Generalized Fixed Point Theorems for Compatible Mapping in Fuzzy 2-Metric Space for Integral Type Mapping

Rasik M. Patel, Ramakant Bhardwaj*
*The Research Scholar of CMJ University, Shillong (Meghalaya)
*Truba Institute of Engineering & Information Technology, Bhopal (M.P)
Email: rasik_maths@yahoo.com, drrkbhardwaj100@gmail.com

Abstract
In this paper, we give some new definitions of compatible mappings of types (I) and (II) in fuzzy-2 metric space and prove some common fixed point theorems for four mappings under the condition of compatible mappings of types (I) and (II) in complete fuzzy-2 metric space. Our results extend, generalize and improve the corresponding results given by many authors.

2010 Mathematics Subject Classification: Primary 47H10, 54H25.

Key words and phrases: Fuzzy metric space, Fuzzy 2-metric space, Compatible mappings, Common fixed point.

1. Introduction and Preliminaries
Impact of fixed point theory in different branches of mathematics and its applications is immense. The first result on fixed points for contractive type mapping was the much celebrated Banach’s contraction principle. Banach’s contraction principle by S. Banach [37] in 1922. In the general setting of complete metric space, this theorem runs as the follows, Theorem 1.1(Banach’s contraction principle) Let (X, d) be a complete metric space, c ∈ (0, 1) and f: X→X be a mapping such that for each x, y ∈ X, d (f(x), f(y)) ≤ c d(x, y) Then f has a unique fixed point a ∈ X, such that for each x ∈ X, limn→∞ fnx = a. After the classical result, R. Kannan [46] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions. In 2002, A. Branciari [1] analyzed the existence of fixed point for mapping f defined on a complete metric space (X,d) satisfying a general contractive condition of integral type.

Theorem 1.2(Branciari) Let (X, d) be a complete metric space, c ∈ (0, 1) and let f: X→X be a mapping such that for each x, y ∈ X, \( \int_0^1 \varphi(t) dt \leq c \int_0^1 \varphi(t) dt \). Where \( \varphi: [0, +\infty) \to [0, +\infty) \) is a Lebesgue integrable mapping which is summable on each compact subset of \([0, +\infty)\) , non negative, and such that for each \( e > 0 \), \( \int_0^e \varphi(t) dt \), then f has a unique fixed point a ∈ X such that for each x ∈ X, \( \lim_{n \to \infty} f^n x = a \). After the paper of Branciari, a lot of a research works have been carried out on generalizing contractive conditions of integral type for a different contractive mapping satisfying various known properties. A fine work has been done by Rhoades [2] extending the result of Branciari by replacing the condition [1.2] by the following \( \int_0^1 \varphi(t) dt \leq \int_0^1 \max \left\{ \frac{d(x,y)}{d(x,x)}d(x,f(x)), d(y,f(y)) \right\} \frac{dt}{(d(x,x)+d(y,y))^2} \). The aim of this paper is to generalize some mixed type of contractive conditions to the mapping and then a pair of mappings, satisfying a general contractive mappings such as R. Kannan type [46], S.K. Chatrerjee type [38], T. Zamfirescu type [48], etc. The concept of fuzzy sets was introduced initially by Zadeh [50] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. For example, Deng [12], Eereg [13], Fang [18], George [19], Kaleva and Seikkala [28], Kramosil and Michalek [29] have introduced the concept of fuzzy metric spaces in different ways. In fuzzy metric spaces given by Kramosil and Michalek [29], Grabiec [20] obtained the fuzzy version of Banach’s contraction principle, which has been improved, generalized and extended by some authors. Sessa [35] defined a generalization of commutativity introduced by Jungck [24], which is called the weak commutativity. Further, Jungck [25] introduced more generalized commutativity, so called compatibility. Mishra et al. [30] obtained some common fixed point theorems for compatible mappings in fuzzy metric spaces. Recently, Jungck et al. [26] introduced the concept of compatible mappings of type (A) in metric spaces, which is equivalent to the concept of compatible mappings under some conditions, and proved common fixed point theorems in metric spaces. Cho [9] introduced the concept of compatible mappings of type (a) in fuzzy metric spaces. Again, Pathak et al. [32] introduced the concept of compatible mappings of type (B) in metric spaces and Cho et al. [10] introduced the concept of compatible mappings of type (b) in fuzzy metric spaces. Pathak et al. [33] introduced the concept of compatible mappings of type (I) and (II) in metric spaces. On the other hand, George and Veeramani [19], Kramosil and
Michalek[29] introduced the concept of fuzzy topological spaces induced by fuzzy metrics which have very important applications in quantum particle physics particularly in connections with both string and e(∞) theory which were given and studied by El Naschie[14–17,49]. Many authors[21,31,39,40,41] have also proved some fixed point theorems in fuzzy (probabilistic) metric spaces (see [2–7,18,20,22,23,36]).

