Interval Valued intuitionistic Fuzzy Homomorphism of BF-algebras

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Abstract
The notion of interval-valued intuitionistic fuzzy sets was first introduced by Atanassov and Gargov as a generalization of both interval-valued fuzzy sets and intuitionistic fuzzy sets. Satyanarayana et. al., applied the concept of interval-valued intuitionistic fuzzy ideals to BF-algebras. In this paper, we introduce the notion of interval-valued intuitionistic fuzzy homomorphism of BF-algebras and investigate some interesting properties.

Keywords: BF-algebras, interval valued intuitionistic fuzzy sets, i-v intuitionistic fuzzy ideals

1. Introduction and preliminaries
For the first time Zadeh (1965) introduced the concept of fuzzy sets and also Zadeh (1975) introduced the concept of an interval-valued fuzzy sets, which is an extension of the concept of fuzzy set. Atanassov and Gargov, 1989 introduced the notion of interval-valued intuitionistic fuzzy sets, which is a generalization of both intuitionistic fuzzy sets and interval-valued fuzzy sets. On other hand, Satyanarayana et al., (2012) applied the concept of interval-valued intuitionistic fuzzy ideals. In this paper we introduce the notion of interval-valued intuitionistic fuzzy homomorphism of BF-algebras and investigate some interesting properties.

By a BF-algebra we mean an algebra satisfying the axioms:
(1) \( x \ast x = 0 \),
(2) \( x \ast 0 = x \),
(3) \( 0 \ast (x \ast y) = y \ast x \), for all \( x, y \in X \)
Throughout this paper, \( X \) is a BF-algebra.

Example 1.1 Let \( R \) be the set of real number and let \( A = (R, \ast, 0) \) be the algebra with the operation \( \ast \) defined by
\[
x \ast y = \begin{cases} 
x, & \text{if } y = 0 \\
y, & \text{if } x = 0 \\
0, & \text{otherwise}
\end{cases}
\]
Definition 1.2 The subset \( I \) of \( X \) is said to be an ideal of \( X \), if (i) \( 0 \in I \) and (ii) \( x \ast y \in I \) and \( y \ast x \in I \).

Definition 1.3 A mapping \( f : X \rightarrow Y \) of BF-algebra is called a homomorphism if \( f(x \ast y) = f(x) \ast f(y) \), for all \( x, y \in X \). Note that if \( f \) is a homomorphism of BF-algebras, then \( f(0)=0 \).

An intuitionistic fuzzy set (shortly IFS) in a non-empty set \( X \) is an object having the form \( A = \{ (x, \mu_A(x), \lambda_A(x)) : x \in X \} \), where the function \( \mu_A : X \rightarrow [0,1] \) and \( \lambda_A : X \rightarrow [0,1] \) denote the degree of membership (namely \( \mu_A(x) \)) and the degree of non membership (namely \( \lambda_A(x) \)) of each element \( x \in X \). For the sake of simplicity we use the symbol form \( A = (X, \mu_A, \lambda_A) \) or \( A = (\mu_A, \lambda_A) \).

By interval number \( D \) we mean an interval \( [a^-, a^+] \) where \( 0 \leq a^- \leq a^+ \leq 1 \). The set of all closed subintervals of \([0,1]\) is denoted by \( D[0,1] \). For interval numbers \( D_1 = [a_1^-, b_1^+] \), \( D_2 = [a_2^-, b_2^+] \).
We define
- \( \min(D_1, D_2) = D_1 \cap D_2 = \min([a_1^-, b_1^+], [a_2^-, b_2^+]) = \min\{a_1^-, a_2^-, b_1^+, b_2^+\} \)
• \( \max(D_1, D_2) = D_1 \cup D_2 = \max([a_1^-, b_1^+], [a_2^-, b_2^+]) \)
  
  \[ = \max \{ a_1^-, a_2^- \}, \max \{ b_1^+, b_2^+ \} \]

• \( D_1 + D_2 = [a_1^- + a_2^- - a_1^-, a_2^+ + b_2^+ - b_1^+] \)

And put

• \( D_1 \leq D_2 \Leftrightarrow a_1^- \leq a_2^- \) and \( b_1^+ \leq b_2^+ \)

• \( D_1 = D_2 \Leftrightarrow a_1^- = a_2^- \) and \( b_1^+ = b_2^+ \)

• \( D_1 < D_2 \Leftrightarrow D_1 \leq D_2 \) and \( D_1 \neq D_2 \)

• \( mD = [ma_1^-, mb_1^+] = [ma_1^-, mb_1^+] \), where \( 0 \leq m \leq 1 \).

