Development of Implicit Rational Runge-Kutta Schemes for Second Order Ordinary Differential Equations

Usman Abdullahi S.¹, Mukhtar Hassan S.²
1.Department of Statistics, Federal Polytechnic Bali, Taraba State – Nigeria
2.GNS Department, Federal Polytechnic Bali, Taraba State – Nigeria
*E-Mail of corresponding author: usasmut@gmail.com

Abstract
In this paper, the development of One – Stage Implicit Rational Runge – Kutta methods are considered using Taylor and Binomial series expansion for the direct solution of general second order initial value problems of ordinary differential equations with constant step length. The basic properties of the developed method were investigated and found to be consistent and convergent.

Keywords: Implicit Rational Runge Kutta scheme, Second Order Equations, Convergence and Consistent

1. Introduction
Consider the numerical approximation first order initial value problems of the form,
\[ y' = f(x, y) \]
\[ y(x_0) = y_0, \quad a \leq x \leq b \]  \hspace{1cm} (1.1)

A Runge-Kutta method is the most important family of implicit and explicit iterative method of approximation of initial value problems of ordinary differential equations. So far many work and schemes have been developed for solving problem (1). The numerical solution of (1.1) is
\[ y_{n+1} = y_n + h\phi(x_n, y_n, h) \] \hspace{1cm} (1.2)

where
\[ \phi(x, y, h) = \sum_{i} c_i k_i \]

\[ k_i = f(x, y) \]

and
\[ \sum_{j=1}^{r} b_{ij} k_j \]

with constraints
\[ a_i = \sum_{j=1}^{r} b_{ij}, \quad i = 1(1)s \]

The derivative of suitable parameters \(a_i, b_i\) and \(c_i\) of higher order term involves a large amount of tedious algebraic manipulations and functions evaluations which is both time consuming and error prone, Julyan and Oreste (1992). The derivation of the Runge – Kutta methods is extensively discussed by Lambert (1973), Butcher (1987), Fatunla (1987). According to Julyan and Oreste (1992) the minimum number of stages necessary for an explicit method to attain order \(p\) is still an open problem. Therefore so many new schemes and approximation formula have been derived this includes the work of Ababneh et al. (2009a), Ababneh et al. (2009b) Faranak and Ismail (2010).

Since the stability function of the implicit Runge-Kutta scheme is a rational function, Butcher (2003); Hong (1982) first proposes rational form of Runge-Kutta method (1.2), then Okunbor (1987) investigate rational form and derived the explicit rational Runge-Kutta scheme:
\[ y_{n+1} = \frac{y_n + h \sum_{i=1}^{r} \left[w_i K_i + \right]}{1 + h y_n \sum_{i=1}^{r} v_i H_i} \] \hspace{1cm} (1.4)

where
\[ k_i = f(x_n + c_i h, y_n + h \sum_{j=1}^{i} a_{i-j} k_j), \quad i = 1(1)r \]

and
\[ H_i = g(x_n + d_i h, z_n + h \sum_{j=1}^{r} b_{i-j} H_j), \quad i = 1(1)r \]

in which
\[ g(x_n, z_n) = -z_n^2 f(x_n, y_n) \]

and \(z_n = \frac{1}{y_n}\) \hspace{1cm} (1.6)

where \(c_i, a_{ij}, b_{ij}, d_i\) are arbitrary constants to be determined.
\( d_t = \sum_{j=1}^{l} b_{ij}, \quad (1.8) \)

is imposed to ensure consistency of the method.

In view of these inadequacies of the explicit schemes and the superior region of absolute stability associated with implicit schemes, Ademuluyi and Babatola (2000) generate implicit rational Runge-Kutta and generates also the parameters so that the resulting numerical approximation method shall be A-stable and will have low bound for local truncation error. Since then many new rational Runge – Kutta schemes have been developed for the solution of first order initial value problems and found out to give better estimates. Among these authors are: Ademuluyi and Babatola (2001), Odekunle (2001), Odekunle et al. (2004), Bolarinwa (2005), Babatola et al. (2007), Bolarinwa et al. (2012) and Abhulimen and Uluko (2012) The schemes are absolutely stable, consistent and convergent and was used to approximate a variety of first order differential equations. However, the methods are presently receiving more attention as efficient schemes for the solutions of various types of first order initial value problems are considered.

