar{\delta} \eta \) imply \( (\mu \lor \rho) \bar{\delta} \eta \). Hence, (P2) is fulfilled. Now, let \( \mu, \rho \in L^X \) such that \( \mu = \bar{0} \) or \( \rho = \bar{0} \). Since 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\bar{0}) \) is the finest L-filter on \( X \) and from the fact \( \varphi_{1,2}, cl \bar{0} = \bar{0} \), we get 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\bar{0}) = \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \bar{0}) \leq (\varphi_{1,2}, cl \rho)' \) holds for all \( \rho \in L^X \). Thus, \( \bar{0} \bar{\delta} \rho \) for all \( \rho \in L^X \). Since \( \mu = \bar{0} \) or \( \rho = \bar{0} \), then we have \( \mu \bar{\delta} \rho \), that is, (P3) is also fulfilled. Since \( \mu \bar{\delta} \rho \) implies 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \mu) \leq (\varphi_{1,2}, cl \rho)' \) which means by the inequality \( (\varphi_{1,2}, cl \mu) \leq \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \mu) \) that \( (\varphi_{1,2}, cl \mu) \leq (\varphi_{1,2}, cl \rho)' \). Because of Proposition 2.1 and the fact that \( \varphi_{1,2}, cl \) is hull operator we get 
\( \mu \leq \varphi_{1,2}, cl \mu \leq (\varphi_{1,2}, cl \rho)' \). Thus, (P4) is fulfilled. Let \( \mu, \rho \in L^X \) such that \( \mu \bar{\delta} \rho \), then 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \mu) \leq (\varphi_{1,2}, cl \rho)' \). Consider, \( F = S_{\varphi_{1,2}, cl \mu} \), hence \( F \in \varphi_{1,2}, C (X) \) and therefore 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (F) \leq (\varphi_{1,2}, cl \rho)' \) holds. Since \( (X, \tau) \) is \( \varphi_{1,2}, R_3 \)-space, then from Theorem 3.2 there exists \( \eta' \in L^X \) with arbitrary choice such that 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (F) \leq \eta' \) and 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \eta) \leq (\varphi_{1,2}, cl \rho)' \) are hold. Therefore, there exists \( \eta \in L^X \) such that 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \mu) \leq (\varphi_{1,2}, cl \eta)' \) and 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \eta) \leq (\varphi_{1,2}, cl \rho)' \), which means that \( \mu \bar{\delta} \eta \) and \( \eta ' \bar{\delta} \rho \). Hence, (P5) is also fulfilled. Consequently, \( \delta \) is L-proximity on \( X \).

Conversely, \( F_1, F_2 \in \varphi_{1,2}, C (X) \) such that \( F_1 \cap F_2 = \phi \). Then, \( F_1 \subseteq F_2 \) and therefore \( \check{\chi}_{F_1} \leq \check{\chi}_{F_2} = \check{\chi}_{F_1} \). Hence because of Lemma 3.1 part (1) we have 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \mu) \leq \check{\chi}_{F_1} \). Since \( F_1, F_2 \in \varphi_{1,2}, C (X) \), then 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \check{\chi}_{F_1}) = \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \mu) \leq \check{\chi}_{F_1} = (\varphi_{1,2}, cl \check{\chi}_{F_1}) \) and therefore \( \check{\chi}_{F_1} \bar{\delta} \check{\chi}_{F_2} \). From (P5), there exists \( \rho \in L^X \) such that 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\varphi_{1,2}, cl \mu) \leq \check{\chi}_{F_1} \leq \rho \) and 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (\rho) \leq \check{\chi}_{F_1} = \check{\chi}_{F_2} \) are hold. Because of Lemma 3.1 part (1), we have 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (F_2) \leq \rho \). Hence, 
\( \mathcal{N}_{\varphi_{1,2}, \rho} (F_1) \land \mathcal{N}_{\varphi_{1,2}, \rho} (F_2)(\eta) \geq \rho(\mu) \land \rho'(\eta) \) holds for all \( \mu, \eta \in L^X \). Consider \( \eta = \check{\chi}_{F_2} \lor x_1 = \rho \) and 
\( \mu = (\varphi_{1,2}, cl \check{\chi}_{F_1} \lor x_1)' = \rho \) for all \( x \in \check{\chi}_{F_1} \setminus \check{\chi}_{F_2} \), then we get 
\( \sup(\mu \lor \eta) = 0 \) and 
\( \mathcal{N}_{\varphi_{1,2}, \rho}(F_1) \land \mathcal{N}_{\varphi_{1,2}, \rho}(F_2)(\eta) \geq 0 \) are fulfilled. Hence, there exist \( \mu, \eta \in L^X \) such that 
\( \mathcal{N}_{\varphi_{1,2}, \rho}(F_1) \land \mathcal{N}_{\varphi_{1,2}, \rho}(F_2)(\eta) \geq \sup(\mu \lor \eta) \), that is, the infimum 
\( \mathcal{N}_{\varphi_{1,2}, \rho}(F_1) \land \mathcal{N}_{\varphi_{1,2}, \rho}(F_2) \) does not exists. Consequently, \( (X, \varphi_{1,2}, int_\delta) \) is characterized