Let \( L \) be a given nonempty set. An interval-valued fuzzy set \( B \) on \( L \) is defined by \( B = \{(x, [\mu_B^-(x), \mu_B^+(x)]: x \in L \} \), where \( \mu_B^-(x) \) and \( \mu_B^+(x) \) are fuzzy sets of \( L \) such that \( \mu_B^-(x) \leq \mu_B^+(x) \) for all \( x \in L \). Let \( \tilde{\mu}_B(x) = [\mu_B^-(x), \mu_B^+(x)] \), then \( B = \{(x, \tilde{\mu}_B(x)): x \in L \} \) where \( \tilde{\mu}_B : L \rightarrow D[0, 1] \).

A mapping \( A = (\tilde{\mu}_A, \tilde{\lambda}_A) : L \rightarrow D[0, 1] \times D[0, 1] \) is called an interval-valued intuitionistic fuzzy set (i-v IF set, in short) in \( L \) if \( 0 \leq \mu_A^+(x) + \lambda_A^+(x) \leq 1 \) and \( 0 \leq \mu_A^-(x) + \lambda_A^-(x) \leq 1 \) for all \( x \in L \) (that is, \( A^+ = (X, \mu_A^+, \lambda_A^+) \) and \( A^- = (X, \mu_A^-, \lambda_A^-) \) are intuitionistic fuzzy sets), where the mappings \( \tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)] : L \rightarrow D[0, 1] \) and \( \tilde{\lambda}_A(x) = [\lambda_A^-(x), \lambda_A^+(x)] : L \rightarrow D[0, 1] \) denote the degree of membership (namely \( \tilde{\mu}_A(x) \)) and degree of non-membership (namely \( \tilde{\lambda}_A(x) \)) of each element \( x \in L \) to \( A \) respectively.

2. MAIN RESULTS

Definition 2.1: An interval-valued IFS \( A = (X, \tilde{\mu}_A, \tilde{\lambda}_A) \) is called interval-valued intuitionistic fuzzy ideal (shortly i-v IF ideal) of BF-algebra \( X \) if it satisfies

(i-v IF1) \( \tilde{\mu}_A(0) \geq \tilde{\mu}_A(x) \) and \( \tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x) \)

(i-v IF2) \( \tilde{\mu}_A(x) \geq \min \{ \tilde{\mu}_A(x * y), \tilde{\mu}_A(y) \} \)

(i-v IF3) \( \tilde{\lambda}_A(x) \leq \max \{ \tilde{\lambda}_A(x * y), \tilde{\lambda}_A(y) \} \), for all \( x, y \in X \).
Example 2.2 Consider a BF-algebra $X = \{0, 1, 2, 3\}$ with following table

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Let $A$ be an interval-valued intuitionistic fuzzy set in $X$ defined by $\mu_A(0) = \tilde{\mu}_A(1) = [0.6, 0.7]$ and $\mu_A(2) = \tilde{\mu}_A(3) = [0.2, 0.3]$, $\lambda_A(0) = \tilde{\lambda}_A(1) = [0.1, 0.2]$, $\lambda_A(2) = \tilde{\lambda}_A(3) = [0.5, 0.7]$. It is easy to verify that $A$ is an interval-valued intuitionistic fuzzy ideal of $X$.

Definition 2.3 Let $f : X \to X'$ be a homomorphism of BF-algebras. For any interval valued intuitionistic fuzzy set $A = (X', \tilde{\mu}_A, \tilde{\lambda}_A)$ in $X'$ we define a new interval valued intuitionistic fuzzy set $A^f = (X, \tilde{\mu}_A^f, \tilde{\lambda}_A^f)$ in $X$, by $\tilde{\mu}_A^f(x) = \tilde{\mu}_A(f(x))$, $\tilde{\lambda}_A^f(x) = \tilde{\lambda}_A(f(x))$ for all $x \in X$.

Theorem 2.4 Let $X$ and $X'$ be BF-algebras and $f$ is a homomorphism from $X$ onto $X'$.

(i). If $A = (X', \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of $X'$, then $A^f = (X, \tilde{\mu}_A^f, \tilde{\lambda}_A^f)$ is an i-v intuitionistic fuzzy ideal of $X$.

(ii). If $A^f = (X, \tilde{\mu}_A^f, \tilde{\lambda}_A^f)$ is an i-v intuitionistic fuzzy ideal of $X$, then $A = (X', \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of $X'$.