**Definition 1.1:** A binary operation * on [0, 1] is a continuous t-norm if it satisfies the following conditions:
1. * is associative and commutative,
2. * is continuous,
3. a * 1 = a for all a ∈ [0, 1],
4. a * b ≤ c * d whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0, 1],
5. M(x, y, .) is non-decreasing with respect to t for all x, y ∈ X and t, s > 0,
6. M(x, y, t) = My, x, t.
7. M(x, y, t) ≤ M(x, z, t) + s whenever M(x, z, t) = 1.

Two typical examples of continuous t-norm are a * b = ab and a * b = min(a, b).

**Definition 1.2:** A 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary (Non-empty) set, * is a continuous t-norm and M is a fuzzy set on X satisfying the following conditions:
1. M(x, y, t) > 0 for all x, y ∈ X and t > 0,
2. M(x, y, t) = 1 if and only if x = y,
3. M(x, y, t) = M(y, x, t),
4. M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s).
5. M(x, y, t) : (0, ∞) → [0,1] is continuous.

Let M(x, y, t) be a fuzzy metric space. For any t > 0, the open ball B(x, r, t) with center x ∈ X and radius 0 < r < 1 is defined by B(x, r, t) = {y ∈ X : M(x, y, t) > 1 − r}. Let (X, M, *) be a fuzzy metric space. Let s be the set of all A ⊂ S with x ∈ A if and only if there exist t > 0 and 0 < r < 1 such that B(x, r, t) ⊂ A. Then s is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence {xn} in X converges to x if and only if M(xn, x, t) → 1 as n → ∞ for all t > 0. It is called a Cauchy sequence if, for any 0 < e < 1 and t > 0, there exists n0 ∈ N such that M(xn, xm, t) > 1 − e for any n, m ≥ n0. The fuzzy metric space (X, M, *) is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists t > 0 and 0 < r < 1 such that M(x, y, t) > 1 − r for all x, y ∈ A.

**Example 1.3 [11]:** Let X = R and denote a * b = ab for all a, b ∈ [0, 1]. For any t ∈ (0, ∞), define M(x, y, t) = 1 − |x − y|/t for all x, y ∈ X. Then M is a fuzzy metric in X.

**Lemma 1.4 [11]:** Let (X, M, *) be a fuzzy metric space. Then M(x, y, t) is non-decreasing with respect to t for all x, y ∈ X.

**Definition 1.5:** Let (X, M, *) be a fuzzy metric space. M is said to be continuous on X^2 × (0, ∞) if

\[
\lim_{n \to \infty} M(x_n, x, t_n) = M(x, y, t)
\]

whenever a sequence \{x_n\} in X^2 × (0, ∞) converges to a point (x, y, t) ∈ X^2 × (0, ∞), i.e.,

\[
\lim_{n \to \infty} M(x_n, x, t_n) = 1 \quad \text{and} \quad \lim_{n \to \infty} M(x, y, t_n) = M(x, y, t)
\]

**Lemma 1.6:** Let (X, M, *) be a fuzzy metric space. Then M is continuous function on X^2 × (0, ∞).

**Proof:** See Proposition 1 of [36].

**Lemma 1.7 [11]:** Let (X, M, *) be a fuzzy metric space. If we define E^\lambda,M : X^2 → R^+ U{0}

\[
E^\lambda,M = \inf \{ t > 0 : M(x, y, t) > 1 - \lambda \} \quad \text{for all } \lambda \in (0,1) \text{ and } x, y \in X.
\]

(1) For all μ ∈ (0, 1), there exists λ ∈ (0, 1) such that

\[
E^\mu,M(x_1, x_n) \leq E^\lambda,M(x_1, x_2) + E^\lambda,M(x_2, x_3) + \cdots + E^\lambda,M(x_{n-1}, x_n), \text{ for all } x_1, x_2, x_3, \ldots, x_n \in X.
\]

(2) The sequence \{x_n\} is convergent in fuzzy metric space (X, M, *) if and only if E^\lambda,M(x_n, x) → 0. Also, the Sequence \{x_n\} is a Cauchy sequence if and only if it is a Cauchy sequence with E^\lambda,M.

**Lemma 1.8 [11]:** Let (X, M, *) be a fuzzy metric space. If a sequence \{x_n\} in X is such that, for any n ∈ N, M(x_n, x_{n+1}, t) ≥ M(x_0, x_1, k^n t) for all k > 1, then the sequence \{x_n\} is a Cauchy sequence.
2. Some definitions of compatible mappings

In this section, we give some definitions of compatible mappings with types, some properties and examples of compatible mappings in fuzzy metric spaces.

**Definition 2.1:** Let A and S be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, \(Ax = Sx\) implies that \(ASx = SAx\).