\( \text{FR}_3 \)-space. □

4. Characterized Compact L-spaces

The notion of \( \varphi_{1,2} \)-compactness of the L-filters and of the topological L-spaces are introduced in [6] by means of the \( \varphi_{1,2} \) L-convergence in the characterized L-spaces. Moreover, the compactness in the characterized L-
spaces is also introduced by means of the $\phi_{1,2}$-compactness of the L-filters and therefore it will be suitable to study here the relation between the characterized compact L-spaces and some of our classes of separation axioms in the characterized L-spaces.

Let $(X, \tau)$ be a topological L-space, $F \subseteq X$ and $\phi_1, \phi_2 \in O_{(L^\tau, \tau)}$. Then $x \in X$ is said to be $\phi_{1,2}$-adherence point for the L-filter $M$ on $X$ ([6]), if the infimum $M \land N_{\phi_{1,2}}(x)$ exists for all $\phi_{1,2}$ L-neighborhood filters $N_{\phi_{1,2}}(x)$ at $x \in X$. As shown in [6], the point $x \in X$ is said to be $\phi_{1,2}$-adherence point for the L-filter $M$ on $X$ if and only if there exists a L-filter $K \in \mathcal{F}_X$ finer than $M$ and $K \rightarrow x$, that is, $K \subseteq M$ and $K \subseteq N_{\phi_{1,2}}(x)$ are hold for some $\mathcal{K} \in \mathcal{F}_X$. The ordinary subset $F$ is said to be $\phi_{1,2}$-closed with respect to $\phi_{1,2}$.int if $M \leq N_{\phi_{1,2}}(x)$ implies $x \in F$ for some $M \in \mathcal{F}_X$. The subset $F$ is said to be $\phi_{1,2}$-compact subset ([6]), if every L-filter on $F$ has a finer $\phi_{1,2}$ L-converging L-filter, that is, every L-filter on $F$ has a $\phi_{1,2}$-adherence point in $F$. Moreover, the topological L-space $(X, \tau)$ is said to be $\phi_{1,2}$-compact if $X$ is $\phi_{1,2}$-compact. Generally, the characterized L-space $(X, \phi_{1,2}, \text{int})$ is said to be compact L-space if the related topological L-space $(X, \tau)$ is $\phi_{1,2}$-compact.

In the following proposition, we show that in the characterized $L^\tau$-space $(X, \phi_{1,2}, \text{int})$, every $\phi_{1,2}$-compact subset is $\phi_{1,2}$-closed with respect to the $\phi_{1,2}$-interior operator $\phi_{1,2}.\text{int}$.

**Proposition 4.1** Let a topological L-space $(X, \tau)$ be fixed and $\phi_1, \phi_2 \in O_{(L^\tau, \tau)}$. Then every $\phi_{1,2}$-compact subset of a characterized $L^\tau$-space is $\phi_{1,2}$-closed.