2. Derivation of the Scheme

Consider the second order initial value problems

\[ y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad \alpha \leq x \leq \beta \quad (2.1) \]

The general \( s \) – stage Runge-Kutta scheme for general second order initial value problems of ordinary differential equations of the form (1) as defined by Jain (1984) is

\[ y_{n+1} = y_n + h y'_n + \sum_{j=1}^{s} c_j k_j, \quad (2.2) \]

and

\[ y'_{n+1} = y'_n + \frac{1}{h} \sum_{j=1}^{s} c'_j k_j, \quad (2.3) \]

where

\[ K_i = \frac{h^2}{2} \left[ x_n + c_i h, y_n + h c_i y'_n + \sum_{j=1}^{s} a_{ij} k_j, y'_n + \frac{1}{h} \sum_{j=1}^{s} b_{ij} k_j \right], \quad i = 1(1)s \]

\[ c_i = \sum_{j=1}^{s} a_{ij} = \frac{1}{2} \sum_{j=1}^{s} b_{ij}, \quad i = 1(1)r \]

with

The rational form of (2.2) and (2.3) can be defined as

\[ y_{n+1} = \frac{y_n + h y'_n + \sum_{j=1}^{s} w_j K_j}{1 + \sum_{j=1}^{s} w_j H_j}, \quad (2.5) \]

\[ y'_{n+1} = \frac{y'_n + \frac{1}{h} \sum_{j=1}^{s} w_j' K_j}{1 + \frac{1}{h} \sum_{j=1}^{s} w_j' H_j}, \quad (2.6) \]

where

\[ K_i = \frac{h^2}{2} \left[ x_n + c_i h, y_n + h c_i y'_n + \sum_{j=1}^{s} a_{ij} K_j, y'_n + \frac{1}{h} \sum_{j=1}^{s} b_{ij} K_j \right], \quad i = 1(1)s \]

\[ H_i = \frac{h^2}{2} \left[ x_n + d_i h, z_n + h d_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h} \sum_{j=1}^{s} \beta_{ij} H_j \right], \quad i = 1(1)s \]

with constraints

\[ c_i = \sum_{j=1}^{s} a_{ij} = \frac{1}{2} \sum_{j=1}^{s} b_{ij}, \quad i = 1(1)r \]
\[
d_{i} = \sum_{j=1}^{i} \alpha_{ij} \beta_{j} = \frac{1}{2} \sum_{j} \beta_{j} \quad i = 1(1)r
\]  

(A)

in which

\[
g(x_{n}, z_{n}, z'_{n}) = -z_{n}^{2}f(x_{n}, y_{n}, y'_{n}) \quad \text{and} \quad z_{n} = \frac{1}{y_{n}}
\]  

(B)

The constraint equations are to ensure consistency of the method, \( h \) is the step size and the parameters \( a_{ij}, b_{ij}, c_{i}, d_{ij}, \beta_{ij} \) are constants called the parameters of the method.

Using Bobatola et. al (2007), the following procedures are adapted.

i. Obtain the Taylor series expansion of \( K_{i} \) and \( H_{i} \) about the point \( (x_{n}, y_{n}, y'_{n}) \) and binomial series expansion of right side of (2.1) and (2.2).

iv. Insert the Taylor series expansion into (2.1) and (2.2) respectively.

v. Compare the final expansion of \( K_{i} \) and \( H_{i} \) about the point \( (x_{n}, y_{n}, y'_{n}) \) to the Taylor series expansion of \( 1 + n_{y} + n_{y}' \) about \( (x_{n}, y_{n}, y'_{n}) \) in the powers of \( h \).

Normally the numbers of parameters exceed the number of equations, these parameters are chosen to ensure that (one or more of the following conditions are satisfied.

iv. Minimum bound of local truncation error exists.

v. The method has maximized interval of absolute stability.

vi. Minimized computer storage facilities are utilized.