**Definition 2.2** [30]: Let A and S be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the mappings are said to be compatible if, for all \(t > 0\),

\[
\lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1
\]

Whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.
\]

**Proposition 2.3** [40]: If the self-mappings A and S of a fuzzy metric space \((X, M, \ast)\) are compatible, then they are weak compatible.

The converse is not true as seen in following example.

**Example 2.4** [11]: Let \((X, M, \ast)\) be a fuzzy metric space, where \(X = [0, 2]\) with t-norm defined \(a \ast b = \min \{a, b\}\), for all \(a, b \in [0,1]\) and \(M(x, y, t) = \frac{t}{t + |x-y|}\) for all \(x, y \in X\). Define the self-mappings A and S on X as follows:

\[
Ax = \begin{cases} 
  2 & \text{if } 0 \leq x \leq 1, \\
  \frac{x}{2} & \text{if } 1 < x \leq 2,
\end{cases} \quad Sx = \begin{cases} 
  2 & \text{if } x = 1, \\
  x + 3 & \text{if } x \neq 1.
\end{cases}
\]

Then we have \(S1 = A1 = 2\) and \(S2 = A2 = 1\). Also \(SA1 = AS1 = 1\) and \(SA2 = AS2 = 2\). Thus \((A, S)\) is weak compatible. Again,

\[
Ax_n = 1 - \frac{1}{4n}, \quad Sx_n = 1 - \frac{1}{10n},
\]

Thus we have \(Ax_n \to 1\), \(Sx_n \to 1\). Further, it follows that \(SAx_n = \frac{4}{5} - \frac{1}{20n}\), \(ASx_n = 2\). Therefore, we have

\[
\lim_{n \to \infty} M(ASx_n, SAx_n, t) = \lim_{n \to \infty} M(2, \frac{4}{5} - \frac{1}{20n}, t) = \frac{t}{t + \frac{6}{5}} < 1 \quad \text{for all } t > 0.
\]

Hence \((A, S)\) is not compatible.

**Definition 2.5** [9]: Let A and S be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. The mappings A and S are said to be compatible of type \((\alpha)\) if, for all \(t > 0\),

\[
\lim_{n \to \infty} M(ASx_n, SSx_n, t) = 1, \quad \lim_{n \to \infty} M(SAx_n, AAx_n, t) = 1
\]

Whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.
\]

**Definition 2.6** [10]: Let A and S be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. The mappings A and S are said to be compatible of type \((\beta)\) if, for all \(t > 0\),

\[
\lim_{n \to \infty} M(AAx_n, SSx_n, t) = 1
\]

Whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.
\]

**Proposition 2.7** [09]: Let \((X, M, \ast)\) be a fuzzy metric space with \(t \ast t = t\) for all \(t \in [0,1]\) and A, S be continuous mappings from X into itself. Then A and S are compatible if and only if they are compatible of type \((\alpha)\).

**Proposition 2.8** [10]: Let \((X, M, \ast)\) be a fuzzy metric space with \(t \ast t = t\) for all \(t \in [0,1]\) and A, S be continuous mappings from X into itself. Then A and S are compatible if and only if they are compatible of type \((\beta)\).

**Proposition 2.9** [10]: Let \((X, M, \ast)\) be a fuzzy metric space with \(t \ast t = t\) for all \(t \in [0,1]\) and A, S be continuous mappings from X into itself. Then A and S are compatible of type \((\alpha)\) if and only if they are compatible of type \((\beta)\).

Now, Turkoglu et al. [45] introduced the following definitions:

**Definition 2.10:** Let A and S be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the pair \((A, S)\) is called A-compatible if, for all \(t > 0\),

\[
\lim_{n \to \infty} M(ASx_n, SSx_n, t) = 1
\]
Whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.
\]

**Definition 2.11:** Let \( A \) and \( S \) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the pair \((A, S)\) is called \( \ast \)-compatible if and only if \((S, A)\) is \( \ast \)-compatible.

**Definition 2.12:** Let \( A \) and \( S \) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the pair \((A, S)\) is said to be compatible of type \((I)\) if, for all \( t > 0 \),
\[
\lim_{n \to \infty} M (ASx_n, x, t) \leq M (Sx, x, t)
\]
Whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.
\]

**Remark 2.13:** in [47], the above Definition was introduced as follows:
\[
\lim_{n \to \infty} M (ASx_n, x, t) \leq M (Sx, x, t)
\]
Whenever \( \lambda \in (0,1] \) and \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.
\]
If \( \lambda = 1 \), then, by Lemma 1.4, it is easily verified that, for all \( \lambda \in (0,1] \) by using above Definition, we have Definition of [48].

**Definition 2.14:** Let \( A \) and \( S \) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the pair \((A, S)\) is said to be compatible of type \((II)\) if and only if \((S, A)\) is compatible of type \((I)\).

**Remark 2.15:** In [25, 26, 33, 34], we can find the equivalent formulations of above Definitions and their examples in metric spaces (see [25, 26, 33, 34]). Such mappings are independent of each other and more general than commuting and weakly commuting mappings (see [25, 35]).