Proof: (i) Suppose $A = (X', \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of $X'$. For $x' \in X'$ there exist $x \in X$ such that $f(x) = x'$. we have

$\tilde{\mu}_A^f(0) = \begin{bmatrix} \mu_A^f(0), \mu_A^f(0) \end{bmatrix}$ and $\tilde{\lambda}_A^f(0) = \begin{bmatrix} \lambda_A^f(0), \lambda_A^f(0) \end{bmatrix}$

$= \begin{bmatrix} \mu_A(f(0)), \mu_A^f(f(0)) \\ \mu_A^+(f(0)) \end{bmatrix}$

$= \begin{bmatrix} \mu_A^+(0'), \mu_A^+(0') \end{bmatrix}$

$\geq \begin{bmatrix} \mu_A^+(x'}, \mu_A^+(x') \end{bmatrix}$

$= \begin{bmatrix} \mu_A(f(x)), \mu_A^+(f(x)) \\ \mu_A^+(f(x)) \end{bmatrix}$

$= \begin{bmatrix} \mu_A^+(x), \mu_A^+(x) \end{bmatrix}$

$= \tilde{\mu}_A^f(x)$

$= \tilde{\lambda}_A^f(x)$
Let \( x, z \in X, y' \in X' \) then there exists \( y \in X \) such that \( f(y) = y' \). We have 
\[
\tilde{\mu}_A^f(x) = \tilde{\mu}_A(f(x)) \geq \min \{ \tilde{\mu}_A(f(x) \ast y'), \tilde{\nu}_A(y') \} \\
= \min \{ \tilde{\mu}_A(f(x \ast y)), \tilde{\nu}_A(f(y)) \} \\
= \min \{ \tilde{\mu}_A^f(x \ast y), \tilde{\nu}_A^f(y) \}
\]
and 
\[
\tilde{\lambda}_A^f(x) = \tilde{\lambda}_A(f(x)) \\
\leq \max \{ \tilde{\lambda}_A(f(x) \ast y'), \tilde{\lambda}_A(y') \} \\
= \max \{ \tilde{\lambda}_A(f(x)\ast f(y)), \tilde{\lambda}_A(f(y)) \} \\
= \max \{ \tilde{\lambda}_A(f(x \ast y)), \tilde{\lambda}_A(f(y)) \} \\
= \max \{ \tilde{\mu}_A^f(x \ast y), \tilde{\nu}_A^f(y) \}
\]
Hence \( A^f = (X, \tilde{\mu}_A^f, \tilde{\nu}_A^f, \tilde{\lambda}_A^f) \) is an i-v intuitionistic fuzzy ideal of \( X \).

(ii) Since \( f : X \rightarrow X' \) is onto, for \( x, y' \in X' \) there exist \( a, b \in X \) such that \( f(a) = x, f(b) = y \).

Now 
\[
\tilde{\mu}_A^f(x) = \tilde{\mu}_A(f(a)) = \tilde{\mu}_A^f(a) \geq \min \{ \tilde{\mu}_A^f((a \ast b)), \tilde{\nu}_A^f(b) \} \\
= \min \{ \tilde{\mu}_A^f(f(a \ast b)), \tilde{\nu}_A^f(f(b)) \} \\
= \min \{ \tilde{\mu}_A^f(f(a) \ast f(b)), \tilde{\nu}_A^f(f(b)) \} \\
= \min \{ \tilde{\mu}_A^f(x \ast y), \tilde{\nu}_A^f(y) \}
\]
and 
\[
\tilde{\lambda}_A^f(x) = \tilde{\lambda}_A(f(a)) = \tilde{\lambda}_A^f(a) \leq \max \{ \tilde{\lambda}_A^f((a \ast b)), \tilde{\nu}_A^f(b) \} \\
= \max \{ \tilde{\lambda}_A^f(f(a \ast b)), \tilde{\nu}_A^f(f(b)) \} \\
= \max \{ \tilde{\lambda}_A^f(f(a) \ast f(b)), \tilde{\nu}_A^f(f(b)) \} \\
= \max \{ \tilde{\mu}_A^f(x \ast y), \tilde{\lambda}_A^f(y) \}
\]
Hence \( A = (X', \tilde{\mu}_A^f, \tilde{\nu}_A^f, \tilde{\lambda}_A^f) \) is an i-v intuitionistic fuzzy ideal \( X' \).

