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The Dynamics of the Fixed Points to Duffing Map

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Abstract
We study the fixed points of Duffing map and the general properties of them , we find the contracting and expanding area of this map .So, we divide the parameter space for many regions to determinate the fixed points attracting , repelling or saddle . Also we determinate the bifurcation points of the parameter space .
Keywords: Duffing map , fixed point , bifurcation point , attracting- expanding area.

1. Introduction

Chaos Theory is a synonym for dynamical systems theory, a branch of mathematics. Dynamical systems come in three flavors: flows (continuous dynamical systems), cascades (discrete, reversible, dynamical systems), and semi-cascades (discrete, irreversible, dynamical systems).The concept of a dynamical system has its origins in Newtonian mechanics. There, as in other natural sciences and engineering disciplines, the evolution rule of dynamical systems is given implicitly by a relation that gives the state of the system only a short time into the future.In a dynamical system, a bifurcation is a period doubling, quadrupling, etc., that accompanies the onset of chaos. It represents the sudden appearance of a qualitatively different solution for a nonlinear system as some parameter is varied. Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behaviour. Bifurcations occur in both continuous systems (described by ordinary differential equations , delay differential equations, or partial differential equations), and discrete systems (described by maps). The name "bifurcation" was first introduced by Henri Poincaré in 1885 in the first paper in mathematics showing such a behavior. Henri Poincaré also later named various types of stationary points and classified them.

The Duffing map is a 2 – D discrete dynamical system , its form is $F_{a,b} = \begin{pmatrix} y \\ -bx + ay - y^3 \end{pmatrix}$. It has fractal attractor studied by Paul Bourke 1998 .Some literatures studied its chaotic properties , Mishra and Mankar [3] constructed encryption method using Duffing map, this method is desined using message embedded scheme and is analyzed for its validity, for plaintext sensitivity , key sensitivity, known plaintext and brute-force attacks. Tarai and Khan[2] applied backstepping method for projective synchronization of Duffing map ,in this case the error goes to zero through oscillation , the numerical simulation results indicate that their approach works very well .In this research we show that the Duffing map is diffeomorphism map and it has three fixed points , in our research we divide the parameter space into six regions to determinate types of the fixed points, and to find the bifurcation parameter points .

2. Preliminaries

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a map .We say F is C^∞, if its mixed k-th partial derivatives exist and are continuous for all $k \in \mathbb{Z}_+$, and it is called a diffeomorphism if it is one-to-one, onto, C^∞ and its inverse is C^∞. Let V be a subset of $\mathbb{R}^2$, and v_0 be any element in $\mathbb{R}^2$ .Consider $F : V \to \mathbb{R}^2$ be a map. Furthermore assume that the first partials of the coordinate maps f and g of F exist at v_0 .The differential of F at v_0 is the linear map $DF(v_0)$ defined on $\mathbb{R}^2$ by $DF(v_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(v_0) & \frac{\partial f}{\partial y}(v_0) \\ \frac{\partial g}{\partial x}(v_0) & \frac{\partial g}{\partial y}(v_0) \end{pmatrix}$, for all v_0 in $\mathbb{R}^2$ .The determinant of $DF(v_0)$ is called the Jacobian of F at v_0 and it is denoted by $JF(v_0) = \det DF(v_0)$, so F is said to be area-contracting at v_0 if $|\det DF(v_0)| < 1$, F is said to be area-expanding at v_0 if $|\det DF(v_0)| > 1$. Let A be an $n \times n$ matrix .The real number λ is called eigenvalue of A if there exists a non zero vector X in $\mathbb{R}^n$ such that $AX = \lambda X$, every non zero vector X satisfying this equation is called an eigen vector of A associated with the eigenvalue λ .The point $\begin{pmatrix} x \\ y \end{pmatrix}$ is called fixed point if $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, it is an attracting fixed point, the eigenvalues of DF at this point if λ_1 and λ_2 are less than one in absolute value , and it is repelling fixed point, if λ_1 and λ_2 are less than one in absolute value . $A \in GL(2,\mathbb{Z})$ with $\det(A) = \pm 1$, is called hyperbolic matrix if $|\lambda_i| \neq 1$ where λ_i are the eigenvalues of A .Gulick [3] defined the bifurcation point at one dimension, we generalized this definition in two dimensions. A parameterized family $F_{a,b}$ has a bifurcation of parameter $\binom{a'}{b'}$ if the number or nature (attracting vs. repelling) of periodic point of the family changes as the parameter a or b passes through $\binom{a'}{b'}$. In this case, $\binom{a'}{b'}$ is said to be a bifurcation point .We denote A^{-1} to the inverse of A.

