

Iterative Algorithms for Nonlinear Equations and Dynamical Behaviors: Applications

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Abstract: Numerical iteration methods for solving the roots of nonlinear transcendental or algebraic model equations (in 1D, 2D or 3D) are useful in most applied sciences (Biology, physics, mathematics, Chemistry…) and in engineering, for example, problems of beam deflections. This article presents new iterative algorithms for finding roots of nonlinear equations applying some fixed point transformation and interpolation. A method for solving nonlinear systems (in higher dimensions, for multi-variables) is also considered. Our main focus is on methods not involving the equation f(x) in problem and or its derivatives. These new algorithm can be considered as the acceleration convergence of several existing methods. For convergence and efficiency proofs and applications, we solve deflection of a beam differential equation and some test experiments in in Matlab. Different (real & complex) dynamical (convergence plane) analyzes are also shown graphically.

Keywords: nonlinear equations, deflection of beam, iterations, dynamical analysis, applications, 2D

1. Introduction

The research work in nonlinear root finding [of f(x)] is not limited to algebraic real polynomial equations in 1D, 2D or 3D. The function f(x) can be any scalar (numerical) function or transcendental function [trigonometric, logarithmic, exponential, rational form or mixtures of such models…]. The functions may appear as a solution of model problems such as boundary value problems of beam differential equation in mechanical engineering.

In the literatures, there are several applications of root solving in science and engineering. Douglas [22] analyzed (stability of) the problem of simple control system design for the isothermal continuously stirred tank reactor (CSTR)), using nonlinear transfer function with proportionality K_c to compute the roots (eigenvalues), s. Where, the types of roots ‘s’ (positive, negative, real or complex) determine the stability conditions. And the Soave-Redlich-Kwong [12, 17, 22] Equation of State to calculate the specific volume **V** of a pure gas, at a given temperature **T** and pressure **P** for constants **a** and **b** depending on temperature and pressure of a particular gas (in Thermodynamics & Chemical Engineering) also involves nonlinear root finding.

In [23], an iterative method based on Newton’s method was used (by Raposo-Pulido & Peláez) to solve the Kepler equation for hyperbolic (HKE) orbits to obtain the value of the hyperbolic anomaly **H** in terms of the mean anomaly **M** and the eccentricity **e**. HKE is most important in the fields of astronomy, celestial mechanics, and astrodynamics. B., Kalantari presented the idea

"Polynomiography" with its applications in arts and sciences. The root finding methods also help us gain more insights and complete geometry of graph of a function and intersection or break points and critical points of curves (models), for instance, in business activities.

The fixed point iteration method FPIM (successive approximation) is one of the numerical techniques. One advantage of this FPIM is that only a single initial guess 'x_0' is required to start the iteration process to approximate a unique solution. The other advantage is that the method does not need calculation of the function and its derivative and is simple to use. The most popular Newton method, Chebyshev's method and Halley's method are basic examples of fixed point methods, from the representations viewpoints of their formula [1-20]. But all involve calculation of the function f(x) and its derivatives.

Here, we present new iterative formula for finding roots of nonlinear equations (for problems in engineering) using some transformation and with a possible extension for solving nonlinear systems (in higher dimension). The methods do not involve the equation f(x) in problem and or its derivatives. These new methods may be considered as the acceleration convergence of the Aitken's Δ^2-method, which is known to accelerate the successive iterations of fixed point method FPIM.

1.2 Materials and methods

We consider a fixed point method. A given fixed point function (FPE) transformation of an equation f(x) and Taylor's theorem in linear interpolation with possible extension for reformulation (to solve systems of equations) in higher dimension is used. For this, a theorem is defined. The method is implemented in matlab with a given initial guess x_0.

2. Main Results

Theorem 2.1: For a fixed point function FPE g(x) of f(x) = 0, the formula in (1a) below is an iterative method.

$$p(x_n) = x(n) + [g(x_n)-x(n)]/[1-g'(x_n)] \quad (1a)$$

And the iterative formula (1a) can be reformulated for solving the system F(x,y) = 0 via its fixed point function G(x.y) using the iterative method in (1b) below (in 2D)

$$\begin{aligned} &\Delta(x_i, y_i)^t = -G'^{-1}{}_i * X_i + G'^{-1}{}_i * g_i, \\ &X_i = (x_i, y_i)^t, g_i = (g_1(x_i, y_i), g_2(x_i, y_i))^t \end{aligned} \quad (1b)$$