**Proof.** Let $(X, \phi_{1,2}, \text{int})$ is characterized $L^\tau$-space and let $F$ is a $\phi_{1,2}$-compact subset of $X$. Then, for all $M \in \mathcal{F}_X$ there exists $\mathcal{K} \subseteq \mathcal{F}_X$ such that $\mathcal{K} \subseteq M$ and $\mathcal{K} \subseteq N_{\phi_{1,2}}(x)$ are hold for some $x \in F$.

Since $\mathcal{K} \subseteq \mathcal{F}_X \leq \mathcal{F}_X$ and $(X, \phi_{1,2}, \text{int})$ is characterized $L^\tau$-space, then $\mathcal{K} \subseteq N_{\phi_{1,2}}(x)$ and $\mathcal{K} \subseteq N_{\phi_{1,2}}(y)$ imply that $x = y$. Therefore, $y \in F$ for some $\mathcal{K} \in \mathcal{F}_X$. Hence, $F$ is $\phi_{1,2}$-closed. □

In the following proposition we give new property for the characterized $L^\tau$-spaces by using the $\phi_{1,2}$ L-neighborhood filters for the L-subsets.

**Proposition 4.2** Let $(X, \tau)$ be a topological L-space and $\phi_1, \phi_2 \in O_{(L^\tau, \tau)}$. Then in the characterized $L^\tau$-space $(X, \phi_{1,2}, \text{int})$, every disjoint $\phi_{1,2}$-compact subsets $F_1$ and $F_2$ of $X$ have the $\phi_{1,2}$ L-neighborhood filters $N_{\phi_{1,2}}(F_1)$ and $N_{\phi_{1,2}}(F_2)$ such that the infimum $N_{\phi_{1,2}}(F_1) \land N_{\phi_{1,2}}(F_2)$ does not exists.

**Proof.** Let $F_1$ and $F_2$ are two $\phi_{1,2}$-compact subsets of the characterized $L^\tau$-space $(X, \phi_{1,2}, \text{int})$ such that $F_1 \cap F_2 = \emptyset$. Then, for all $M \in \mathcal{F}_X$, there exists $\mathcal{K}_i \subseteq \mathcal{F}_X$ such that $\mathcal{K}_i \subseteq M$ and $\mathcal{K}_i \subseteq N_{\phi_{1,2}}(x_i)$ for some $x_i \in F_i$, where $i \in \{1, 2\}$. Since $\mathcal{F}_X \leq \mathcal{F}_X$ for all $i \in \{1, 2\}$, then we can say that $\mathcal{K}_i \subseteq N_{\phi_{1,2}}(x_i) \subseteq N_{\phi_{1,2}}(F_i)$ and therefore there is $\mathcal{K} = (\mathcal{K}_1 \land \mathcal{K}_2) \subseteq \mathcal{F}_X$ such that $\mathcal{K} \subseteq N_{\phi_{1,2}}(x_i)$ for some $x_i \in F_i$. Since $(X, \phi_{1,2}, \text{int})$ is $L^\tau$-space, then $x_1 = x_2$ which contradicts $F_1 \cap F_2 = \emptyset$. Hence, for every $L \in \mathcal{F}_X$ we get $L \not\subseteq N_{\phi_{1,2}}(F_1)$ or $L \not\subseteq N_{\phi_{1,2}}(F_2)$ which means that the infimum $N_{\phi_{1,2}}(F_1) \land N_{\phi_{1,2}}(F_2)$ does not exists. Hence, $F_1$ and $F_2$ can be separated by two.
disjoint $\phi_{1,2}$ L-neighborhood filters. □

The notion of compactness for the characterized L-spaces fulfills the following property which will be used in the prove of the important result given in Proposition 4.3.

**Lemma 4.1** Let $(X, \tau)$ be a topological L-space and $\phi_1, \phi_2 \in O_{(L^\tau, \tau)}$. Then every $\phi_{1,2}$-closed subset of a characterized compact L-space $(X, \phi_{1,2}, \text{int})$ is $\phi_{1,2}$-compact.