To derive a One – stage scheme, we set \( s = 1 \) in equations (2.5), (2.6), (2.7) and (2.8) to have

\[
y_{n+1} = y_{n} + h y'_{n} + w_{1} K_{1} \quad 1 + y_{n}' H_{1} \quad (2.9)
\]

and

\[
y_{n+1}' = y_{n}' + \frac{1}{h} w_{1} K_{1} \quad \frac{1}{1 + y_{n}' H_{1}} \quad (2.10)
\]

with constraints

\[
k_{1} = \frac{h^{2}}{2} \left( x_{n} + c_{1} h, y_{n} + h c_{2} y'_{n} + a_{11} K_{1}, y'_{n} + \frac{1}{h} b_{11} K_{1} \right), i = 1(1)s
\]  

(2.11a)

and

\[
H_{1} = \frac{h^{2}}{2} g \left( x_{n} + d_{1} h, z_{n} + h d_{2} z'_{n} + a_{11} H_{1}, z'_{n} + \frac{1}{h} \beta_{11} H_{1} \right), i = 1(1)s
\]  

(2.11b)

where \( c_{1}, a_{11}, b_{11}, d_{1}, a_{11}, \beta_{11}, w_{1}', v_{1} \) and \( v_{1}' \) are all constants to be determined.

Equation (2.9) can be written as

\[
y_{n+1} = \left( y_{n} + h y'_{n} + w_{1} k_{1} \right) \left( 1 + y_{n}' H_{1} \right)^{-1}
\]

(2.13)

Expanding the bracket and neglecting 2\(^{nd}\) and higher orders gives

\[
y_{n+1} = \left( y_{n} + h y'_{n} + w_{1} k_{1} \right) \left( 1 - y_{n}' H_{1} \right)
\]

(2.14)

Expanding (2.14) and re-arranging, gives

\[
y_{n+1}' = y_{n}' + h y'_{n}' - \left( y_{n}' v_{1} + h y_{n} y'_{n} v_{1} \right) H_{1} + \left( w_{1} - y_{n} v_{1} H_{1} w_{1} \right) K_{1}
\]

(2.15)

Equation (2.10) can be written as

\[
y_{n+1}' = \left( y_{n}' + \frac{1}{h} w_{1} k_{1} \right) \left( 1 + \frac{1}{h} y_{n}' v_{1} H_{1} \right)^{-1}
\]

(2.16)

Expanding the binomial and re-arranging also gives

\[
y_{n+1}' = y_{n}' + \frac{1}{h} w_{1} k_{1} - \left( \frac{1}{h} y_{n}' v_{1} \right) K_{1} - \left( \frac{1}{h} y_{n}' v_{1} \right) \left( w_{1} - y_{n} v_{1} H_{1} w_{1} \right) K_{1}
\]

(2.17)