The iteration process in 3D is given by

$$\begin{aligned} &\Delta(x_i, y_i, z_i)^t = -G'^{-1}{}_i * X_i + G'^{-1}{}_i * g_i, X_i = (x_i, y_i, z_i)^t, \\ &g_i = (g_1(x_i, y_i, z_i), g_2(x_i, y_i, z_i), g_i(x_i, y_i, z_i))^t. \end{aligned} \quad (1c)$$

Proof: Since it is more general, let's show the formula for the higher dimension. Consider the Taylor's series of a function of two variables f = f(X) = f(x,y) at (x_0,y_0),

$$f(x_0+h, y_0+k) = \sum_{i=0}^{n} \frac{1}{i!}\left(h\frac{\partial^i}{\partial x^i} + k\frac{\partial^i}{\partial y^i}\right) f(x_0, y_0) + R_n \tag{1d}$$

$$\left(h\frac{\partial^m}{\partial x^m} + k\frac{\partial^m}{\partial y^m}\right) f(x_0, y_0) = \sum_{i=0}^{m} \binom{m}{i} h^i k^{m-i} \frac{\partial^m f(x_0, y_0)}{\partial x^i \partial y^{m-i}}; \Delta x = h, \Delta y = k$$

R_n is the remainder error of the series estimation.
Linearization of G(x,y)=X- P(X) through Taylor's expansion near initial guess $X_0 = (x_0,y_0)^t$ and a FPE P(X) = (g1(X), g2(X)) of the system F(X) = F(x,y) using (1d) gives $\Delta(x,y)^t = -G'^{-1} * G_0$. With G'^{-1} is inverse of the Jacobian of G , and $G_0 = (x_0\text{-}g_10,\ y_0\text{-}g_20)^t$, i.e., $G' = \begin{bmatrix} 1-g_{1x} & g_{1y} \\ g_{2x} & 1-g_{2y} \end{bmatrix}$

We may rewrite this method as

$$\begin{aligned} &\Delta(x,y)^t = -G'^{-1} * X_0 + G'^{-1} * g_0, \\ &X_0 = (x_0, y_0)^t, g_0 = (g_1(x_0, y_0), g_2(x_0, y_0))^t \end{aligned} \tag{1e}$$

The iteration process in 2D is then given by

$$\begin{aligned} &\Delta(x_i, y_i)^t = -G'^{-1}{}_i * X_i + G'^{-1}{}_i * g_i, \\ &X_i = (x_i, y_i)^t, g_i = (g_1(x_i, y_i), g_2(x_i, y_i))^t \end{aligned} \tag{1f}$$

We can observe that the iterative process does not depend on F(x,y) & F'(x,y), which is a success, when F' is an ill-conditioned. Notice also that (1f) is an extension of (1a). Hence proof of (1a) in 1D is clear. For that, we let f(x) = x-g(x) and apply linearization, where g(x) is a FPE of f(x). Some necessary conditions for (1f) to hold are:

(A) $|g_{1x}| + |g_{2y}| < 1$, or $|g_{1x} g_{2y}| < 1$,

(B) 1-[$g_{1x} + g_{2y} + g_{2x}g_{1y} - g_{2y}g_{1x}$] $\neq 0$.

The iteration process in 3D is given by

$$\Delta(x_i, y_i, z_i)^t = -G'^{-1}{}_i * X_i + G'^{-1}{}_i * g_i, X_i = (x_i, y_i, z_i)^t,$$
$$g_i = (g_1(x_i, y_i, z_i), g_2(x_i, y_i, z_i), g_i(x_i, y_i, z_i))^t.$$

Now we shall show convergence order p of (1a).

Proof: Applying Taylor's series at x = x_n, error of computation $e_n = |x_n - x_{n-1}|$

$$g(x_n+e_n) = g(x_n)+g'(x_n)e_n+0.5g''(x_n)e_n^2\ldots(*)$$
$$g'(x_n+e_n) = g'(x_n)+g''(x_n)e_n+g'''(x_n)e_n^2\ldots(**)$$
$$x_{n+1} = x_n-(g(x_n)-x_n)/(g'(x_n)-1) \qquad (a*)$$

Using (*) & (**) in (a*) we obtain

$$e_{n+1} = -(x_n-g(x_n+e_n))/(1-g'(x_n+e_n)),$$
$$e_{n+1} = ce_n^p \qquad (b*)$$

And p > = 2. The method is at least quadratically convergent.