**Proof.** Let $F$ is $\phi_{1,2}$-closed subset of a characterized compact L-space $(X, \phi_{1,2}, \text{int})$ and let $M \in \mathcal{F}_L F$. Then, $\mathcal{M} \leq \mathcal{N}_{\phi_{1,2}}(x)$ implies that $x \in F$. Since $\mathcal{F}_L F \leq \mathcal{F}_L X$, then $M \in \mathcal{F}_L X$ and hence there exists $K \in \mathcal{F}_L X$ such that $K \leq M$ and $K \leq \mathcal{N}_{\phi_{1,2}}(x)$ are hold. Since $M \in \mathcal{F}_L F$ and $K \leq M$, then $K \in \mathcal{F}_L F$. Thus, for all $M \in \mathcal{F}_L F$ we get $K \leq M$ such that $K \leq \mathcal{N}_{\phi_{1,2}}(x)$. Therefore, $x \in F$ is $\phi_{1,2}$-adherence point of $M$, that is, $F$ is $\phi_{1,2}$-compact. □

The following proposition give an important relation between the characterized compact $FT_2$-spaces and the characterized $FT_4$-spaces.

**Proposition 4.3** Let $(X, \tau)$ be a topological L-space and $\phi_1, \phi_2 \in O_{(L^\tau, \tau)}$. Then every characterized compact $FT_2$-space $(X, \phi_{1,2}, \text{int})$ is characterized $FT_4$-space.

**Proof.** Follows directly from Lemma 4.1 and Proposition 4.2. □

**Lemma 4.2** Let $(X, \tau)$ and $(X, \sigma)$ are two topological L-spaces such that $\tau$ is finer than $\sigma$. If $\phi_1, \phi_2 \in O_{(L^\tau, \tau)}$ and $\psi_1, \psi_2 \in O_{(L^\sigma, \sigma)}$ and $(X, \psi_{1,2}, \text{int})$ is characterized compact L-space, then $(X, \phi_{1,2}, \text{int})$ is also characterized compact L-space.

**Proof.** Let $\mathcal{N}_{\phi_{1,2}}(x)$ and $\mathcal{N}_{\psi_{1,2}}(x)$ are the $\phi_{1,2}$ L-neighborhood and $\psi_{1,2}$ L-neighborhood at $x \in X$ with respect to $\phi_{1,2}, \text{int}$ and $\psi_{1,2}, \text{int}$ respectively. Since $\tau$ is finer than $\sigma$, then $\mathcal{N}_{\phi_{1,2}}(x) \leq \mathcal{N}_{\psi_{1,2}}(x)$ for all $x \in X$. Because of $(X, \psi_{1,2}, \text{int})$ is characterized compact L-space, then for all $M \in \mathcal{F}_L X$, there exists $K \in \mathcal{F}_L X$ such that $K \leq M$ and $K \leq \mathcal{N}_{\psi_{1,2}}(x)$ are hold for all $x \in X$, therefore $K \leq \mathcal{N}_{\phi_{1,2}}(x)$ holds for all $x \in X$. Hence, $(X, \phi_{1,2}, \text{int})$ is characterized compact L-space. □

**Proposition 4.4** [2] If $(X, \phi_{1,2}, \text{int})$ is characterized $FT_2$-space and $\phi_{1,2}, \text{int}$ is finer than $\psi_{1,2}, \text{int}$, then $(X, \psi_{1,2}, \text{int})$ is also characterized $FT_2$-space.

**Proposition 4.5** Let $(X, \tau)$ and $(X, \sigma)$ are two topological L-spaces such that $\tau$ is finer than $\sigma$, $\phi_1, \phi_2 \in O_{(L^\tau, \tau)}$ and $\psi_1, \psi_2 \in O_{(L^\sigma, \sigma)}$. If $(X, \psi_{1,2}, \text{int})$ is characterized compact L-space and $(X, \phi_{1,2}, \text{int})$ is characterized $FT_2$-space, then $\phi_{1,2}, \text{int}$ and $\psi_{1,2}, \text{int}$ are isomorphic.