**Definition 2.5** Let \( f \) be a mapping on set \( X \) and \( A = (X, \tilde{\mu}_A, \tilde{\nu}_A, \tilde{\lambda}_A) \) be an i-v IFS in \( X \). Then the i-v fuzzy sets \( \tilde{\mu} \) and \( \tilde{\nu} \) on \( f(X) \) is defined by 
\[
\tilde{\mu}(y) = \sup_{x \in f^{-1}(y)} \tilde{\mu}_A(x) \quad \text{and} \quad \tilde{\nu}(y) = \inf_{x \in f^{-1}(y)} \tilde{\lambda}_A(x)
\]
for all \( y \in f(X) \) is called image of \( A \) under \( f \). If \( \tilde{\mu} \) and \( \tilde{\nu} \) are i-v fuzzy sets in \( f(X) \), then the fuzzy set \( \tilde{\mu} = \tilde{\mu} \circ f \) and \( \tilde{\nu} = \tilde{\nu} \circ f \) is called the pre-image of \( \tilde{\mu} \) and \( \tilde{\nu} \) respectively under \( f \).

**Definition 2.6** An i-v IFS \( A = (X, \tilde{\mu}_A, \tilde{\lambda}_A) \) in \( X \) is said to satisfy the “sup-inf” property if for any sub -set \( T \subseteq X \) there exist \( x_0, y_0 \in T \) such that 
\[
\tilde{\mu}_A(x_0) = \sup_{t \in T} \tilde{\mu}_A(t) \quad \text{and} \quad \tilde{\lambda}_A(y_0) = \inf_{s \in T} \tilde{\lambda}_A(s)
\]
**Theorem 2.7** Let \( f : X \to X' \) be onto homomorphism of BF-algebras. If \( A = (X, \tilde{\mu}_A, \tilde{\lambda}_A) \) is an i-v intuitionistic fuzzy ideal of \( X \) with “sup-inf” property. Then the image of \( A \) under \( f \) is also an i-v intuitionistic fuzzy ideal of \( X' \).

**Proof:** For any \( X \in X \) we have \( \tilde{\mu}_A(0) \geq \tilde{\mu}_A(x) \) and \( \tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x) \).

Suppose \( A = (X, \tilde{\mu}_A, \tilde{\lambda}_A) \) is an i-v intuitionistic fuzzy ideal of \( X \) with “sup-inf” property. The image of \( A \) under \( f \) is defined by \( \tilde{u} : X' \to [0,1] \) by \( \tilde{u}(y') = \sup_{x \in f^{-1}(y')} \tilde{\mu}_A(x) \) for all \( y' \in X' \) and \( \tilde{v} : X' \to [0,1] \) by \( \tilde{v}(y') = \inf_{x \in f^{-1}(y')} \tilde{\lambda}_A(x) \) for all \( y' \in X' \).

Since \( A = (X, \tilde{\mu}_A, \tilde{\lambda}_A) \) is an i-v intuitionistic fuzzy ideal of \( X \).

Thus \( \tilde{u}(0') = \sup_{t \in f^{-1}(0')} \tilde{\mu}_A(t) = \tilde{\mu}_A(0) \geq \tilde{\mu}_A(x) \) for all \( x \in X \).

Therefore \( \tilde{u}(0') \geq \tilde{\mu}_A(x) \) for all \( x \in X \).

Further more we have \( \tilde{u}(x') = \sup_{t \in f^{-1}(x')} \tilde{\mu}_A(t) \) for all \( x' \in X' \).

Hence \( \tilde{u}(0') \geq \sup_{t \in f^{-1}(x')} \tilde{\mu}_A(t) = \tilde{\mu}(x') \). Therefore \( \tilde{u}(0') \geq \tilde{u}(x') \) for all \( x' \in X' \).

And \( \tilde{v}(0') = \inf_{t \in f^{-1}(0')} \tilde{\lambda}_A(t) = \tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x) \) for all \( x \in X \).

Therefore \( \tilde{v}(0') \leq \tilde{\lambda}_A(x) \) for all \( x \in X \). Further more we have \( \tilde{v}(x') = \inf_{t \in f^{-1}(x')} \tilde{\lambda}_A(t) \) for all \( x' \in X' \).

Hence \( \tilde{v}(0') \leq \inf_{t \in f^{-1}(x')} \tilde{\lambda}_A(t) = \tilde{v}(x') \), \( \forall x' \in X' \). Thus \( \tilde{v}(0') \leq \tilde{v}(x') \), \( \forall x' \in X' \).