Here are several higher order fixed point methods (containing f(x) and its derivatives) for solving f(x) = 0.

$$x_{n+1} = \varphi(x_n) = x_n - 3\frac{2f(x_n)[f'(x_n)]^2 + [f(x_n)]^2 f''(x_n)}{6[f'(x_n)]^3 + [f(x_n)]^2 f'''(x_n)}. \quad (2)$$

$$x_{n+1} = \xi(x_n) = x_n - 3\frac{2f(x_n)[f'(x_n)]^2 + [f(x_n)]^2 f''(x_n)}{6[f'(x_n)]^3 + \frac{[f(x_n)f''(x_n)]^2}{f'(x_n)}}. \quad (3)$$

$$x_{n+1} = \theta(x_n) = x_n - 3\frac{2f(x_n)[f'(x_n)]^2 + [f(x_n)]^2 f''(x_n)}{6[f'(x_n)]^3 + f\ (x_n)f'(x_n)f''(x_n)} \quad (4)$$

$$\phi(x_{k+1}) = x_k - \frac{6f(x_k)[f'(x_k)]^2}{6[f'(x_k)]^3 - 3f(x_k)f'(x_k)f''(x_k) + [f(x_k)]^2 f'''(x_k)}. \quad (5)$$

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k) - f(x_k)}{f'(x_k) + f'^2(x_k)} : k = 0,1,\ldots,n \quad (6)$$

$$\psi(x) = x - m\frac{f(x)}{f'(x)} -$$
$$m/2[1 - \frac{2m(m+1)}{2m+1}(1 - f(x)f''(x)/[f'(x)]^2)]\frac{f(x)}{f'(x) - f(x)}. \quad (6a)$$

Here, in (6a), m is the multiplicity of a root r. For more information and many other methods one may refer [14, 15, 16]. Also the FPIM is

$$x(n) = g(x_n) \qquad (7)$$

The **Aitken's** method is

$$x_{i+2} = x_{i+1}-(\Delta x_i)^2/\Delta^2 x_{i-1} \qquad (8)$$

3. Experiments: applications on test equation models

In Civil and or Mechanical engineering, Euler–Bernoulli fourth order ordinary differential beam equation (ODBE) is used to compute or estimate the deflection or amount of bending. Here, we solve for the critical values and or stability points of the deflection w = y(x) of the beam boundary value problem (BVP) of the fourth order ODBE with the form:

$$\begin{aligned} &\frac{d^4y}{dx^4} = w'''' = f(x) = x, \\ &w(0) = w(1) = 0, \\ &w''(0) = w'(1) = 0; 0 \le x \le 1. \end{aligned} \qquad (9)$$

Using symbolic matlab dsolve, the exact solution of (9) is: w(x) = y_e = $x^5/120 - x^3/60 + x/120$. The critical points are roots of the first derivative of y(x), the gradient g(x) of w = y(x). Hence,

$$g(x) = dy/dx = x^4/24 – x^2/20+1/120 \qquad (10)$$

To approximate the roots of g(x) in (10), i.e, and critical points of w(x) in (9), one solves using the root finding method. With x0 = 2 and x0 = 0, the critical points or values of the deflection are computed to be at x1 = 1.0000, x2 = 0.4472 with minimum deflection is w1=y(1) = 0 and the maximum is w2 = y(0.4472) = 0.0024.

Here are some more test problems.

1. f1(x) = cosx-3x+1 = 0,

g(x) = (cosx + 1)/3; x_0 = 0;

r = 0.6071.

2. f2(x) = x+e^x = 0, g(x) = -e^x;
3. x_0 = 0; r = -0.5671.
4. f3(x) = x^2 – 20 = 0, or, x - $\sqrt{20}$ = 0 , g(x) = 20/x; x_0 = 4; r = 4.4721.
5. (The examples below are for showing dynamical analysis)
6. f4 = x^3-1=0, r = 1.
7. f5 = x^4-1=0, r = -1. g(x) = 1/x^3

7. $f_6(x) = 3x^3 - 3 = 0, x_o = 0$;r=1.

Table-1: Numerical Results of iterations

Eq.	**FPIM**	**Aitken**		**(1a)**
f1	X0=0 X1=0.6667 X2=0.5953 X3=0.6093…. …x6=0.6071	X0=0 X1=0.6667 X2=0.5953 X3=0.6022 X4=0.6016	X1=0.6667 X2=0.5953 X3=0.6093 X4=0.6071	X0=0 X1=0.6667 X2=0.6075 X3=0.6071
f2	**FPIM**	**Aitken**		**(1a)**
	X0=0 X1=-1 X2=-0.3679 X3= -0.6922 …x16=-0.5671 x17=-0.5672 (oscillates)	X0=0 X1=-0.5 X2=-0.5663 X3=-0.5764 X4=-0.5782	X1=-1 X2=-0.36 X3= -0.6922 X4= -0.2579 ….	X0=0 X1=-0.5000 X2=-0.5663 X3=-0.5671
f3	**FPIM**	**Aitken**		**(1a)**
	X0=4 X2n=4 X2n-1=5 Oscillates fast	X0=4 X1=4.4444 X2=4.4720 X3=4.9458		X0=4 X1=4.4444 X2=4.4720 X3=4.4721