**Proof.** Since $\tau$ is finer than $\sigma$, then $\psi_{1,2}, \text{int} \leq \phi_{1,2}, \text{int}$. Hence, because of Proposition 4.4, $(X, \psi_{1,2}, \text{int})$ is characterized $FT_2$-space. From Lemma 4.2, we have $(X, \phi_{1,2}, \text{int})$ is characterized compact L-space. Hence, we can find the identity mapping $\text{id}_X : (X, \phi_{1,2}, \text{int}) \rightarrow (X, \psi_{1,2}, \text{int})$ which is bijective $\phi_{1,2}, \psi_{1,2}$ L-continuous and its inverse is $\psi_{1,2}, \phi_{1,2}$ L-continuous, that is, $\phi_{1,2}, \psi_{1,2}$ L-isomorphism. Consequently, $\phi_{1,2}, \text{int}$ and $\psi_{1,2}, \text{int}$ are isomorphic. □
5. Characterized Uniform L-spaces

In this section, we go to investigate the notion of characterized uniform L-space by the notion of uniform L-spaces introduced in [17]. Moreover, the relation between the separated uniform L-spaces, the associated characterized uniform FT\textsubscript{0}\textsuperscript{0} -spaces and the F\textsubscript{1,2}ϕ\textsubscript{0} -spaces which introduced in [2] are investigated.

By an L-relation on a set \( X \) we mean a mapping \( R : X \times X \rightarrow L \), that is, an L-subset of \( X \times X \). For each L-relation \( R \) on \( X \), the inverse \( R^{-1} \) of \( R \) is the L-relation on \( X \) defined by \( R^{-1}(x, y) = R(y, x) \) for all \( x, y \in X \). Let \( \mathcal{U} \) be an L-filer on \( X \times X \). The inverse \( \mathcal{U}^{-1} \) of \( \mathcal{U} \) is an L-filter on \( X \times X \) defined by:

\[
\mathcal{U}^{-1}(R) = \mathcal{U}(R^{-1})
\]

for all \( R \in L_{X \times X} \).

The composition \( R_1 \circ R_2 \) of two L-relations \( R_1 \) and \( R_2 \) on a set \( X \) is the L-relation on \( X \) defined by:

\[
(R_1 \circ R_2)(x, y) = \bigvee_{x \in X} (R_2(x, z) \wedge R_1(z, y))
\]

for all \( x, y \in X \). For each pair \((x, y)\) of elements \( x \) and \( y \) of \( X \), the mapping \((x, y) : L_{X \times X} \rightarrow L \) defined by:

\[
(x, y)(R) = R(x, y)
\]

for all \( R \in L_{X \times X} \) is a homogeneous L-filter on \( X \times X \).

Let \( \mathcal{U} \) and \( \mathcal{V} \) are L-filers on \( X \times X \) such that \((x, y) \leq \mathcal{U}\) and \((y, z) \leq \mathcal{V}\) hold for some \( x, y, z \in X \). Then the composition \( \mathcal{V} \circ \mathcal{U} \) of \( \mathcal{V} \) and \( \mathcal{U} \) is an L-filter ([17]) on \( X \times X \) defined by:

\[
(\mathcal{V} \circ \mathcal{U})(R) = \bigvee_{R_2 \circ R_1 \leq R} (\mathcal{U}(R_1) \wedge \mathcal{V}(R_2))
\]

for all \( R \in L_{X \times X} \).

By the uniform L-structure \( \mathcal{U} \) on a set \( X \), we mean an L-filter ([17]) on \( X \times X \) such that the following axioms are fulfilled:

\[
\text{(U1)} \quad (x, x) \leq \mathcal{U} \quad \text{for all } x \in X .
\]

\[
\text{(U2)} \quad \mathcal{U} = \mathcal{U}^{-1} .
\]

\[
\text{(U3)} \quad \mathcal{U} \circ \mathcal{U} \leq \mathcal{U} .
\]

The pair \((X, \mathcal{U})\) is called uniform L-space.

A uniform L-structure \( \mathcal{U} \) on a set \( X \) is called separated if for all \( x, y \in X \) with \( x \neq y \) there is \( R \in L_{X \times X} \) such that \( \mathcal{U}(R) = 1 \) and \( R(x, y) = 0 \). In this case the uniform L-space \((X, \mathcal{U})\) is called separated uniform L-space.

