

Partial Ordering in Soft Set Context

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Abstract

In [1], [2], [3], [4], [5], [6] and [7] basic introduction of soft set is discussed with examples. The main aim of this paper is develop partial ordering in soft set context.

1. Introduction

Soft set theory was first proposed in [1]. It is a general mathematical tool for dealing with uncertainties and not clearly defined objects. Traditional tools for formal modeling, reasoning and computing are crisp, deterministic and precise in nature. However, in most cases, complicated problems in economics, engineering, environment, social science, medical science, etc., that involved data which are not always all crisp cannot be successfully dealt with using classical methods because of various types of uncertainties present in these problems. Soft set is one of the various non-classical methods that can be considered as a mathematical tool for dealing with uncertainties.

Various potential applications of soft set in many areas like, in smoothness of functions, game theory, operations research, Riemann-integration, probability theory, theory of measurement and so on are highlighted in [2]. Following [1], [2], [3], [4], [5], [6] and [7] we present various operations on soft sets.

2. Concept of Soft set and Basic Definitions

Definition 2.1 (See [2] and [6]) A pair (F, E) is called a soft set over a given universal set U , if and only if F is a mapping of a set of parameters E , into the power set of U . That is, $F : E \rightarrow P(U)$. Clearly, a soft set over U is a parameterized family of subsets of a given universe U . Also, for any $p \in E$, $F(p)$ is considered as the set of p -approximate element of the soft set (F, E) .

Example 1 (See [2])

- (i) Let (X, τ) be a topological space, that is, X is a set and τ is a topology (a family of subsets of X called the open sets of X). Then, the family of open neighbourhoods $T(x)$ of point x , where $T(x) = \{V \in \tau \mid x \in V\}$ may be considered as the soft set $(T(x), \tau)$.
- (ii) Also, fuzzy set is a special case of soft set; let A be a fuzzy set and μ_A be the membership function of the fuzzy set A , that is, μ_A is a mapping of U into $[0, 1]$, let $F(\alpha) = \{x \in U \mid \mu_A(x) \geq \alpha\}$, $\alpha \in [0, 1]$ be a family of α -level sets for function μ_A . If the family F is known $\mu_A(x)$ can be found by means of the definition: $\mu_A(x) = \sup_{x \in F(\alpha)} \alpha$. Hence every fuzzy set A may be considered as the soft set $(F, [0, 1])$.

Definition 2.2 A soft set (F, E) over a universe U is said to be null soft set denoted by $\tilde{\emptyset}$, if $\forall e \in E, F(e) = \emptyset$.

Definition 2.3 A soft set (F, A) over a universe U is called absolute soft set denoted by (F, \widetilde{A}) , if $\forall e \in E, F(e) = U$.

Definition 2.4 Let $E = \{e_1, e_2, e_3, \dots, e_n\}$ be a set of parameters. The not-set of E denoted by $\neg E$ is defined as $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \dots, \neg e_n\}$.

Definition 2.5 The complement of a soft set (F, E) denoted by $(F, E)^c$ is defined as $(F, E)^c = (F^c, \neg E)$.

Where: $F^c: \neg A \rightarrow P(U)$ is a mapping given by $F^c(\alpha) = U - F(\neg\alpha)$, $\forall \alpha \in \neg A$

We call F^c the soft complement function of F .

3 Soft set relations

Definition 3.1 [4] Let (F, A) and (G, B) be two soft sets over U , then the Cartesian product of (F, A) and (G, B) is define as $(F, A) \times (G, B) = (H, A \times B)$, where $A \times B \rightarrow P(U \times U)$ and $H(a, b) = F(a) \times G(b)$ where $(a, b) \in A \times B$. i.e. $H(a, b) = \{(h_i, h_j); \text{ where } h_i \in F(a) \text{ and } h_j \in G(b)\}$

Definition 3.2 [4] Let (F, A) and (G, B) be two soft sets over U , then a relation from (F, A) to (G, B) is a soft subset of $(F, A) \times (G, B)$. A relation from (F, A) to (G, B) is of the form (H_1, S) where $S \subset A \times B$ and $H_1(a, b) = H(a, b) \forall a, b \in S$. Where $(H, A \times B) = (F, A) \times (G, B)$ as defined above. (i.e. Cartesian product of soft sets). Any subset of $(F, A) \times (F, A)$ is called a relation on (F, A) . In an equivalent way, we can define the relation R on the soft set (F, A) in the parameterized form as follows.

If $(F, A) = \{F(a), F(b), \dots\}$, then $F(a)RF(b)$ if and only if $F(a) \times F(b) \in R$.