Since \( f \) is onto mapping then for any \( x', y' \in X' \). Since \( X' = f(X) \), then there exist \( x, y \in X \) such that \( x' = f(x), y' = f(y) \). Let \( x_0 \in f^{-1}(x') \) be such that \( \tilde{\mu}_A(x_0) = \sup_{t \in f^{-1}(x')} \tilde{\mu}_A(t) \) and

hence \( \tilde{u}(x') = \tilde{u}(f(x)) = \sup_{t \in f^{-1}(f(x))} \tilde{\mu}_A(t) \)

\[ = \tilde{\mu}_A(x_0) \]

\[ \geq \min \{ \tilde{\mu}_A((x_0 * y)), \tilde{\mu}_A(y) \} \]

\[ = \min \{ \tilde{u}(f(x_0 * y)), \tilde{u}(f(y)) \} \]

\[ = \min \{ \tilde{u}(f(x_0) * f(y)), \tilde{u}(f(y)) \} \]

\[ = \min \{ \tilde{u}(x' * y'), \tilde{u}(y') \} \]

Therefore \( \tilde{u}(x') \geq \min \{ \tilde{u}(x' * y'), \tilde{u}(y') \} \) for all \( x', y' \in X \).
Let $x_0 \in f^{-1}(x')$ be such that $\lambda_A(x_0) = \inf_{t \in f^{-1}(x')} \lambda_A(t)$.

Now $\tilde{\nu}(x') = \tilde{\nu}(f(x)) = \inf_{t \in f^{-1}(f(x))} \lambda_A(t)
= \lambda_A(x_0)
\leq \max \{ \lambda_A(x_0, y), \lambda_A(y) \}
= \max \{ \tilde{\nu}(f(x_0, y)), \tilde{\nu}(f(y)) \}
= \max \{ \tilde{\nu}(f(x_0)*f(y)), \tilde{\nu}(y) \}
= \max \{ \tilde{\nu}(x'*y'), \tilde{\nu}(y') \}$

Therefore $\tilde{\nu}(x') \leq \max \{ \tilde{\nu}(x'*y'), \tilde{\nu}(y') \}$ for all $x', y' \in X$

Thus $A = (X', \mu_A, \lambda_A)$ is an i-v intuitionistic fuzzy ideal of $X'$.

**Definition 2.8** Let $A = (X, \mu_A, \lambda_A)$ be an i-v IFS in $X$ and let $\alpha, \beta \in [0, 1]$ be such that $\alpha + \beta \leq 1, 1]$. Then the set $X_A^{(\alpha, \beta)} = \{ x \in X / \mu_A(x) \geq \alpha, \lambda_A(x) \leq \beta \}$ is called an $(\alpha, \beta)$-level sub set of $A = (X, \mu_A, \lambda_A)$.

**Theorem 2.9** Let $A = (X, \mu_A, \lambda_A)$ be an interval-valued intuitionistic fuzzy ideal of $X$. Then $X_A^{(\alpha, \beta)}$ is an ideal of $X$, for every $(\alpha, \beta) \in \text{Im}(\mu_A) \times \text{Im}(\lambda_A)$ with $\alpha + \beta \leq 1, 1]$

**Proof:** Let $x \in X_A^{(\alpha, \beta)}$, then $x \in X, \mu_A(x) \geq \alpha$ and $\lambda_A(x) \leq \beta$.

Let $x, y \in X$ be such that $x*y$ and $y \in X_A^{(\alpha, \beta)}$ then $\mu_A(x*y) \geq \alpha, \lambda_A(x*y) \leq \beta$ and $\mu_A(y) \geq \alpha, \lambda_A(y) \leq \beta$. It follows from (i-v IF2) and (i-v IF3) that

$$\mu_A(x) \geq \min \{ \mu_A(x*y), \mu_A(y) \} \geq \min \{ \alpha, \alpha \} = \alpha$$

and

$$\lambda_A(x) \leq \max \{ \lambda_A(x*y), \lambda_A(y) \} \leq \max \{ \beta, \beta \} = \beta$$

So that $x \in X_A^{(\alpha, \beta)}$. Hence $X_A^{(\alpha, \beta)}$ is an ideal of $X$.

**Theorem 2.10** Let $A = (X, \mu_A, \lambda_A)$ be an i-v IFS in $X$ such that $X_A^{(\alpha, \beta)}$ is an ideal of $X$. If $(\alpha, \beta) \in \text{Im}(\mu_A) \times \text{Im}(\lambda_A)$ with $\alpha + \beta \leq 1, 1]$, then $A = (X, \mu_A, \lambda_A)$ is an interval-valued IF ideal of $X$.