**Proposition 2.16:** Let \( A \) and \( S \) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Suppose that the pair \((A, S)\) is compatible of type \((I)\) (respectively, \((II)\)) and \( A z = S z \) for some \( z \in X \). Then, for all \( t > 0 \),
\[
M (Az, Sz, t) \geq M (Az, Sz, t) (\text{Respectively, } M (Sz, Az, t) \geq M (Sz, Sz, t)).
\]

**Proof:** See Proposition 7 of [47].

**Definition 2.17:** (S.Gäahler [43]) Let \( X \) be a nonempty set. A real valued function \( d \) on \( X \times X \times X \) is said to be a 2-metric on \( X \) if
(1) Given distinct elements \( x, y, z \) of \( X \), there exists an element \( z \in X \) such that \( d(x, y, z) \neq 0 \),
(2) \( d(x, y, z) = 0 \) when at least two of \( x, y, z \in X \) are equal,
(3) \( d(x, y, z) = d(y, z, x) = d(z, x, y) \) for all \( x, y, z \in X \),
(4) \( d(x, y, z) \leq d(x, y, w) + d(y, w, z) + d(w, z, x) \) for all \( x, y, z, w \in X \).

The pair \((X, d)\) is called a 2-metric space.

**Example 2.18** [27]: Let \( X = \mathbb{R}^3 \) and let \( d(x, y, z) := \) the area of the triangle spanned by \( x, y \) and \( z \) which may be given explicitly by the formula, \( d(x, y, z) = |x_1(y_2z_3 - y_3z_2) - x_2(y_1z_3 - y_3z_1) + x_3(y_1z_2 - y_2z_1)| \), where \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3), z = (z_1, z_2, z_3) \). Then \((X, d)\) is a 2-metric space.

**Definition 2.19:** (S. Sharma [44]) the 3-tuple \((X, M, \ast)\) is called a fuzzy 2-metric space if \( X \) is an arbitrary set, \( \ast \) is a continuous \( \ast \)-norm of H-type and \( M \) is a fuzzy set in \( H \). For each \( h, m, n \in (0,1] \), \( \lim_{n \to \infty} M (x_n, x, t) \leq M (x, x, t) \) for all \( n \geq n_0 \) and \( a \in X \), \( M (x_n, x, a, t) > 1 - \lambda \). That is
\[
\lim_{n \to \infty} M (x_n, x, a, t) = 1 \quad \text{for all } a \in X \text{ and } t > 0.
\]
(2) A sequence \(\{x_n\}\) in fuzzy-2 metric space \(X\) is called a Cauchy sequence, if for any \(\lambda \in (0,1)\) and \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(m, n \geq n_0\) and \(a \in X\),

\[ M(x_n, x_m, a, t) > 1 - \lambda. \]

(3) A fuzzy-2 metric space in which every Cauchy sequence is convergent is said to be complete.

**Definition 2.22** [27]: Self mappings \(A\) and \(B\) of a fuzzy-2 metric space \((X, M, \ast)\) is said to be compatible, if

\[ \lim_{n \to \infty} M(ABx_n, BAx_n, a, t) = 1 \quad \text{for all } a \in X \text{ and } t > 0, \]

whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \quad \text{for some } z \in X. \]

**Lemma 2.23** [27]: Let \((X, M, \ast)\) be a fuzzy-2 metric space. If there exists \(q \in (0,1)\) such that

\[ M(x, y, z, qt) \geq M(x, y, z, t) \quad \text{for all } x, y, z \in X \text{ with } z \neq x, z \neq y \text{ and } t > 0, \]

then \(x = y\).

### 3. The main results

In this section, we prove a fixed point theorem for four mappings under the condition of compatible mappings of types (I) and (II).

Let \(\varphi\) be the set of all continuous and increasing functions \(\varphi: [0,1] \to [0,1]\) in any coordinate and \(\varphi(t, t, t, t, t, t) > t\) for all \(t \in [0,1]\).

**Theorem 3.1:** Let \((X, M, \ast)\) be a complete fuzzy-2 metric space with \(t \ast t = t\) for all \(t \in [0,1]\). Let \(A, B, S\) and \(T\) be mappings from \(X\) into itself such that

(i) \(A(X) \subseteq T(X), B(X) \subseteq S(X)\),

(ii) There exists a constant \(k \in (0,1)\) such that

\[ M(Sx, Ty, a, t), M(Ax, Sy, a, t), M(By, Ty, a, t), M(Ax, Tx, a, t), M(By, Sx, a, (2-\alpha)t), M(Ax, Tx, a, t), M(Ax, Sx, a, t) \]

\[ \zeta(t) \]

\[ \alpha \in (0,2), \quad t > 0 \text{ and } \varphi \in \varphi. \]