3.The General Properties of Duffing Map

In this section , we find the fixed point and we study the general properties of Duffing map , for example : $F_{a,b}$ is one to one , onto , C^{∞} and invertible so it is diffeomorphism, and we find the parameter which $F_{a,b}$ has area contracting or expanding .

Proposition 3-1 :

Let $F_{a,b}: \mathbb{R}^2 \to \mathbb{R}^2$ be a Duffing map , if $a > b+1$ then $F_{a,b}$ has three fixed points : $\binom{0}{0}$, $\binom{\sqrt{a-b-1}}{\sqrt{a-b-1}}$ and $\binom{-\sqrt{a-b-1}}{-\sqrt{a-b-1}}$.

Proof : By definition of fixed point , $F_{a,b}\binom{x}{y} = \binom{y}{-bx+ay-y^3} = \binom{x}{y}$ then $x = y$ and $y = -bx + ay - y^3$, so $x = -bx + ax - x^3$, therefore $x(-b + a - x^2 - 1) = 0$ then $P^0 = \binom{0}{0}$ is the first fixed point , and $-b + a - x^2 - 1 = 0$ that is $x^2 = a - b - 1$,then $x = \pm\sqrt{a-b-1}$ and $y = \pm\sqrt{a-b-1}$ therefore $P^+ = \binom{\sqrt{a-b-1}}{\sqrt{a-b-1}}$ …….(3.1) and $P^- = \binom{-\sqrt{a-b-1}}{-\sqrt{a-b-1}}$ ……(3.2) are the second and third fixed points . ❒

Proposition 3-2 :

Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be the Duffing map then the Jacobian of $F_{a,b}$ is b .

Proof : $DF_{a,b}\binom{x}{y} = \begin{pmatrix} 0 & 1 \\ -b & a-3y^2 \end{pmatrix}$, so $JF_{a,b}\binom{x}{y} = det\begin{pmatrix} 0 & 1 \\ -b & a-3y^2 \end{pmatrix} = b$. ❒

Proposition 3-3 :

The eigenvalues of $DF_{a,b}\binom{x}{y}$ are $\lambda_{1,2} = \frac{-(3y^2-a)\pm\sqrt{(3y^2-a)^2-4b}}{2}$; $\forall \binom{x}{y} \in \mathbb{R}^2$.

Proof : If λ is eigen value of $DF_{a,b}\binom{x}{y}$ then must be satisfied the characteristic equation $\begin{vmatrix} \lambda & -1 \\ b & \lambda - a + 3y^2 \end{vmatrix} = 0$ then $\lambda^2 - a\lambda + 3\lambda y^2 + b = 0$ and the solutions of this equation are $\lambda_{1,2}$ where $\lambda_{1,2} = \frac{-(3y^2-a)\pm\sqrt{(3y^2-a)^2-4b}}{2}$. ❒

Remark :

1. It is clear that if $|3y^2 - a| > 2\sqrt{b}$,$b > 0$ then the eigenvalues of $DF_{a,b}\binom{x}{y}$ are real .
2. The eigenvalues of $DF_{a,b}\binom{x}{y}$ at fixed points are $\lambda^0 = \frac{a\pm\sqrt{a^2-4b}}{2}$ for fixed point P^0 and $\lambda_{1,2}^{+} = \frac{-(2a-3b-3)\pm\sqrt{(2a-3b-3)^2-4b}}{2}$ for fixed point P^+ and $\lambda_{1,2}^{-} = \frac{-(2a-3b-3)\pm\sqrt{(2a-3b-3)^2-4b}}{2}$ for fixed points P^- .