Now, the Taylor’s series expansion of \( y_{n+1}' \) about \( x'_{n} \) is given as
\[
y_{n+1} = y_n + h y'_n + \frac{h^2 y''_n}{2!} + \frac{h^3 y'''_n}{3!} + \frac{h^4 y^{(4)}_n}{4!} + \ldots
\]
and
\[
y'_{n+1} = y'_n + h y''_n + \frac{h^2 y'''_n}{2!} + \frac{h^3 y^{(4)}_n}{3!} + \ldots
\]
where
\[
y_i = f(x_n, y_n, y'_n) = f_n
\]
\[
y''_n = f_x + y'_n f_y + f_n f_{yx} = \Delta f_n
\]
\[
y^{(4)}_n = f_{xx} + y''_n f_{yy} + f^2_{fyy} + 2y'_n f_{fyy} + 2f_n f_{fyy} + f_y \Delta f_n
\]
Substituting (2.20)
\[
y^{(4)}_n = \Delta^2 f_n + f_y \Delta f_n + f_n f_y
\]
Since \[\Delta = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + f_n \frac{\partial}{\partial y}
\]
Using the Taylor’s series of the function of three variables we have from (2.21)
\[
K_1 = \frac{h}{2} \left[ b_{11} K_{fyy} + a_{11} b_{11} K_{fyy} + \frac{h^2}{2} \left( f_n + a_{11} K_f + c_1 b_{11} K_{fxy} + a_{11} K_{fyy} + c_1 y'_n b_{11} K_{fyy} \right) \right]
+ \frac{h^3}{2} \left[ c_1 f_x + c_1 y'_n f_y + c_1 a_{11} y'_n f_{fyy} + c_1 a_{11} y_n f_{fyy} \right] + \frac{h^4}{4} \left[ c_1^2 f_{xxx} + c_1^2 y'_n f_{fyy} + c_1^2 y_n f_{fyy} \right] + O(h^5)
\]
Simplifying further and arranging the equation in powers of \( h \) gives,
\[
K_1 = \frac{h}{2} \left[ b_{11} (hA_1 + h^2 B_1 + h^3 C_1) f_{fyy} + a_{11} b_{11} (hA_1 + h^2 B_1) f_{fyy} + a_{11} b_{11} (hA_1 + h^2 B_1) f_{fyy} + a_{11} b_{11} (hA_1 + h^2 B_1) f_{fyy} \right]
+ \frac{h^2}{2} \left[ f_n + a_{11} K_f + c_1 b_{11} (hA_1 + h^2 B_1) f_{fyy} + a_{11} K_{fyy} + c_1 y'_n b_{11} K_{fyy} \right]
+ \frac{h^3}{2} \left[ c_1 f_x + c_1 y'_n f_y + c_1 a_{11} y'_n f_{fyy} + c_1 a_{11} y_n f_{fyy} \right]
+ \frac{h^4}{4} \left[ c_1^2 f_{xxx} + c_1^2 y'_n f_{fyy} + c_1^2 y_n f_{fyy} \right] + O(h^5)
\]
Substituting equation (2.23) into (2.22) gives
\[
K_1 = \frac{h}{2} \left[ b_{11} (hA_1 + h^2 B_1 + h^3 C_1) f_{fyy} + a_{11} b_{11} (hA_1 + h^2 B_1) f_{fyy} + a_{11} b_{11} (hA_1 + h^2 B_1) f_{fyy} + a_{11} b_{11} (hA_1 + h^2 B_1) f_{fyy} \right]
+ \frac{h^2}{2} \left[ f_n + a_{11} K_f + c_1 b_{11} (hA_1 + h^2 B_1) f_{fyy} + a_{11} K_{fyy} + c_1 y'_n b_{11} K_{fyy} \right]
+ \frac{h^3}{2} \left[ c_1 f_x + c_1 y'_n f_y + c_1 a_{11} y'_n f_{fyy} + c_1 a_{11} y_n f_{fyy} \right]
+ \frac{h^4}{4} \left[ c_1^2 f_{xxx} + c_1^2 y'_n f_{fyy} + c_1^2 y_n f_{fyy} \right] + O(h^5)
\]
On equating powers of \( h \) from equation (2.22) and (2.23), gives
\[
A_1 = 0, \quad B_1 = \frac{1}{2} f_n, \quad C_1 = \frac{1}{2} \left( c_1 f_x + c_1 y_n f_y + 1/2 b_{11} f_{fyy} \right) = \frac{1}{2} c_1 \Delta f_n, \quad \text{since } c_1 = \frac{1}{2} b_{11}
\]
\[
D_1 = \frac{1}{4} \left( c_1^2 \Delta f_n + b_{11} \Delta f_{fyy} + a_{11} f_{fyy} \right)
\]
Substituting \( A_1, B_1, C_1 \) and \( D_1 \) into (2.23) gives,
\[
K_1 = \frac{h^2}{2} f_n + \frac{h^3}{2} c_1 \Delta f_n + \frac{h^4}{4} \left( c_1^2 \Delta f_n + b_{11} \Delta f_{fyy} + a_{11} f_{fyy} \right)
\]
(2.26)
Similarly, expanding \( H_1 \) in Taylor’s series about \( (x_n, z_n, z'_n) \), from (2.11b), we have
\[
H_1 = \frac{h}{2} \left[ \beta_{11} H_{1, g_{2x}} + a_{11} \beta_{11} H_{2, g_{2x}} + \frac{h^2}{2} \left[ g_n + a_{11} H_{1, g_{xx}} + d_1 \beta_{11} H_{1, g_{xx}} + a_{11} H_{2, g_{xx}} + d_1 a_{11} \beta_{11} H_{1, g_{xx}} \right] \right]
+ \frac{h^3}{2} \left[ d_1 g_{xx} + d_1 z_n g_{xx} + d_1 a_{11} H_{1, g_{xx}} + d_1 a_{11} \beta_{11} H_{1, g_{xx}} \right] + \frac{h^4}{4} \left[ c_1^2 g_{xx} + d_1^2 \beta_{11}^2 g_{xx} + d_1^2 \beta_{11}^2 \beta_{11} g_{xx} \right]
+ O(h^5)
\]
Equation (2.27) is also implicit which cannot be proceed by successive substitution. Assuming a solution of the equation is of the form
\[
H_1 = h L + h^2 M + h^3 N + h^4 R + 0(h^5)
\]
Substituting the values of \( H_1 \) in (2.28) into equation (2.27) and equating powers of \( h \) of the equation, we can get the following after substitutions:
\[ L_1 = 0, \quad M_1 = \frac{1}{2} g_n, \quad N_1 = \frac{1}{2} d_1 \Delta g_n \quad \text{and} \quad R_1 = \frac{1}{4} (d_1^2 \Delta^2 g_n + \beta_{11} \Delta g_n g_{x1} + \alpha_{11} g_n g_{x2}) \]  