3.1.Graphical analysis

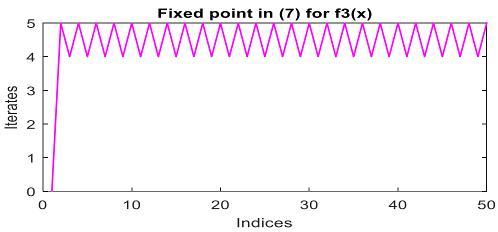

Fig-1: Oscillation of FPIM (7) for f3(x)

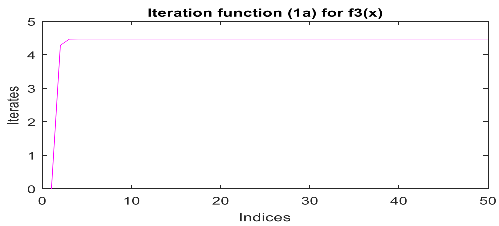

Fig-2: Convergence of method (1a) for f3(x)

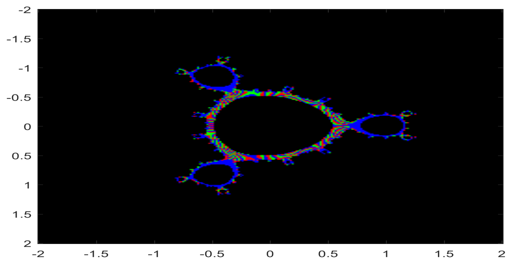

Fig-3: Regions of basins of attraction of (7) for f5

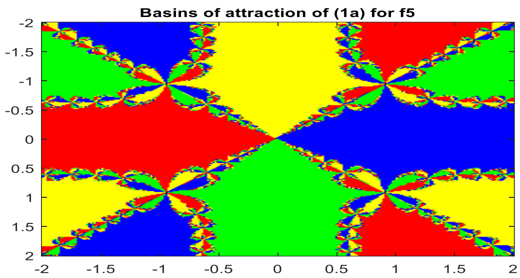

Fig-3: Regions of basins of attraction of (1a) for f5 = z^4-1 [there are 4 roots, and 4 branches]

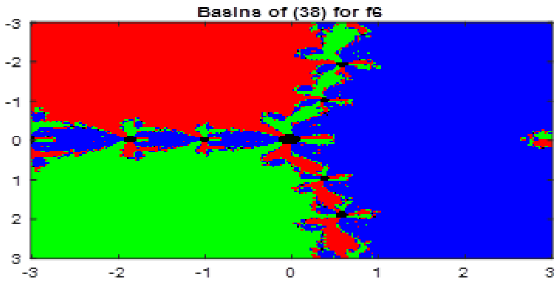

Fig-4: Regions of basins of attraction of (6a) for f6 [r =1, m = 1]

3.2. Discussions

Some improvements and or drawbacks:

One can derive several formulae for fixed point equations for a single nonlinear transcendental equation f(x). The new method can be applied to several of FPMs with convergence. There is a possibility that these methods converge faster even if successive approximation FPIM does not converge at all as shown in table above. The convergence speed of the new methods may be dependent on the form of the FPM. The new methods are acceleration convergence of FPIM and even can be faster convergent than Aitken's method. The methods converge faster even when the FPIM oscillates or too slow. The methods converge faster than Aitken's method if the same initial guess x0 is used and when the other two next estimations (x1 and x2) for Aitken's are obtained from FPIM. And the new methods do not need more than one initial guesses. The method could be reformulated for system of equations.

The convergence of the methods is investigated graphically and regions of basins of attractions highly depend on the fixed point equation of f(x). These graphs and the regions demonstrate the very behaviors of the new methods [See figures].

4. Conclusions and future work

In this work, we developed new real root finding methods for nonlinear equations. We have examined different graphical analyses, resulting in beautiful arts. The main advantage is that the methods are simple to implement as they need only one single starting guess x_0. There is no evaluation of f(x) or its derivatives. The other advantage is that the the possibility of offering algorithms with different complexities (since there can be many fixed point equations) related to the natural representation of f(x). With the same x_0, they improve convergence speed of Aitken's method which is a known acceleration convergence of FPIM, and some other methods. We have shown possible reformulation for solving system of nonlinear equations in higher dimensions. We discussed some existing methods and areas of applications (also in beam deflection BVP). The contemporary works look our good ingenuity and hope that further analysis will follow in our future.

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