If the mappings \(A, B, S\) and \(T\) satisfy any one of the following conditions:

(iii) The pairs \((A, S)\) and \((B, T)\) are compatible of type (II) and \(A\) or \(B\) is continuous,

(iv) The pairs \((A, S)\) and \((B, T)\) are compatible of type (I) and \(S\) or \(T\) is continuous,

then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

**Proof:** Let \(x_0 \in X\) be an arbitrary point. Since \(A(X) \subseteq T(X), B(X) \subseteq S(X)\), there exist \(x_1, x_2 \in X\) such that

\[ A x_0 = T x_1, B x_1 = S x_2. \]

Inductively, construct the sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[ y_{2n} = A x_{2n} = T x_{2n+1}, \quad y_{2n+1} = B x_{2n+1} = S x_{2n+2} \]

for \(n = 0, 1, 2, \ldots\) Then by \(\alpha = 1 - q\) and \(q \in (\frac{1}{2}, 1)\), if we set \(d_m(t) = M(y_m, y_{m+1}, a, kt)\) for \(t > 0\), then we prove that \(\{d_m(t)\}\) is increasing with respect to \(m\). Setting \(m = 2n\), then we have
\[ \int_0^{d_{2n}(t)} \zeta(t) dt = \int_0^M (y_{2n}, x_{2n+1}, a, t) \zeta(t) dt \]
\[ = \int_0^M (x_{2n}, y_{2n+1}, a, t) \zeta(t) dt \]
\[ \geq \int_0^\phi \min \left\{ \begin{array}{l}
M(x_{2n}, y_{2n+1}, a, t), M(x_{2n+1}, y_{2n}, a, t), M(x_{2n}, y_{2n+1}, a, t), M(x_{2n}, y_{2n+1}, a, t), \end{array} \right\} \zeta(t) dt \]
\[ = \int_0^\phi \min \left\{ \begin{array}{l}
M(x_{2n+1}, y_{2n}, a, (1-q)t), M(x_{2n}, y_{2n+1}, a, t), \end{array} \right\} \zeta(t) dt \]
\[ = \int_0^\phi \left\{ \begin{array}{l}
\min \{d_{2n-1}(t), d_{2n}(t), d_{2n}(t), 1, d_{2n-1}(t), d_{2n-1}(t), d_{2n-1}(t), d_{2n-1}(t) \} \end{array} \right\} \zeta(t) dt. \]

The above inequality is true if \( \emptyset \) is an increasing function.

We claim that for all \( n \in \mathbb{N} \),
\[ \int_0^{d_{2n}(t)} \zeta(t) dt \geq \int_0^{d_{2n-1}(t)} \zeta(t) dt. \]

In fact, if
\[ \int_0^{d_{2n}(t)} \zeta(t) dt < \int_0^{d_{2n-1}(t)} \zeta(t) dt. \]

We have
\[ \int_0^{d_{2n}(kt)} \zeta(t) dt \geq \int_0^{d_{2n}(qt)} \zeta(t) dt \]

That is
\[ \int_0^{d_{2n}(kt)} \zeta(t) dt > \int_0^{d_{2n}(qt)} \zeta(t) dt \]

which is a contradiction.

Hence
\[ \int_0^{d_{2n}(t)} \zeta(t) dt \geq \int_0^{d_{2n-1}(t)} \zeta(t) dt \]

for all \( n \in \mathbb{N} \) and \( t > 0 \).

Similarly, for \( m = 2n+1 \), we have
\[ \int_0^{d_{2n+1}(t)} \zeta(t) dt \geq \int_0^{d_{2n}(t)} \zeta(t) dt \]

and so \( \{d(m)\} \) is an increasing in \([0, 1]\).

Similarly, for \( m = 2n+1 \), we have
\[ \int_0^{d_{2n+1}(t)} \zeta(t) dt \geq \int_0^{d_{2n}(t)} \zeta(t) dt \]

and so
\[ \int_0^{d_{2n+1}(t)} \zeta(t) dt \geq \int_0^{d_{2n}(t)} \zeta(t) dt \]

for all \( n \in \mathbb{N} \), that is
\[ \int_0^M (y_n, y_{n+1}, a, t) \zeta(t) dt \geq \int_0^M (y_{n-1}, y_n, a, q/k) \zeta(t) dt \geq \ldots \geq \int_0^M (y_0, y_1, a, \left(\frac{q}{k}\right)^n) \zeta(t) dt. \]

Hence \( \{y_n\} \) is a Cauchy sequence and, by the completeness of \( X \), \( \{y_n\} \) converges to a point \( z \) in \( X \). Let
\[
\lim_{n \to \infty} y_n = z. \quad \text{Hence we have}
\]
\[
\lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} B_{x_{2n+1}} = \lim_{n \to \infty} S_{x_{2n+2}} = z.
\]