(2.29)

Substituting equation (2.29) into equation (2.28) gives

\[ H_1 = \frac{h^2}{2} g_n + \frac{h^4}{2} d_1 \Delta g_n + \frac{h^4}{2} (d_1^2 \Delta^2 g_n + \beta_{11} \Delta g_n g_{x1} + \alpha_{11} g_n g_{x2}) \]  

(2.30)

Using equations (2.23) and (2.28) into equations (2.15) and (2.17) respectively gives

\[
y_{n+1} = y_n + h y_n' - (y_n^2 v_1 + h y_n y_n' v_1) (h^2 M_1 + h^3 N_1 + h^4 R_1) + [w_1 - y_n v_1 w_1 (h^2 M_1 + h^3 N_1 + h^4 R_1)] (h^2 B_1 + h^3 C_1 + h^4 D_1)
\]

Expanding the brackets and re-arranging in powers of \( h \) gives

\[
y_{n+1} = y_n + h y_n' + h^2 (w_1 B_1 - y_n^2 v_1 M_1) + h^3 (w_1 C_1 - y_n^2 v_1 N_1 - y_n y_n' v_1 M_1) + 0 (h^4)
\]

Also for \( y_{n+1} \) gives

\[
y_{n+1}' = y_n' + \frac{1}{h} \Delta^1 (h^2 B_1 + h^3 C_1 + h^4 D_1)
\]

Expanding the brackets and re-arrange in powers of \( h \) gives

\[
y_{n+1} = y_n + h (w_1 B_1 - y_n^2 v_1 M_1) + h^2 (w_1 C_1 - y_n^2 v_1 N_1 - y_n y_n' v_1 M_1) + h^3 (w_1 D_1 - y_n^2 v_1 R_1 - y_n y_n' v_1 B_1 N_1 - y_n y_n' v_1' C_1 M_1) + 0 (h^4)
\]

(2.32)

Comparing the corresponding powers in \( h \) of equations (2.31) and (2.32) with equations (2.18) and (2.19) we obtain

\[
\begin{aligned}
\frac{1}{2} w_1 f_n - \frac{1}{2} y_n v_1 g_n &= \frac{1}{2} f_n \\
\frac{1}{2} w_1 c_1 \Delta f_n - \frac{1}{2} y_n^2 v_1^2 d_1 \Delta g_n - \frac{1}{2} y_n v_1 g_n &= \frac{1}{6} \Delta f_n \\
\frac{1}{2} w_1 f_n - \frac{1}{2} y_n v_1 g_n &= f_n \\
\frac{1}{2} w_1 c_1 \Delta f_n - \frac{1}{2} y_n^2 v_1^2 d_1 \Delta g_n - \frac{1}{2} y_n v_1 f_n (\frac{1}{2} g_n) &= \frac{1}{2} \Delta f_n
\end{aligned}
\]

(2.33)