Proposition 3-4 :

Let $F_{a,b}: \mathbb{R}^2 \to \mathbb{R}^2$ be a Duffing map , If $b \neq 0$ then $F_{a,b}$ is diffeomorphism .

Proof : To prove $F_{a,b}$is diffeomorphism , we must prove :

1) $F_{a,b}$ is one-to-one map : Let $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in \mathbb{R}^2$ such that $F_{a,b}\binom{x_1}{y_1} = F_{a,b}\binom{x_2}{y_2}$ then $\binom{y_1}{-bx_1+ay_1-{y_1}^3} = \binom{y_2}{-bx_2+ay_2-{y_2}^3}$ so $y_1 = y_2$ and $-bx_1 + ay_1 - {y_1}^3 = -bx_2 + ay_2 - {y_2}^3$ hence $-bx_1 = -bx_2$ and $b \neq 0$ so $x_1 = x_2$, that is , $F_{a,b}$ is one-to-one.

2) $F_{a,b}$ is $\mathbb{C}^{\infty}$: $F\binom{x}{y} = \binom{y}{-bx+ay-y^3}$ then all first partial derivatives exist and continuous. Note that $\frac{\partial^n f(x,y)}{\partial x^n} = 0\ \forall n \in \mathbb{N}$, and $\frac{\partial^n f(x,y)}{\partial y^n} = 0$, $\frac{\partial^n g(x,y)}{\partial x^n} = 0$, $\forall n \geq 2$, and $\frac{\partial^n g(x,y)}{\partial y^n} = 0$, $\forall n \geq 4$ .We get that all its $F_{a,b}$ ixed k-th partial derivatives exist and continuous for all k. From definition of diffeomorfism , $F_{a,b}$ is $\mathbb{C}^{\infty}$.

3) $F_{a,b}$ is onto : Let $\binom{v}{w}$ any element in $\mathbb{R}^2$ such that $y = v$ and $x = \frac{w+v^3-av}{-b}$. Then there exist $\begin{pmatrix} \frac{w+v^3-av}{-b} \\ v \end{pmatrix} \in \mathbb{R}^2 \ni F_{a,b}\binom{x}{y} = \binom{v}{w}$ .Then $F_{a,b}$ is onto .

4) $F_{a,b}$ has an inverse : Let $F_{a,b}^{-1}\binom{v}{w} = \begin{pmatrix} \frac{w+v^3-av}{-b} \\ v \end{pmatrix}$ such that $F_{a,b} \circ F_{a,b}^{-1}\binom{v}{w} = F_{a,b}^{-1} \circ F_{a,b}\binom{v}{w} = \binom{v}{w}$. Note that , $F_{a,b}^{-1} \circ F_{a,b}\binom{v}{w} = F_{a,b}^{-1}\begin{pmatrix} w \\ -bv + aw - w^3 \end{pmatrix} = \begin{pmatrix} \frac{-bv+aw-w^3+w^3-aw}{-b} \\ w \end{pmatrix} = \binom{v}{w}$, and $F_{a,b} \circ F_{a,b}^{-1}\binom{v}{w} = F_{a,b}\begin{pmatrix} \frac{w+v^3-av}{-b} \\ v \end{pmatrix} = \begin{pmatrix} v \\ -b\left(\frac{w+v^3-av}{-b}\right) + av - v^3 \end{pmatrix} = \binom{v}{w}$. Then $F_{a,b}$ has an inverse and it is invertible . ❒

It is clear that if $\lambda_1\lambda_2 = b \neq 1$ or -1 then $DF_{a,b}\binom{x}{y}$ is a hyperbolic matrix , the question what about $\lambda_1\lambda_2 = 1$, these proposition answer this question .

Remark 3-5 : If $b = 0$ then $F_{a,0} = \begin{pmatrix} y \\ ay - y^3 \end{pmatrix}$, so $ker(F_{a,b}) = \left\{\binom{x}{y} : x \in \mathbb{R}\right\}$ then $F_{a,0}$ is not one to one, that is , $F_{a,0}$ is not diffeomorphism .