Now, suppose that \( T \) is continuous and the pairs \((A, S), (B, T)\) are compatible of type \((I)\). Hence we have
\[
\lim_{n \to \infty} s_{n} = h, \quad \lim_{n \to \infty} u_{n} = h, \quad \lim_{n \to \infty} e_{n} = h.
\]

Now, for \( c = 1 \), setting \( x = x_{2n} \) and \( y = x_{2n+1} \) in the inequality (3.1), we have
\[
\int_{0}^{\infty} M(A_{x_{2n}}, B_{x_{2n+1}}, a, t) \zeta(t) dt \geq \int_{0}^{\infty} \min \left\{ M(S_{x_{2n}}, T_{x_{2n+1}}, a, t), M(A_{x_{2n}}, S_{x_{2n+1}}, a, t), M(B_{x_{2n+1}}, T_{x_{2n+1}}, a, t) \right\} \zeta(t) dt.
\]

Letting \( n \to \infty \), we have
\[
\int_{0}^{\infty} M(x_{2n}, \lim_{n \to \infty} B_{x_{2n+1}}, a, t) \zeta(t) dt \geq \int_{0}^{\infty} \min \left\{ M(S_{x_{2n}}, T_{x_{2n+1}}, a, t), M(A_{x_{2n}}, S_{x_{2n+1}}, a, t), M(B_{x_{2n+1}}, S_{x_{2n+1}}, a, t) \right\} \zeta(t) dt.
\]

Now using (2.22) and (2.23), we have
\[
\int_{0}^{\infty} M(x_{2n+1}, a, t) \zeta(t) dt \geq \int_{0}^{\infty} \min \left\{ M(S_{x_{2n}}, T_{x_{2n+1}}, a, t), M(A_{x_{2n}}, S_{x_{2n+1}}, a, t), M(B_{x_{2n+1}}, S_{x_{2n+1}}, a, t) \right\} \zeta(t) dt.
\]

Therefore,
\[
\int_{0}^{\infty} M(x_{2n+1}, a, t) \zeta(t) dt \geq \int_{0}^{\infty} \min \left\{ M(S_{x_{2n}}, T_{x_{2n+1}}, a, t), M(A_{x_{2n}}, S_{x_{2n+1}}, a, t), M(B_{x_{2n+1}}, S_{x_{2n+1}}, a, t) \right\} \zeta(t) dt.
\]

So it follow that \( T_{z} = z \).

Again, replacing \( x \) by \( x_{2n} \) and \( y \) by \( z \) in (1), for all \( a = 1 \), we have
\[
\int_{0}^{\infty} M(A_{x_{2n}}, B_{z}, a, t) \zeta(t) dt \geq \int_{0}^{\infty} \min \left\{ M(S_{x_{2n}}, T_{z}, a, t), M(A_{x_{2n}}, S_{z}, a, t), M(B_{z}, T_{z}, a, t), M(A_{x_{2n}}, T_{x_{2n+1}}, a, t), M(B_{z}, S_{x_{2n+1}}, a, t), M(A_{x_{2n}}, S_{x_{2n+1}}, a, t) \right\} \zeta(t) dt.
\]

And so, letting \( n \to \infty \), we have
\[
\int_{0}^{\infty} M(B_{z}, x, a, t) \zeta(t) dt \geq \int_{0}^{\infty} \min \left\{ M(S_{x_{2n}}, T_{z}, a, t), M(A_{x_{2n}}, S_{z}, a, t), M(B_{z}, T_{z}, a, t), M(A_{x_{2n}}, T_{x_{2n+1}}, a, t), M(B_{z}, S_{x_{2n+1}}, a, t), M(A_{x_{2n}}, S_{x_{2n+1}}, a, t) \right\} \zeta(t) dt.
\]

Which implies that \( B_{z} = z \). Since \( B(X) \subseteq S(X) \), there exist \( u \in X \) such that \( S_{u} = z = B_{z} \). So, for \( a = 1 \), we have
\[
\int_{0}^{\infty} M(A_{u}, B_{z}, a, t) \zeta(t) dt \geq \int_{0}^{\infty} \min \left\{ M(S_{u}, T_{z}, a, t), M(A_{u}, S_{z}, a, t), M(B_{z}, T_{z}, a, t), M(A_{u}, T_{u}, a, t), M(B_{z}, S_{u}, a, t), M(A_{u}, S_{u}, a, t) \right\} \zeta(t) dt.
\]

And so,
\[
\int_{0}^{\infty} M(A_{u}, x, a, t) \zeta(t) dt \geq \int_{0}^{\infty} \min \left\{ M(S_{u}, T_{z}, a, t), M(A_{u}, S_{z}, a, t), M(B_{z}, T_{z}, a, t), M(A_{u}, T_{u}, a, t), M(B_{z}, S_{u}, a, t), M(A_{u}, S_{u}, a, t) \right\} \zeta(t) dt.
\]