(By using the equations in (2.25) and (2.29))

Since from equation (1.7)

\[
\begin{aligned}
g_n &= -\frac{f_n}{y_n} \\
g_x &= -\frac{f_x}{y_n} \\
g_z &= -2 \frac{f_z}{y_n} + f_y' \\
g_{x'} &= -2 \frac{f_{x'}}{y_n} + f_{y'} \\
z_n' &= -\frac{y_n'}{y_n}
\end{aligned}
\]

and

\[
\Delta g_n = g_n + z_n' g_x + g_{y} g_{x},
\]

Using those equations into equation (2.33), we get the following simultaneous equations

\[
\begin{aligned}
w_1 + v_1 &= 1 \\
w_1 c_1 + v_1 d_1 &= \frac{1}{3} \\
w_1^2 + v_1' &= 2 \\
w_1 c_1 + v_1' d_1 &= 1
\end{aligned}
\]

(2.35)

Equation (2.35) has (4) equations with (6) unknowns; there will not be a unique solution for (2.35).

There will be a family of one-stage scheme of order four.

i. Choosing the parameters

\[
\begin{aligned}
w_1 &= \frac{1}{3}, & v_1 &= \frac{2}{3}, & c_1 &= a_{11} = b_{11} = 0, & w_1' &= 0, & v_1' &= 2, & d_1 &= \alpha_{11} = \frac{1}{2}, & \beta_{11} = 1
\end{aligned}
\]

arbitrarily the following scheme is obtain.

\[
y_{n+1} = \frac{h y_n' + \frac{1}{3} K_1}{1 + \frac{2}{3} y_n H_1}
\]

(2.36)

and

\[
y_{n+1}' = \frac{y_n'}{1 + \frac{2}{h} H_1}
\]

(2.37)
where
\[ K_1 = \frac{h^2}{2} f(x_n, y_n, y'_n) \]
\[ H_1 = \frac{h^2}{2} f\left(x_n + \frac{1}{2} h, y_n + \frac{1}{2} h y'_n + \frac{1}{2} h z'_n + \frac{1}{2} H_1, z'_n + \frac{1}{2} H_1 \right), \]
since \( d_1 = \alpha_{11} = \frac{1}{2} \beta_{11} \)

ii. Choosing the parameters

From (2.35) setting

\[ \frac{w_1}{v_i} = \frac{1}{2}, \quad c_i = a_{11} = \frac{1}{2}, \quad d_i = \alpha_{11} = \frac{1}{6}, \quad w'_i = 2, \quad v'_i = 0, \quad b_{11} = 1, \quad \beta_{11} = \frac{1}{3} \]

From (2.35) setting

\[ y_{n+1} = \frac{y_n + h y'_n + \frac{1}{2} K_1}{1 + \frac{1}{2} y_n H_1} \]  
(2.38)

and

\[ y'_{n+1} = y'_n + \frac{2}{h} K_1 \]  
(2.39)

where

\[ K_1 = \frac{h^2}{2} f\left(x_n + \frac{1}{2} h, y_n + \frac{1}{2} h y'_n + \frac{1}{2} h K_1, y'_n + \frac{1}{2} H_1 \right), \]
since \( c_i = a_{11} = \frac{1}{2} b_{11} \)

and

\[ H_1 = \frac{h^2}{2} g\left(x_n + \frac{1}{6} h, z'_n + \frac{1}{6} H_1 \right), \]
since \( d_i = \alpha_{11} = \frac{1}{2} \beta_{11} \)

3. CONVERGENCE

A numerical method is said to be convergent if the numerical solution approaches the exact solution as the step size tends to zero.

Convergent = \( \lim_{h \to 0} |y(x_{n+1}) - y_{n+1}| \)

In other words, if the discretization error at \( x_{n+1} \) tends to zero as \( h \to \infty \), i.e if

\[ e_{n+1} = |y(x_{n+1}) - y_{n+1}| \to 0 \quad \text{as} \quad n \to \infty \]  
(3.1)

From equation (2.5),

\[ y'_{n+1} = \frac{y_n + \frac{1}{h} \sum_{r=0}^{n} w'_r K_r}{1 + \frac{1}{h} y_n \sum_{r=0}^{n} v'_r H_r} \]  
(3.2)

while the exact solution \( y'(x_{n+1}) \) seems to satisfy the equation of the form

\[ y'(x_{n+1}) = \frac{y(x_n) + \frac{1}{h} \sum_{r=0}^{n} w'_r K_r}{1 + \frac{1}{h} y(x_n) \sum_{r=0}^{n} v'_r H_r} + T_{n+1} \]  
(3.3)

Where \( T_{n+1} \) is a local truncation error.