Proposition 3-6 :

1. $DF_{a,b}\binom{x}{y}$ is non hyperbolic matrix if $y = \pm\sqrt{\frac{a-2}{3}}$ and $b = 1$. Also , it is non hyperbolic matrix if $y = \pm\sqrt{\frac{a}{3}}$ and $b = -1$.
2. If $b = -1$ and $a = 2b + 2$ then one of eigenvalues for every fixed point of $F_{a,b}$ is -1 .

Proof : 1. let $y = \pm\sqrt{\frac{a-2}{3}}$ then $3y^2 = a - 2$ and $4 + 4(3y^2 - a) + (3y^2 - a)^2 = (3y^2 - a)^2 - 4$ hence $\left|\frac{-(3y^2-a)\pm\sqrt{(3y^2-a)^2-4}}{2}\right| = 1$ so $\lambda_{1,2} = 1$. By follow the same steps we get $DF_{a,b}\binom{x}{y}$ is non hyperbolic matrix if $y = \pm\sqrt{\frac{a}{3}}$ and $b = -1$.

2. since $\lambda^+ = \frac{-(2a-3b-3)\pm\sqrt{(2a-3b-3)^2-4b}}{2}$ and $a = 2b + 2$ then $2\lambda^+ = -(b+1) \pm (b-1)$ hence $\lambda_1^+ = -1$, also $\lambda_1^- = -1$, and we have $\lambda^0 = \frac{a\pm\sqrt{a^2+4}}{2}$ then $\lambda_2^0 = b + 1 - \sqrt{b^2+b+1}$,since $b = -1$ so $\lambda_2^0 = -1$.❒

Proposition 3-7 :

For all $\binom{x}{y} \in \mathbb{R}^2$, $y \neq \pm\sqrt{\frac{a-2}{3}}$ and $y \neq \pm\sqrt{\frac{a}{3}}$, and $|3y^2 - a| > 2\sqrt{b}$, $DF_{a,b}\binom{x}{y}$ is a hyperbolic matrix if and only if $|b| = 1$.

Proof : ⇒) since $D\,F_{a,b}\binom{x}{y} \in GL(2,\mathbb{Z})$ and $det\left(D\,F_{a,b}\binom{x}{y}\right) = |b|$, then $b = 1$ or -1 .

⇐) Let $b = 1$. Then since $\det\left(D\,F_{a,b}\binom{x}{y}\right) = b$ we have $\lambda_1\lambda_2 = b$, that is $\lambda_1\lambda_2 = 1$.

Then $\lambda_1 = \frac{1}{\lambda_2}$. If $\lambda_1 > 1$ therefore $\lambda_2 < 1$, or $\lambda_1 < 1$ so $\lambda_2 > 1$.

So $D\,F_{a,b}\binom{x}{y}$ is a hyperbolic matrix . ❒

Proposition 3-8 :

If $|b| < 1$ then $F_{a,b}$ is area-contracting and its area-expanding if $|b| > 1$.

Proof: If $|b| < 1$ then , by proposition 3-2 , $\left|det\left(D\,F_{a,b}\binom{x}{y}\right)\right| < 1$ that is $F_{a,b}\binom{x}{y}$ is area-contracting map and if $|b| > 1$ then $\left|det\left(D\,F_{a,b}\binom{x}{y}\right)\right| > 1$ this implies that $F_{a,b}\binom{x}{y}$ is an area-expanding map. ❒

4.The Fixed Point's Properties of Duffing Map

we divide the parameter space into six regions to determinate types of the fixed points such that where $|b| < 1$ and $b + 1 < a < 2b + 2$ two of these fixed point are attracting and the other one is repelling ,so all of fixed points are repelling if $|b| > 1$ and $a < 2b + 2$ or $a > 2b + 2$, and all of them are saddle where $|b| = 1$. Finally , if $a > 2b + 2$ the fixed points are repelling were $0 \le b < 1$ and two are repelling and one is saddle were $-1 < b < 0$.