Which implies that \( A_{u} = z \). Since the pair \((A, S)\) is compatible of type \((I)\) and \( A_{u} = S_{u} = z \), by proposition 2.16, we have...
\[ \int_0^\infty \zeta(t) dt > \int_0^\infty \zeta(t) dt. \]

And so,
\[ \int_0^\infty \zeta(t) dt > \int_0^\infty \zeta(t) dt. \]

Again, for \( \alpha = 1 \), we have
\[ \int_0 M(\alpha z, \alpha t) \zeta(t) dt \geq \int_0 \phi \left\{ \min \left\{ M(Sz, Tz, a, t), M(\alpha z, Sz, a, t), M(Bz, Sz, a, t), M(\alpha z, Tz, a, t), M(Az, Sz, a, t) \right\} \right\} \zeta(t) dt \]

Hence we have
\[ \int_0 \zeta(t) dt > \int_0 M(\alpha z, a, t) \zeta(t) dt. \]

And so \( A z = z \). Therefore, \( A z = B z = S z = T z = z \) and \( z \) is a common fixed point of the self-mapping \( s A, B, S \) and \( T \). The uniqueness of a common fixed point of the mappings \( A, B, S \) and \( T \) be easily verified by using (3.1)

In fact, if \( z' \) be another fixed point for \( A, B, S \) and \( T \), then, for \( \alpha = 1 \), we have
\[ \int_0 M(z, z', a, kt) \zeta(t) dt = \int_0 M(Az, Bz', a, kt) \zeta(t) dt \]
\[ \int_0 \phi \left\{ \min \left\{ M(Sz, Tz, a, t), M(Az, Sz, a, t), M(Bz, Sz, a, t), M(Az, Tz, a, t), M(Az, Sz, a, t) \right\} \right\} \zeta(t) dt \]
\[ \geq \int_0 \phi \left\{ \min \left\{ M(Sz, Tz, a, t), M(Az, Sz, a, t), M(Bz, Sz, a, t), M(Az, Tz, a, t), M(Az, Sz, a, t) \right\} \right\} \zeta(t) dt \]

so \( z = z' \).

**Example 3.2:** Let \( (X, d) \) be a fuzzy-2 metric space and denote \( *(a, b) = ab \) for all \( a, b \in [0, 1] \). For each \( h, m, n \in \mathbb{R}^+ \) and \( t > 0 \) and \( x, y, a \in X \) define \( M(x, y, a, t) = \frac{h^n}{ht^n + md(xy, a)} \). Define the self-mappings \( A, B, S, T \) on \( X \) by
\[ A x = Bx = 1, \]
\[ S x = \frac{3-x}{2}, T x = \frac{2x+3}{5} \]

If we define a sequence \( \{x_n\} \) in \( X \) by \( 1 - \frac{1}{n} \), then we have
\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = 1, \quad \lim_{n \to \infty} M(SAx_n, 1, a, t) \leq M(A1, 1, a, t) = 1. \]
\[ \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = 1, \quad \lim_{n \to \infty} M(TBx_n, 1, a, t) \leq M(B1, 1, a, t) = 1. \]

This is the pairs \( (A, S), (B, T) \) are compatible of type (II) and \( A, B \) are continuous. Consider a function
\[ \Phi: [0, 1]^7 \to [0, 1] \text{ defined by } \Phi(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (\min \{x_i\})^h \text{ for some } 0 < h < 1. \]

Then we have
\[ M(Ax, By, a, t) \geq \Phi(x_1, x_2, x_3, x_4, x_5, x_6, x_7). \]

Therefore, all the conditions of theorem 3.1 are satisfied and so \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 3.3:** Let \( (X, M, *) \) be a complete fuzzy-2 metric space with \( t \ast t = t \) for all \( t \in [0,1] \). Let \( A, B, S \) and \( T \) be mappings from \( X \) into itself such that
Mathematical Theory and Modeling

Vol.3, No.6, 2013-Selected from International Conference on Recent Trends in Applied Sciences with Engineering Applications

(i) \( A(X) \subseteq T(X), B(X) \subseteq S(X) \),

(ii) There exists a constant \( k \in \{0, \frac{1}{2}\} \) such that

\[
\int_0^1 M(Ax, By, a, t) \zeta(t) dt \geq \int_0^1 \left\{ \min\left\{ a_1(t) M(Ax, Ty, a, t) + a_2(t) M(AX, Sx, a, t) + a_3(t) M(By, Ty, a, t) + a_4(t) M(AX, TX, a, t) \right\} \right\}^\frac{1}{2} \zeta(t) dt
\]

for all \( x, y, a \in X, a \in (0,2), t > 0 \) and \( \{a_i\} \subseteq R^+ \) such that \( \sum_{i=1}^4 a_i(t) = 1 \).