Subtracting equation (3.3) from (3.2) gives

\[ y'_{n+1} - y'(x_{n+1}) = \frac{y_n + \frac{1}{h} \sum_{r=0}^{n} w'_r K_r}{1 + \frac{1}{h} y_n \sum_{r=0}^{n} v'_r H_r} - \frac{y(x_n) + \frac{1}{h} \sum_{r=0}^{n} w'_r K_r}{1 + \frac{1}{h} y(x_n) \sum_{r=0}^{n} v'_r H_r} + T_{n+1} \]  
(3.4)

Adopting equation (3.4) gives
Continuing in this manner, we get the following

\[ e_{n+1} = \left( 1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r'H_r \right) \left[ y_n + \frac{1}{h} \sum_{r=1}^{s} w_r K_r \right] - \left( 1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r'H_r \right) \left( y(x_n) + \frac{1}{h} \sum_{r=1}^{s} w_r K_r \right) + T_{n+1} \]  

Expanding the brackets and re-arranging gives

\[ e_{n+1} = \left( 1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r'H_r \right) \left[ 1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r'H_r \right] + T_{n+1} \]

This implies that

\[ e_{n+1} = \left( 1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r'H_r \right) \left[ 1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r'H_r \right] + T_{n+1} \]

From equations (3.7), setting

\[ A_n = \left( 1 + \frac{1}{h} \sum_{r=1}^{s} w_r K_r \sum_{r=1}^{s} v_r'H_r \right) \quad B_n = \left[ 1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r'H_r \right] \quad C_n = \left[ 1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r'H_r \right] \]

and \( T_{n+1} = 7 \)

Then

\[ e_{n+1} = \frac{A_n}{B_n C_n} e_n + T \]

(3.8)

Let \( B = \max B_n > 0 \), \( C = \max C_n > 0 \) and \( A = \max A_n < 0 \) then (3.8) becomes,

\[ e_{n+1} \leq \frac{A}{BC} e_n + T \]

Set \( \frac{A}{BC} = K < 1 \), then

\[ e_{n+1} \leq Ke_n + T \]

(3.9)

If \( n = 0 \), then from (3.9),

\[ e_1 = Ke_0 + T \]

\[ e_2 = Ke_1 + T = K^2 e_0 + KT + T \] by substituting the value of \( e_1 \)

\[ e_3 = Ke_2 + T = K^3 e_0 + K^2 T + T \]

Continuing in this manner, we get the following

\[ e_{n+1} = K^{n+1} e_0 + \sum_{t=0}^{n+1} K^t T \]

(3.10)

Since \( \frac{A}{BC} = K < 1 \), then one can see that as \( n \to \infty \), \( e_{n+1} \to 0 \). This proves that the scheme converges.

**7. CONSISTENCY**

A scheme is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve as the step size ends to zero. To prove if equation (2.5) is consistent, subtract \( y_n \) from both side of (2.5), then

\[ y_{n+1} - y_n = \frac{y_n + hy' + \sum_{r=1}^{s} w_r K_r}{1 + y \sum_{r=1}^{s} v_r'H_r} - y_n \]  

(4.1)

\[ y_{n+1} - y_n = \frac{y_n + hy' + \sum_{r=1}^{s} w_r K_r - y_n - y^2 \sum_{r=1}^{s} v_r'H_r}{1 + y \sum_{r=1}^{s} v_r'H_r} \]  