**Proof:** Let $A(x) = (\alpha, \beta)$ for all $x \in X$, that is, $\mu_A(x) = \alpha$ and $\lambda_A(x) = \beta$ for all $x \in X$. Since $0 \in X_A^{(\alpha, \beta)}$, we have $\mu_A(0) \geq \alpha = \mu_A(x)$ and $\lambda_A(0) \leq \beta = \lambda_A(x)$ for all $x \in X$. Let $x, y \in X$ be
such that $A(x * y) = \left(\tilde{\mu}_A, \tilde{\lambda}_A\right)$ and $A(y) = \left(\tilde{\mu}_A, \tilde{\lambda}_A\right)$, that is, $\tilde{\mu}_A(x * y) = \tilde{\mu}_A(x) * \tilde{\mu}_A(y)$ and $\tilde{\lambda}_A(y) = \tilde{\lambda}_A(y)$. Then $x * y \in X(\tilde{\mu}_A, \tilde{\lambda}_A)$ and $y \in X(\tilde{\mu}_A, \tilde{\lambda}_A)$. We may assume that $(\tilde{\mu}_A, \tilde{\lambda}_A) \subseteq (\tilde{\mu}_A, \tilde{\lambda}_A)$, that is, $\tilde{\mu}_A \leq \tilde{\mu}_A$ and $\tilde{\lambda}_A \geq \tilde{\lambda}_A$, with out loss of generality. It follows that $X(\tilde{\mu}_A, \tilde{\lambda}_A)$ is an ideal of $X$. Hence we have the following corollary.

**Corollary 2.11** Let $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in $X$. Then $A$ is an i-v intuitionistic fuzzy ideal of $X$ if and only if $U(\tilde{\mu}_A, \tilde{\lambda}_A)$ and $L(\tilde{\mu}_A, \tilde{\lambda}_A)$ are ideals of $X$, for every $\alpha \in [0, \tilde{\mu}_A(0)]$ and $\beta \in [\tilde{\lambda}_A(0), 1]$ with $\tilde{\alpha} + \tilde{\beta} \leq [1, 1]$.

**Theorem 2.12** Let $I \subseteq X$ and $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in $X$ defined by

$$\tilde{\mu}_A(x) = \begin{cases} \tilde{\alpha}_0 & \text{if } x \in I \\ \tilde{\alpha}_1 & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{\lambda}_A(x) = \begin{cases} \tilde{\beta}_0 & \text{if } x \in I \\ \tilde{\beta}_1 & \text{otherwise} \end{cases}$$

for all $x \in X$ where $0 \leq \tilde{\alpha}_0 < \tilde{\alpha}_1$, $0 \leq \tilde{\beta}_0 < \tilde{\beta}_1$ and $\tilde{\alpha}_i + \tilde{\beta}_i \leq 1$ for $i = 0, 1$.

Then the following conditions are equivalent:

1. $A$ is an i-v intuitionistic fuzzy ideal of $X$.
2. $I$ is an ideal of $X$.

**Proof:** Assume (1), that is, $A$ is an i-v intuitionistic fuzzy ideal of $X$.

Let $x, y \in I$. Now $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x) = \tilde{\alpha}_0$ and so $\tilde{\mu}_A(0) \geq \tilde{\alpha}_0$ implies $0 \in I$.

Let $x, y \in X$ be such that $x * y$ and $y \in I$. We have

$$\tilde{\mu}_A(x) = \min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\} = \min\{\tilde{\alpha}_0, \tilde{\alpha}_0\} = \tilde{\alpha}_0$$

and so $x \in I$. Hence $I$ is an ideal of $X$.

Assume (2), Let $x \in X$. If $x \in I$ implies $\tilde{\mu}_A(x) = \tilde{\alpha}_0$, since $0 \in I$ we have $\tilde{\mu}_A(0) = \tilde{\alpha}_0$ and so $\tilde{\mu}_A(0) = \tilde{\mu}_A(x)$. Also $\tilde{\lambda}_A(x) = \tilde{\beta}_0$ and so $\tilde{\lambda}_A(x) = \tilde{\lambda}_A(x)$. If $x \notin I$ implies $\tilde{\mu}_A(x) = \tilde{\alpha}_1$ and $\tilde{\lambda}_A(x) = \tilde{\beta}_1$. Now $\tilde{\mu}_A(0) = \tilde{\alpha}_0 > \tilde{\alpha}_1 = \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) = \tilde{\beta}_0 < \tilde{\beta}_1 = \tilde{\lambda}_A(x)$.