Proposition 4-1 : The Duffing map has only one fixed point $\binom{0}{0}$ if $a \le b + 1$ $\forall b \in \mathbb{R}$. This fixed point is attracting if $|b| \le 1$ and saddle if $|b| > 1$.

Proof : by (3.1) and (3.2) if $a \le b + 1$ then $P^+, P^- \notin \mathbb{R}^2$, that mean the Duffing map has one fixed point $\binom{0}{0}$. So , if $a = b + 1$ the equation $\sqrt{a-b-1} = 0$, then Duffing map has one fixed point $\binom{0}{0}$. To show type of this point , since we have $\lambda^0_{1,2} = \frac{a\pm\sqrt{a^2-4b}}{2}$ and $a \le b + 1$ then $\lambda^0_{1,2} < \frac{b+1\pm(b-1)}{2}$ so $\lambda^0_1 < b$ and $\lambda^0_2 < 1$. Then $\binom{0}{0}$ is attracting if $|b| \le 1$ and saddle if $|b| > 1$ ❒

From Proposition 4-1 we conclude that if $a \le b + 1$ then $b = 1$ is bifurcation point , since the map has attracting fixed point $|b| \le 1$ and saddle if $|b| > 1$.

Proposition 4-2 :

If $(3y^2 - a)^2 - 4b \ge 0$ then the region $|b| < 1$, $b + 1 < a < 2b + 2$ get $F_{a,b}$ has two attracting fixed points P^+ and P^- , and has one repelling fixed point P^0 .

Proof :

since $b + 1 < a$ so $-a + b + 1 < 0$,therefore $-a + (b + 1) < b + 1 < a$, then we get $-a + b < b < a - 1$ and $-2a + 2b < 2b < 2a - 2$ then $-2a - b < -b < 2a - 2 - 3b$, so $-(2a - 3b - 3) + 1 - 4b - 4 < -b < (2a - 3b - 3) + 1$

Therefore $-(2a - 3b - 3) + 1 < -b < (2a - 3b - 3) + 1$ and $-4(2a - 3b - 3) + 4 < -4b < 4(2a - 3b - 3) + 4$, by adding $(2a - 3b - 3)^2$ for both sides we get

$[(2a - 3b - 3) - 2]^2 < (2a - 3b - 3)^2 - 4b < [(2a - 3b - 3) + 2]^2$, we have

$-2 < (2a - 3b - 3) - 2 < \pm\sqrt{(2a - 3b - 3)^2 - 4b} < (2a - 3b - 3) + 2 < 2$ then

$-1 < \frac{-(2a-3b-3)\pm\sqrt{(2a-3b-3)^2-4b}}{2} < 1$, from proposition 2 $|\lambda^\pm| < 1$, then P^+ and P^- are attracting fixed points . To show that P^0 is a repelling fixed point , since $b + 1 < a$ we have $-a < -(b + 1)$ so $-a + 2(b + 1) < b + 1$ therefore $-a < b + 1 < a$ that is $4a + 4 < -4b < 4 - 4a$, by adding for both sides we get $(-a - 2)^2 < a^2 - 4b < (2 - a)^2$ $\Rightarrow -a - 2 < \pm\sqrt{a^2 - 4b} < 2 - a$ by adding a for both sides we have $-2 < a \pm \sqrt{a^2 - 4b} < 2$. That is $-1 < \frac{a\pm\sqrt{a^2-4b}}{2} < 1$, so $|\lambda^0| < 1$. ❒

From Propositions 4-1 and 4-2 , we conclude that if $|b| \le 1$ then $a = b + 1$ is bifurcation point , since the map has one fixed point if $a < b + 1$ and three fixed points if $a > b + 1$.

Proposition 4-3 :

If $|b| > 1$, $b + 1 < a < 2b + 2$ and $(3y^2 - a)^2 - 4b \ge 0$ then the fixed points of $F_{a,b}$ are repelling .