(4.2)
\[
y_{n+1} - y_n = \frac{hy'_n + \sum_{i=1}^{n} w_i K_i - y'_n \sum_{i=1}^{n} v_i H_i}{1 + y_n \sum_{i=1}^{n} v_i H_i}
\]

but

\[
K_i = \frac{h^2}{2} \left( x_n + c_i h, y_n + h c_i, y'_n + \sum_{j=1}^{n} a_{ij} K_j, y''_n + \frac{1}{h} \sum_{j=1}^{n} b_{ij} K_j \right)
\]

and

\[
H_i = \frac{h^2}{2} g \left( x_n + d_i h, z_n + h d_i, z'_n + \sum_{j=1}^{n} \alpha_{ij} H_j, z''_n + \frac{1}{h} \sum_{j=1}^{n} \beta_{ij} H_j \right)
\]

Then (4.2) becomes

\[
y_{n+1} - y_n = \frac{hy'_n + \sum_{i=1}^{n} w_i K_i - y'_n \sum_{i=1}^{n} v_i H_i}{1 + y_n \sum_{i=1}^{n} v_i H_i}
\]

Dividing the above equation throughout by \( h \) and taking the limit as \( h \) tends to zero on both sides gives

\[
\lim_{h \to 0} \frac{y_{n+1} - y_n}{h} = y'_n
\]

Again recall that from (2.6), subtracting \( y'_n \) on both sides gives

\[
y'_{n+1} - y' = \frac{y'_n + \frac{1}{h} \sum_{i=1}^{n} w_i K_i}{1 + \frac{1}{h} y'_n \sum_{i=1}^{n} v_i H_i} - y'_n
\]

Simplify further gives

\[
y'_{n+1} - y' = \frac{1}{h} \sum_{i=1}^{n} w_i K_i - \frac{1}{h} y''_n \sum_{i=1}^{n} v_i H_i
\]

Substituting the values of \( K_i \) and \( H_i \), (4.4) becomes

\[
y'_{n+1} - y' = \frac{\frac{1}{h} \sum_{i=1}^{n} w_i K_i}{\frac{1}{h} \sum_{i=1}^{n} v_i H_i} \left( x_n + c_i h, y_n + h c_i, y'_n + \sum_{j=1}^{n} a_{ij} K_j, y''_n + \frac{1}{h} \sum_{j=1}^{n} b_{ij} K_j \right) - \frac{1}{h} y''_n \sum_{i=1}^{n} v_i H_i
\]

Dividing all through by \( h \) and taking the limit as \( h \) tends to zero on both sides gives

\[
y'_{n+1} - y' = \frac{1}{2} \left( x_n + c_i h, y_n + h c_i, y'_n + \sum_{j=1}^{n} a_{ij} K_j, y''_n + \frac{1}{h} \sum_{j=1}^{n} b_{ij} K_j \right) - \frac{1}{2} g \left( x_n + d_i h, z_n + h d_i, z'_n + \sum_{j=1}^{n} \alpha_{ij} H_j, z''_n + \frac{1}{h} \sum_{j=1}^{n} \beta_{ij} H_j \right)
\]

but by definition

\[
\lim_{h \to 0} \frac{y_{n+1} - y_n}{h} = \frac{1}{2} \left( x_n + d_i h, z_n + h d_i, z'_n + \sum_{j=1}^{n} \alpha_{ij} H_j, z''_n + \frac{1}{h} \sum_{j=1}^{n} \beta_{ij} H_j \right)
\]
\[ f_n = y_n^2 g(x_n, z_n, z'_n) \]

hence the above equation becomes

\[ \lim_{h \to 0} \frac{y_{n+1} - y_n}{h} = f_n \]

Hence, the numerical method is consistent.

**Conclusion**

The new numerical schemes derived follows the techniques of rational form of Runge–Kutta methods proposed by Hong (1982) which was adopted by Okunbor (1987) and Ademiluyi and Babatola (2000) by using Taylor and Binomial expansion in stages evaluation. The order condition obtained in this research is up to five (5) and the stage is up to three (3). This is an improvement on the work of earlier authors.

Due to convergence and consistency of the new schemes, the scheme will be of high accuracy for direct numerical solution of general second order ordinary differential equations. The steps to the derivation of the new schemes are presented in the methodology while the analysis of the schemes proved to be consistent, convergent.

The implementation of the schemes will be highlighted in the forthcoming paper.

**References**


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