To each uniform L-structure \( \mathcal{U} \) on a set \( X \) is associated a stratified L-topology \( \tau_{\mathcal{U}} \). Consider \( \varphi_1, \varphi_2 \in O_{L^x}(\mathcal{U}) \), then the set of all \( \varphi_1,2 \)-open L-subsets of \( X \) related to \( \tau_{\mathcal{U}} \) forms a characterized stratified L-topology on \( X \) generated by the \( \varphi_1,2 \)-interior operator with respect to \( \tau_{\mathcal{U}} \) denoted by \( \varphi_1,2 \cdot \text{int}_{\mathcal{U}} \) and \((X, \varphi_1,2 \cdot \text{int}_{\mathcal{U}})\) is a characterized stratified L-space. The characterized stratified L-space \((X, \varphi_1,2 \cdot \text{int}_{\mathcal{U}})\) will be called associated characterized uniform L-space which is stratified. The related \( \varphi_1,2 \)-interior operator \( \varphi_1,2 \cdot \text{int}_{\mathcal{U}} \) is given by:

\[
\varphi_1,2 \cdot \text{int}_{\mathcal{U}}(\mu) = \mathcal{U}(\hat{x})(\mu)
\]

for all \( x \in X \) and \( \mu \in L^x \).
Proposition 5.1 Let $X$ be non-empty set, $\mathcal{U}$ is a uniform $L$-structure on $X$ and $\varphi_1, \varphi_2 \in O_{(x, \tau_{\mathcal{U}})}$. Then the uniform $L$-space $(X, \mathcal{U})$ is separated if and only if the associated characterized uniform $L$-space $(X, \varphi_{1,2}\cdot \mathrm{int}_{\mathcal{U}})$ is $\mathrm{FT}_{0}$-space.

Proof. Let $(X, \mathcal{U})$ is separated and let $x, y \in X$ such that $x \neq y$. Then, there exists $R \in L^{X \times X}$ such that $\mathcal{U}(R) = 1$ and $R(x, y) = 0$. Consider $\mu = R[y_1]$ for which $\mu(x) = R[y_1](x) = \bigvee_{z \in X} R(z, x) \land y_1(z) = 0$ and

$$(\varphi_{1,2}\cdot \mathrm{int}_{\mathcal{U}})(y)(\mu)(y) = \mathcal{U}[y](\mu) = \bigvee_{R(\eta) \leq \mu} \mathcal{U}(R) \land \eta(y) = 1$$

for all $\eta \in L^{X}$. Hence, there exists $\mu \in L^{X}$ and $\alpha \in L_{\mu}$ such that $\mu(x) < \alpha \leq (\varphi_{1,2}\cdot \mathrm{int}_{\mathcal{U}})(y)$, that is, $(X, \varphi_{1,2}\cdot \mathrm{int}_{\mathcal{U}})$ is $\mathrm{FT}_{0}$-space.

Conversely, let $(X, \varphi_{1,2}\cdot \mathrm{int}_{\mathcal{U}})$ is $\mathrm{FT}_{0}$-space and let $x \neq y$ in $X$. Then, there exists $\mu \in L^{X}$ and $\alpha \in L_{\mu}$ such that $\mu(x) < \alpha \leq (\varphi_{1,2}\cdot \mathrm{int}_{\mathcal{U}})(y)$. This means that $\bigvee_{R(\eta) \leq \mu} \mathcal{U}(R) \land \eta(y) > \mu(x)$ holds for all $\eta \in L^{X}$. Hence, there is $R \in L^{X \times X}$ for which $R(x, y) = (\varphi_{1,2}\cdot \mathrm{int}_{\mathcal{U}})(y)$ if $x = y$ and $R(x, y) = \mu(x)$ if $x \neq y$ such that $R(x, y) = 0$ and $\mathcal{U}(R) = 1$. Thus, $(X, \mathcal{U})$ is separated. □

Corollary 5.1 Let $X$ be non-empty set, $\mathcal{U}$ is a uniform $L$-structure on $X$ and $\varphi_1, \varphi_2 \in O_{(x, \tau_{\mathcal{U}})}$. Then the uniform $L$-space $(X, \mathcal{U})$ is separated if and only if the associated stratified topological $L$-space $(X, \tau_{\mathcal{U}})$ is $\mathrm{F}_{1,2}L$- space.