Therefore, in either cases $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$ for all $x \in X$. 

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Let \( x, y \in X \) be such that \( x \ast y \) and \( y \in X \). If \( x \ast y \in I \) and \( y \in I \) since \( I \) is an ideal of \( X \). We have that \( x \in I \) and so \( \tilde{\mu}_A(x) = \tilde{a}_0 = \min\{\tilde{a}_0, \tilde{a}_1\} = \min\{\tilde{\lambda}_A(x \ast y), \tilde{\lambda}_A(y)\} \) and
\[
\tilde{\lambda}_A(x) = \tilde{\beta}_0 = \max\{\tilde{\beta}_0, \tilde{\beta}_1\} = \max\{\tilde{\lambda}_A(x \ast y), \tilde{\lambda}_A(y)\}
\]
If \( x \ast y \in I \) and \( y \notin I \) or so \( \tilde{\mu}_A(x) = \tilde{a}_i = \min\{\tilde{a}_0, \tilde{a}_1\} = \min\{\tilde{\mu}_A(x \ast y), \tilde{\mu}_A(y)\} \) and
\[
\tilde{\lambda}_A(x) = \tilde{\beta}_i = \max\{\tilde{\beta}_0, \tilde{\beta}_1\} = \max\{\tilde{\lambda}_A(x \ast y), \tilde{\lambda}_A(y)\}
\]
If \( x \ast y \notin I \) and \( y \in I \) or so \( \tilde{\mu}_A(x) = \tilde{a}_i = \min\{\tilde{a}_0, \tilde{a}_1\} = \min\{\tilde{\mu}_A(x \ast y), \tilde{\mu}_A(y)\} \)
\[
\tilde{\lambda}_A(x) = \tilde{\beta}_i = \max\{\tilde{\beta}_0, \tilde{\beta}_1\} = \max\{\tilde{\lambda}_A(x \ast y), \tilde{\lambda}_A(y)\}
\]
Therefore \( \tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(x \ast y), \tilde{\mu}_A(y)\} \) and \( \tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A(x \ast y), \tilde{\lambda}_A(y)\} \) for all \( x, y \in X \).
Hence \( A = (X, \tilde{\mu}_A, \tilde{\lambda}_A) \) is an \( i-v \) intuitionistic fuzzy ideal of \( X \).

**Corollary 2.13** Let \( I \subseteq X \) and \( A = (X, \tilde{\mu}_A, \tilde{\lambda}_A) \) be an \( i-v \) IF in \( X \) defined by
\[
\tilde{\mu}_A(x) = \begin{cases} \tilde{1}, & \text{if } x \in I \\ \tilde{0}, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{\lambda}_A(x) = \begin{cases} \tilde{1}, & \text{if } x \in I \\ \tilde{0}, & \text{otherwise} \end{cases}, \text{for all } x \in X.
\]
Then the following conditions are equivalent:
(1) \( A \) is an \( i-v \) intuitionistic fuzzy ideal of \( X \).
(2) \( I \) is an ideal of \( X \).

**Proposition 2.14** Let \( A = (X, \tilde{\mu}_A, \tilde{\lambda}_A) \) be an \( i-v \) intuitionistic fuzzy ideal of \( X \) and
\[
(\tilde{a}_1, \tilde{\beta}_1), (\tilde{a}_2, \tilde{\beta}_2) \in \text{Im}(\tilde{\mu}_A) \times \text{Im}(\tilde{\lambda}_A) \text{ with } \tilde{a}_i + \tilde{\beta}_i \leq 1 \text{ for } i = 1, 2.
\]
Then
\[
X_A(\tilde{a}_1, \tilde{\beta}_1) = X_A(\tilde{a}_2, \tilde{\beta}_2)
\]
if and only if \( (\tilde{a}_1, \tilde{\beta}_1) = (\tilde{a}_2, \tilde{\beta}_2) \).