Definition 3.3 [4] Let R be a soft set relation from (F, A) to (G, B) , then the domain of R ($dom R$) is defined as the soft set (D, A_1) where

$$A_1 = \{a \in A : H(a, b) \in R \text{ for some } b \in B\} \text{ and } D(a_1) = F(a_1), \forall a_1 \in A.$$

The range of R ($ran R$) is defined as the soft set (RG, B_1) , where $B_1 \subset B$ and $B_1 = \{b \in B : H(a, b) \in R \text{ for some } a \in A\}$ and $RG(b_1) = G(b_1) \forall b_1 \in B_1$.

Definition 3.4 [4] The identity relation $I_{(F,A)}$ on any soft set (F, A) is defined as follows $F(a)I_{(F,A)}F(b)$ iff $a = b$.

Example 2 $I_{(F,A)} = \{F(a) \times F(a), F(b) \times F(b), F(c) \times F(c)\}$ (For more example, see [3] and [4])

4. Composition of soft set relation

Definition 4.1 [3] Let (F, A) , (G, B) and (H, C) be three soft sets. Let R be a soft set relation from (F, A) to (G, B) and S be another soft set relation from (G, B) to (H, C) , then the composition of R and S is a new soft set relation from (F, A) to (H, C) expressed as SoR and is defined as follows;

If $F(a)$ is in (F, A) and $H(c)$ is in (H, C) then $F(a)SoRH(c)$ iff there is some $G(b)$ in (G, B) such that $F(a)RG(b)$ and $G(b)RH(c)$. (For more example, see [3] and [4])

Remark Composition of soft set relation is not commutative.

Types of soft set Relation (see[3])

Let R be a relation on (F, A) , then

- (i) R is reflexive if $H_1(a, a) \in R, \forall a \in A$.
- (ii) R is symmetric if $H_1(a, b) \in R \leftrightarrow H_1(b, a) \in R, \forall (a, b) \in A \times A$.
- (iii) R is anti-symmetric if whenever $H_1(a, b) \in R$ and $H_1(b, a) \in R$ then $a = b, \forall a, b \in A \times A$
- (iv) R is transitive if $H_1(a, b) \in R, H_1(b, c) \in R \rightarrow H_1(a, c) \in R, \forall a, b, c \in A$.

Equivalence relation and partition on soft sets (see[3])

Definition 4.2 A soft set relation R on a soft set (F, A) is called an equivalence relation if it is reflexive, symmetric and transitive.

Example 3 Consider a soft set (F, A) over $U = \{x_1, x_2, \dots, x_9\}$. $A = \{m_1, m_2\}$ where

$$F(m_1) = \{x_1, x_2, x_5, x_6\}, F(m_2) = \{x_3, x_4, x_7, x_8, x_9\}. \text{ Consider a relation } R \text{ defined on } (F, A) \text{ as } \{F(m_1) \times F(m_2), F(m_2) \times F(m_1), F(m_1) \times F(m_1), F(m_2) \times F(m_2)\}$$

Definition 4.3 Let (F, A) be a soft set, then the equivalence class of $F(a)$ denoted by $[F(a)]$ is defined as $[F(a)] = \{F(b) : F(b)RF(a)\}$.

In example 3 above, $[F(m_1)] = \{F(m_1), F(m_2)\} = [F(m_2)]$.

Definition 4.4 [3] The inverse of a soft set relation R denoted as R^{-1} is defined by

$R^{-1} = \{F(b) \times F(a) : F(a)RF(b)\}$. It is clear from the above definition that the inverse of R is defined by reversing the order of every pair belonging to R .

5. Partial Ordering Relations in Soft Set Theory Context

Order and precedence relationship appears in many different places in mathematics and computer science.

Definition 5.1 A relation R on a soft set (F, A) is called a partial ordering of (F, A) or a partial order on (F, A) if it has the following three properties ie R is (i) Reflexive (ii) Antisymmetric and (iii) Transitive.

Suppose R is a relation on a soft set (F, A) satisfying the following

[P₁] (Reflexive): for any $F(a) \in (F, A)$, we have $F(a)RF(a)$

[P₂] (Antisymmetric): if $F(a)RF(b)$ and $F(b)RF(a)$, then $F(a) = F(b)$

[P₃] (Transitive): if $F(a)RF(b)$ and $F(b)RF(c)$, then $F(a)RF(c)$.

Then R is called a partial order or simply an order relation, and R is said to define a partial ordering of (F, A) .

The soft set (F, A) with the partial ordering R is called a partially ordered soft set or simply an ordered soft set.

The most familiar order relation called the usual order, is the relation \subseteq (reads "set inclusion") on soft sets, for this reason, a partial ordering relation is frequently denoted by

\preceq

With this notation, the above three properties of a partial order appear in the following form.