Proof. Immediate from Proposition 5.1 and Theorem 2.1 in [2]. □

In Table(1), we give some special choices for the operations $\varphi_1$ and $\varphi_2$ to obtained some special classes of the $\varphi_{1,2}$-open $L$-sets, $\varphi_{1,2}$-neighborhood filters, characterized proximity $L$-spaces, characterized compact $L$-spaces and the characterized uniform $L$-spaces.

6. Conclusion

In this paper, we introduced and studied three new types of spaces which are named characterized proximity $L$-spaces, characterized compact $L$-spaces and characterized uniform $L$-spaces. The relation between such spaces with the characterized $\mathrm{FT}_{s}$-spaces $\mathrm{F}_{\varphi_{1,2}}\cdot \mathrm{T}_{s}$ spaces, characterized $\mathrm{FR}_{k}$-spaces and $\mathrm{F}_{\varphi_{1,2}}$-spaces are investigated for $s \in \{0, 2, 4\}$ and $k \in \{2, 3\}$. Some new properties for the characterized $\mathrm{FT}_{s}$-spaces, $\mathrm{F}_{\varphi_{1,2}}\cdot \mathrm{T}_{s}$ spaces, characterized $\mathrm{FR}_{k}$-spaces and $\mathrm{F}_{\varphi_{1,2}}$-spaces will be added by applied these new spaces. Many new special classes from the $\varphi_{1,2}$-open $L$-sets, $\varphi_{1,2}$-neighborhood filters, characterized proximity $L$-spaces, characterized compact $L$-spaces and the characterized uniform $L$-spaces are listed in Table (1).
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<td>$\varphi_1 = \text{cl} \circ \text{int}$</td>
<td>$\varphi_2 = S \circ \text{cl}$</td>
<td>$\tau_{(S, \theta)}$</td>
<td>$(S, \theta)$-fuzzy nbd. filter</td>
<td>$(S, \theta)$-proximity L-space</td>
</tr>
<tr>
<td>9</td>
<td>$\varphi_1 = \text{cl} \circ \text{int}$</td>
<td>$\varphi_2 = S \circ \text{int} \circ \text{cl}$</td>
<td>$\tau_{(S, \delta)}$</td>
<td>$(S, \delta)$-fuzzy nbd. filter</td>
<td>$(S, \delta)$-proximity L-space</td>
</tr>
<tr>
<td>10</td>
<td>$\varphi_1 = \text{cl} \circ \text{int} \circ \text{cl}$</td>
<td>$\varphi_2 = 1_{L^x}$</td>
<td>$\beta OF (X)$ [9]</td>
<td>$\beta$-fuzzy nbd. filter</td>
<td>$\beta$-proximity L-space</td>
</tr>
<tr>
<td>11</td>
<td>$\varphi_1 = \text{int} \circ \text{cl} \circ \text{int}$</td>
<td>$\varphi_2 = 1_{L^x}$</td>
<td>$\lambda OF (X)$ [14]</td>
<td>$\lambda$-fuzzy nbd. filter</td>
<td>$\lambda$-proximity L-space</td>
</tr>
<tr>
<td>12</td>
<td>$\varphi_1 = S \circ \text{cl} \circ \text{int}$</td>
<td>$\varphi_2 = 1_{L^x}$</td>
<td>$fOF (X)$</td>
<td>Feebly-fuzzy nbd. filter</td>
<td>Feebly proximity L-space</td>
</tr>
</tbody>
</table>

Table (1) : Some special classes of $\varphi_{1,2}$-open L-sets, $\varphi_{1,2}$ L-neighborhood filters, characterized proximity L-spaces, characterized compact L-spaces, characterized uniform L-spaces.

References