**Proof:** If \( (\tilde{a}_1, \tilde{\beta}_1) = (\tilde{a}_2, \tilde{\beta}_2) \) then clearly \( X_A(\tilde{a}_1, \tilde{\beta}_1) = X_A(\tilde{a}_2, \tilde{\beta}_2) \). Assume that
\[
X_A(\tilde{a}_1, \tilde{\beta}_1) = X_A(\tilde{a}_2, \tilde{\beta}_2).\text{ Since } (\tilde{a}_1, \tilde{\beta}_1) \in \text{Im}(\tilde{\mu}_A) \times \text{Im}(\tilde{\lambda}_A) \text{ then there exist } x \in X \text{ such that}
\]
\[
\tilde{\mu}_A(x) = \tilde{a}_i, \tilde{\lambda}_A(x) = \tilde{\beta}_i.\text{ It follows that } x \in X_A(\tilde{a}_1, \tilde{\beta}_1) = X_A(\tilde{a}_2, \tilde{\beta}_2), \text{ so that}
\]
\[
\tilde{a}_1 = \tilde{\mu}_A(x) \geq \tilde{\alpha}_i \text{ and } \tilde{\beta}_1 = \tilde{\lambda}_A(x) \leq \tilde{\beta}_2.\text{ Similarly we have } \tilde{\alpha}_1 \leq \tilde{\alpha}_2 \text{ and } \tilde{\beta}_1 \geq \tilde{\beta}_2.\text{ Hence}
\]
\[
(\tilde{a}_1, \tilde{\beta}_1) = (\tilde{a}_2, \tilde{\beta}_2).
\]

**Theorem 2.15** Let \( A = (X, \tilde{\mu}_A, \tilde{\lambda}_A) \) be an \( i-v \) IF in \( X \) and \( \text{Im}(A) = \{\tilde{a}_0, \tilde{\beta}_0\}, \tilde{a}_1, \tilde{\beta}_1\}, \ldots, \tilde{a}_k, \tilde{\beta}_k\} \) where \( (\tilde{a}_1, \tilde{\beta}_1) < (\tilde{a}_j, \tilde{\beta}_j) \) whenever \( i > j \). Let \( \{G_r/r = 0, 1, 2, \ldots, k\} \) be family of \( i-v \) ideals of \( X \) such that \( G_0 \subset G_1 \subset \ldots \subset G_k = X \) and \( A(G^r_r) = (\tilde{a}_r, \tilde{\beta}_r) \), that is, \( \tilde{\mu}_A(G^r_r) = \tilde{\alpha}_r \) and \( \tilde{\lambda}_A(G^r_r) = \tilde{\beta}_r \).
where $G_r^* = G_r \setminus G_{r-1}$ and $G_{r-1} = \emptyset$ for $r = 0, 1, 2, 3, \ldots, k$. Then $A = (X, \mu_A, \lambda_A)$ is an i-v intuitionistic fuzzy ideal of $X$.

**Proof:** Since $0 \in G_0$, we have $\tilde{\mu}_A(0) = \tilde{\alpha}_0 \geq \mu_A(x)$ and $\tilde{\lambda}_A(0) = \tilde{\beta}_0 \leq \lambda_A(x)$ for all $x \in X$. Let $x, y \in X$. To prove that $A = (X, \mu_A, \lambda_A)$ satisfies the conditions (i - v IF 2) and (i - v IF 3). We discuss the following cases:

If $x \neq y \in G_r^*$ and $y \in G_r^*$, and $x \in G_{r-1}$, because $G_r$ is an ideal of $X$.

Thus $\mu_A(x) \geq \tilde{\alpha}_r = \min \{ \mu_A(x*y), \mu_A(y) \}$ and $\lambda_A(x) \leq \tilde{\beta}_r = \max \{ \lambda_A(x*y), \lambda_A(y) \}$.

If $x \neq y \in G_r^*$ and $y \notin G_r^*$, then the following four cases will arise:

1. $x \neq y \in X \setminus G_r^*$ and $y \notin X \setminus G_r^*$,
2. $x \neq y \in G_r \setminus G_{r-1}$ and $y \in G_{r-1}$,
3. $x \neq y \in X \setminus G_r^*$ and $y \in G_r \setminus G_{r-1}$,
4. $x \neq y \in G_r \setminus G_{r-1}$ and $y \notin X \setminus G_r^*$.

But, in either case, we know that $\mu_A(x) \geq \min \{ \mu_A(x*y), \mu_A(y) \}$ and $\lambda_A(x) \leq \max \{ \lambda_A(x*y), \lambda_A(y) \}$.

If $x \neq y \in G_r^*$ and $y \notin G_r^*$, then by similar processes, we have $\mu_A(x) \geq \min \{ \mu_A(x*y), \mu_A(y) \}$ and $\lambda_A(x) \leq \max \{ \lambda_A(x*y), \lambda_A(y) \}$ for all $x, y \in X$. Thus $A = (X, \mu_A, \lambda_A)$ is an interval-valued intuitionistic fuzzy ideal of $X$.

**References**


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