Proof : let $\lambda^\pm = \frac{-(2a-3b-3)\pm\sqrt{(2a-3b-3)^2-4b}}{2}$, since $a > b + 1$ then $2\lambda^\pm > b + 1 \pm \sqrt{(b-1)^2}$ that is $\lambda^\pm > \frac{b+1\pm(b-1)}{2}$ so $\lambda^\pm > 1$. Then P^+ and P^- are repelling fixed points.

To show that P^0 is a repelling fixed point , we have $\lambda^0 = \frac{a\pm\sqrt{a^2-4b}}{2}$ and $a > b + 1$ then $2\lambda^0 > b + 1 \pm \sqrt{(b+1)^2 - 4b}$, by follows the above proof then we get $\lambda^0 > 1$.❒

From Propositions 4-2 and 4-3 , we conclude that if $b + 1 < a < 2b + 2$ then $b = 1$ is bifurcation point , since the map has two attracting fixed points and one repelling if $|b| < 1$ and all of them are repelling if $|b| > 1$.

Proposition 4-4 :

If $|b| > 1$, $a > 2b + 2$ and $(3y^2 - a)^2 - 4b \ge 0$ then the fixed points of $F_{a,b}$ are repelling .

Proof : since $\lambda^\pm = \frac{-(2a-3b-3)\pm\sqrt{(2a-3b-3)^2-4b}}{2}$ and $a > b + 1$ then we get

$2\lambda^\pm > -b - 1 \pm \sqrt{[-(b+1)]^2 - 4b}$ so $\lambda^\pm > \frac{-b-1\pm(b-1)}{2}$, since $|b| > 1$ then $\lambda^\pm > 1$ hence P^+ is repelling fixed point . by follows the above proof then we get P^- is repelling fixed point . Now , $\lambda^0 = \frac{a\pm\sqrt{a^2-4b}}{2}$ and $a > 2b + 2$ then $2\lambda^0 > 2(b + 1) \pm \sqrt{4(b^2 + b + 1)}$, hence $\lambda^0_1 > b + 1 + \sqrt{(b-1)^2 + 3b}$ since $|b| > 1$, then $\lambda^0_1 > b + 1 \pm (b - 1)$ so $\lambda^0_1 > 1$ then we have $|\lambda^0_1| > 1$. also $\lambda^0_2 > b + 1 - \sqrt{(b+2)^2 - 5(b+1)}$ then $\lambda^0_2 > b + 1 - (b + 2)$ that is $\lambda^0_2 > -1$ so $|\lambda^0_2| > 1$, then we get P^0 is a repelling fixed point .❒

Proposition 4-5 :
If $b=-1$ and $a>0$ then the fixed points of $F_{a,b}$ are saddle .
Proof :
By hypothesis $P^+=\begin{pmatrix}\sqrt{a}\\ \sqrt{a}\end{pmatrix}$ and $P^-=\begin{pmatrix}-\sqrt{a}\\ -\sqrt{a}\end{pmatrix}$ so $\lambda^\pm=a\pm\sqrt{a^2+1}$. It is clear , $a>0$ and $a^2+1>0$ hence $\lambda^\pm$ are distinct real number .
Let $|\lambda^\pm|\ge 1$ so $|a\pm\sqrt{a^2+1}|\ge 1$, since $\sqrt{a^2+1}>\sqrt{a^2}$ $\forall a>0$ then $\sqrt{a^2+1}-a\ge 1$ so $a^2+1\ge 1+2a+a^2$ that is $2a\le 0$, this contradiction with $a>0$, hence $|\lambda_1^+|<1$ and we have $\lambda_1^+\lambda_2^+=b$ then $|\lambda_1^+\lambda_2^+|=1$ so $|\lambda_2^+|=\frac{1}{|\lambda_1^+|}$ but $|\lambda_1^+|<1$ then $|\lambda_2^+|>1$ that mean P^+ is saddle fixed point , so as P^- . To show that P^0 is a saddle fixed point , since $a>0$ then $a^2+4>4-4a+a^2$ and $\sqrt{a^2+4}>2-a$ so $\frac{a+\sqrt{a^2+4}}{2}>1$ that is $\lambda_1^0>1$. Also we have $\lambda_1^0\lambda_2^0=b$ then $|\lambda_2^0|<1$, hence P^0 is a saddle fixed point .❒