[P₁] (Reflexive): for any $F(a) \in (F, A)$, we have $F(a) \preceq F(a)$

[P₂] (Antisymmetric): if $F(a) \preceq F(b)$ and $F(b) \preceq F(a)$, then $F(a) = F(b)$

[P₃] (Transitive): if $F(a) \preceq F(b)$ and $F(b) \preceq F(c)$, then $F(a) \preceq F(c)$.

Definition 5.2 A soft set (F, A) together with a partial ordering R is called a partially ordered soft set or posset.

An ordered soft set consist of two things, a soft set (F, A) and the partial ordering \preceq . Here we denoted ordered soft set as $((F, A), \preceq)$.

Suppose (F, A) is an ordered soft set. Then the statement

$F(a) \preceq F(b)$ is read “ $F(a)$ precedes $F(b)$ ”.

In this context we also writes

$F(a) < F(b)$ means $F(a) \preceq F(b)$ and $F(a) = F(b)$ reads “ $F(a)$ strictly precedes $F(b)$ ”

$F(b) \succcurlyeq F(a)$ means $F(a) \preceq F(b)$ reads “ $F(b)$ succeeds $F(a)$ ”.

$F(b) > F(a)$ means $F(a) < F(b)$ reads “ $F(b)$ strictly succeeds $F(a)$ ”.

Remark $\nprec, \prec, \succcurlyeq$ and \preccurlyeq means negation of the above

Example 4 Consider a soft set (F, A) over $U = \{x_1, x_2, \dots, x_5\}$. $A = \{a_1, a_2\}$ where $F(a_1) = \{x_1, x_2, x_5\}$, $F(a_2) = \{x_3, x_4, x_5\}$. Consider a relation R defined on (F, A) as $\{F(a_1) \times F(a_1), F(a_2) \times F(a_2), F(a_1) \times F(a_2), F(a_2) \times F(a_1)\}$
 This relation is a partial order relation.

Definition 5.3 Let \preceq be any partial ordering of a soft set (F, A) . The relation \succcurlyeq , that is $F(a)$ succeeds $F(b)$, is also a partial ordering of (F, A) , it is called the *dual order*. Observe that $F(a) \preceq F(b)$ if and only if $F(b) \succcurlyeq F(a)$; hence the dual order \succcurlyeq is the inverse of the relation \preceq , that is $\succcurlyeq = \preceq^{-1}$.

Definition 5.4 Let (F, A_1) be a soft subset of an ordered soft set (F, A) and suppose that $F(a), F(b) \in (F, A_1)$. Then the order in (F, A) induces an order in (F, A_1) in the following natural way. $F(a) \preceq F(b)$ as an element of (F, A_1) whenever $F(a) \preceq F(b)$ as an element of (F, A) .

More precisely, if R is a partial ordering on (F, A) , then the relation

$$R_{(F, A_1)} = R \cap [(F, A) \times (F, A)]$$

is a partial ordering of (F, A_1) called the *induced order* on (F, A_1) or the restriction of R to (F, A_1) . The soft subset (F, A_1) with the induced order is called an ordered soft subset of (F, A) .

Definition 5.5 suppose $<$ is a relation on soft set (F, A) satisfying the following two properties

[Q_1] (Irreflexive): for any $F(a) \in (F, A)$, we have $F(a) \not\prec F(a)$

[Q_2] (Transitive): if $F(a) \preceq F(b)$ and $F(b) \preceq F(c)$, then $F(a) \preceq F(c)$.

Then $<$ is called a *quasi-order* on (F, A) .

Remark

There is a close relationship between *partial orders* and *quasi-orders*, specifically, if \preceq is a partial order on a soft set (F, A) and we define $F(a) < F(b)$ mean $F(a) \preceq F(b)$ but $F(a) \neq F(b)$, then $<$ is a quasi-order on (F, A) . Conversely, if $<$ is a quasi-order on a soft set (F, A) and we define $F(a) \preceq F(b)$ to mean $F(a) < F(b)$ or $F(a) = F(b)$, then \preceq is a partial order on (F, A) . This allows us to switch back and forth between a partial order and its corresponding quasi-order using whichever is more convenient.

Definition 5.6 Suppose $F(a), F(b)$ are distinct element in a partially ordered soft set (F, A) . We say $F(a)$ and $F(b)$ are comparable if

$$F(a) < F(b) \text{ or } F(b) < F(a).$$

That is if one of them precedes the other. Thus $F(a)$ and $F(b)$ are non comparable, written

$$F(a) \parallel F(b).$$

If $F(a) \not\prec F(b)$ and $F(b) \not\prec F(a)$.