Proposition 4-6 :
If $b=1$ and $a>2$ then the fixed points of $F_{a,b}$ are saddle .
Proof : By hypothesis $P^+=\begin{pmatrix}\sqrt{a-2}\\ \sqrt{a-2}\end{pmatrix}$ and $P^-=\begin{pmatrix}-\sqrt{a-2}\\ -\sqrt{a-2}\end{pmatrix}$ so $\lambda^\pm=-(a-3)\pm\sqrt{a^2-6a+8}$ and $\lambda^0=\frac{a\pm\sqrt{a^2-4}}{2}$. since $a>2$ we get $2a>4$ and $(2-a)>a^2-6a+8$ so $2-a-\sqrt{a^2-6a+8}>0$ that is $-(a-3)-\sqrt{a^2-6a+8}>1$ then $\lambda_2^+>1$. Since $\lambda_1^+\lambda_2^+=1$ so $|\lambda_1^+|=\frac{1}{|\lambda_2^+|}$ then $|\lambda_1^+|<1$ that mean P^+ is saddle fixed point , so as P^- . Now by follows the same above steps we get $\lambda_1^0>1$ and $|\lambda_2^0|<1$ then P^0 is a saddle fixed point .❒

Proposition 4-7 : If $|b|<1$ and $a>2b+2$ then :
1. The fixed points of $F_{a,b}$ are repelling if $0\le b<1$.
2. Two fixed points P^+,P^- of $F_{a,b}$ are saddle and P^0 is repelling if $-1<b<0$

Proof : since $\lambda^+=\frac{-(2a-3b-3)\pm\sqrt{(2a-3b-3)^2-4b}}{2}$ and $a>2b+2$ then $2\lambda^+>-(b+1)\pm\sqrt{(b+1)^2-4b}$,so $\lambda_1^+>\frac{-2}{2}$ hence $|\lambda_1^+|>1$ for all b , and $2\lambda_2^+>-(b+1)-\sqrt{(b+1)^2-4b}$,since $-\sqrt{(b+1)^2}<-\sqrt{(b+1)^2-4b}$ then $\lambda_2^+>-(b+1)$. Now if $0\le b<1$ we get $|\lambda_2^+|>1$ and P^+ is repelling fixed point , so as P^+ . If $-1<b<0$ we get $0<-b-1-b+1<2$ so $0<\frac{-(b+1)-\sqrt{(b+1)^2-4b}}{2}<1$ then $|\lambda_2^+|<1$ P^+ is saddle fixed point , so as P^+ .To show P^0 is repelling fixed point if for all b , since we have $2\lambda^0>a\pm\sqrt{a^2-4b}$ then $\lambda_2^0>(b+1)-\sqrt{b^2+b+1}$ so $\lambda_2^0>b+1-\sqrt{(b+2)^2-5(b+1)}$ that is $\lambda_2^0>b+1-\sqrt{(b+2)^2}$ hence $|\lambda_2^0|>1$, by hypothesis $\lambda_1^0>\lambda_2^0$ then $|\lambda_1^0|>1$.❒

From Proposition 4-7 , we conclude that if $a>2b+2$ then $b=0$ is bifurcation point , since the map has two saddle fixed points and one repelling if $b<0$ and all of them are repelling if $b>0$.

References
[1] Gulick D., "Encounters with Chaos", McGraw Hill, Inc, USA, 1992 .
[2] Tarai A. and Khan M. A. , "Projective Synchronization of Chaotic Systems Via Backstepping Design ", International Journal of Applied Mathematical Research , 4(2012),531-540.
[3] Misra M. and Mankar V. H. ,"Message Embedded Cipher Using 2-D Chaotic Map ",International Journal of Chaos ,Control, Modelling and simulation (IJCCMS), 1(2012),13-24 .