Linear Ordered Soft Set

The word “Partial” is used in defining a partially ordered soft set (F, A) , since some of the element of (F, A) need not be comparable. Suppose on the other hand, every pair of element of (F, A) are comparable. Then (F, A) is said to be *linearly ordered* or *totally ordered*. Although an ordered soft set (F, A) may not be linearly ordered, it is still possible for a subsoftset (F, A_1) of (F, A) to be linearly ordered. Such a linearly subsoftset (F, A_1) of an ordered soft set (F, A) is called a *chain* in (F, A) . Clearly every subsoftset of a linearly ordered softset (F, A) must also be linearly ordered.

Definition 5.7 if $((F, A), \preceq)$ is a partially ordered soft set and every two elements of (F, A) are comparable, (F, A) is called *totally ordered* or *linearly ordered soft set*, and \preceq is called a total order or a linear order. A totally ordered soft set is also called a *chain*.

Example 5 Consider the soft set (F, A) ordered by set inclusion with $A = \{a_1, a_2\}$ and $U = \{x_1, x_2, \dots, x_5\}$ is not linearly ordered. Suppose $F(a_1), F(a_2)$ are non-comparable.

Observed that the empty soft set $\emptyset, F(a_1)$ and (F, A) do form a chain in $P(U)$, since $\emptyset \subseteq F(a_1) \subseteq (F, A)$. Similarly, $\emptyset, F(a_2)$ and (F, A) form a chain in $P(U)$.

Product Soft Set and Order

Here we discuss the different ways of defining an order on a soft set which is constructed from ordered soft set. There are a number of ways to define an order relation on the Cartesian product of given ordered soft set. Two of these ways follow

(i) Product Order: Suppose (F, A) and (G, B) are ordered soft set. Then the following is an order relation on the product $(F, A) \times (G, B)$ called the product order

$$\{F(a), F(b)\} \preceq \{G(a'), G(b')\} \text{ if } \{F(a) \preceq G(a')\} \text{ and } \{F(b) \preceq G(b')\}$$

Example 6 Suppose (F, A) and (G, B) are ordered soft set. We show that the product order $(F, A) \times (G, B)$, defined by $\{F(a_1), F(a_2)\} \preceq \{G(b_1), G(b_2)\}$ if $F(a_1) \subseteq G(b_1)$ and $F(a_2) \subseteq G(b_2)$ is a partial ordering of $(F, A) \times (G, B)$.

Goal: we show that \preceq is (a) reflexive (b) antisymmetric (c) transitive

(a) Since $F(a) = F(a)$ and $G(b) = G(b)$, we have $F(a) \subseteq F(a)$ and $G(b) \subseteq G(b)$. Hence $\{F(a_1), F(a_2)\} \subseteq \{G(b_1), G(b_2)\}$ and \preceq is reflexive.

(b) Suppose $\{F(a_1), F(a_2)\} \subseteq \{G(b_1), G(b_2)\}$ and $\{G(b_1), G(b_2)\} \subseteq \{F(a_1), F(a_2)\}$ then $\{F(a_1) \subseteq G(b_1)\}$ and $\{F(a_2) \subseteq G(b_2)\}$, also $\{G(b_1) \subseteq F(a_1)\}$ and $\{G(b_2) \subseteq F(a_2)\}$. Thus, $F(a_1) = G(b_1)$ and $F(a_2) = G(b_2)$. Hence, $\{F(a_1), F(a_2)\} = \{G(b_1), G(b_2)\}$ and is antisymmetric.

(c) Suppose $\{F(a_1), F(a_2)\} \preceq \{G(b_1), G(b_2)\}$ and $\{G(b_1), G(b_2)\} \preceq \{H(c_1), H(c_2)\}$ by transitivity, we have $\{F(a_1), F(a_2)\} \preceq \{H(c_1), H(c_2)\}$, hence the proof.

(ii) Lexicographic Order: Suppose (F, A) and (G, B) are linearly ordered soft set. Then the following is an order relation on the product set $(F, A) \times (G, B)$, called the lexicographic or dictionary order.

$$\{F(a_1), F(a_2)\} \preceq \{G(b_1), G(b_2)\} \begin{cases} \text{if } F(a_1) < G(b_1) \\ \text{or if } F(a_1) = G(b_1) \text{ and } F(a_2) < G(b_2). \end{cases}$$

Thus order can be extended to $(F_1, A_1), (F_1, A_1), \dots, (F_k, A_k)$ as follows

$$\{F(a_1), F(a_2), \dots, F(a_n) < G(b_1), G(b_2), \dots, G(b_n)\}$$

$$\text{If } F(a_1) = G(b_1), F(a_2) = G(b_2), \dots, F(a_{k-1}) = G(b_{k-1}) \text{ but } F(a_k) < G(b_k)$$

NB: Lexicographical order is also linear

6. Conclusion

Soft set has vital applications in many areas as highlighted in [1] and [2]. Partial ordering also plays an important role in mathematics hence developing partial ordering relation in soft set context is of immense benefit.

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