If the mappings \( A, B, S \) and \( T \) satisfy any one of the following conditions:

(iii) The pairs \( (A, S) \) and \( (B, T) \) are compatible of type (II) and \( A \) or \( B \) is continuous,

(iv) The pairs \( (A, S) \) and \( (B, T) \) are compatible of type (I) and \( S \) or \( T \) is continuous, then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

Proof: By Theorem 3.1, if we define

\[
\Phi = \left\{(a_1(t)x_1 + a_2(t)x_2 + a_3(t)x_3 + a_4(t)x_4 + a_5(t)x_5 + a_6(t)x_6 + a_7(t)x_7) \right\}^\frac{1}{2},
\]

Then we have conclusion.

**Corollary 3.4**: Let \( (X, M, \ast) \) be a complete fuzzy-2 metric space with \( t \ast t = t \) for all \( t \in [0,1] \). Let \( A, B, R, S, H \) and \( T \) be mappings from \( X \) into itself such that

(i) \( A(X) \subseteq TH(X), B(X) \subseteq SR(X) \),

(ii) There exists a constant \( k \in \{0, \frac{1}{2}\} \) such that

\[
\int_0^1 M(Ax, By, a, t) \zeta(t) dt \geq \int_0^1 \left\{ \min\left\{ M(AX, Ty, a, t), M(AX, Sx, a, t), M(By, Ty, a, t), M(AX, TX, a, t), M(By, SRx, a, t), M(AX, TX, a, t) \right\} \right\}^\frac{1}{2} \zeta(t) dt
\]

for all \( x, y, a \in X, a \in (0,2), t > 0 \) and \( \emptyset \in \varphi \).

If the mappings \( A, B, S \) and \( T \) satisfy any one of the following conditions:

(iii) The pairs \( (A, SR) \) and \( (B, TH) \) are compatible of type (II) and \( A \) or \( B \) is continuous,

(iv) The pairs \( (A, SR) \) and \( (B, TH) \) are compatible of type (I) and \( SR \) or \( TH \) is continuous,

(v) \( TH = HT, AR = RA, BH = HB \) and \( SR = RS \), then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

Proof: By theorem 3.1 \( A, B, TH \) and \( SR \) have a unique common fixed point in \( X \). That is, there exists \( z \in X \) such that \( A z = B z = THz = SRz = z \). Now, we prove that \( Rz = z \). In fact, by the condition (ii), it follows that

\[
\int_0^1 M(ARz, Bz, a, t) \zeta(t) dt \geq \int_0^1 \left\{ \min\left\{ M(ARz, THz, a, t), M(ARz, SRz, a, t), M(Bz, THz, a, t), M(ARz, THz, a, t), M(ARz, SRz, a, t) \right\} \right\}^\frac{1}{2} \zeta(t) dt
\]

For \( a = 1 \), we have

\[
\int_0^1 M(Rz, z, a, t) \zeta(t) dt \geq \int_0^1 \left\{ \min\left\{ M(Rz, z, a, t), M(Rz, z, a, t), M(Rz, z, a, t), M(Rz, z, a, t) \right\} \right\}^\frac{1}{2} \zeta(t) dt
\]

Which is a contradiction. Therefore, it follows that \( Rz = z \). Hence \( S z = SRz = z \). Similarly, we get \( Tz = Hz = z \).

**Corollary 3.5**: Let \( (X, M, \ast) \) be a complete fuzzy-2 metric space with \( t \ast t = t \) for all \( t \in [0,1] \). Let \( S \) and \( T \) be mappings from \( X \) into itself such that
(i) There exists a constant \( k \in (0, \frac{1}{2}) \) such that

\[
\int_0^\infty M(x, y, a, kt) \zeta(t) dt \geq \int_0^\infty \left\{ \min \left\{ M(Sx, Ty, a, t), M(Ax, Sy, a, t), M(At, Py, a, t), M(At, Aa, t) \right\} \right\} \zeta(t) dt
\]

for all \( x, y, a \in X, k \in (0,2), t > 0 \) and \( \emptyset \in \varphi \). Then \( A, B, S, T \) and \( A \) have a unique common fixed point in \( X \).

Proof: If we get \( A = B = I \) (the identity mapping) in theorem 3.1, then it is easy to check that the pairs \((I, S)\) and \((I, T)\) are compatible of type (II) and the identity mapping \( I \) is continuous. Hence, by theorem 3.1, \( T \) and \( S \) have a unique common fixed point in \( X \).

Corollary 3.6: Let \((X, M, *)\) be a complete fuzzy-2 metric space with \( t * t = t \) for all \( t \in [0,1] \). Let \( A \) and \( S \) be mappings from \( X \) into itself such that

(i) There exists a constant \( k \in (0, \frac{1}{2}) \) such that

\[
\int_0^\infty M(x, y, a, kt) \zeta(t) dt \geq \int_0^\infty \left\{ \min \left\{ M(Sx, Ty, a, t), M(Ax, Sy, a, t), M(Ay, Ty, a, t), M(Ax, Tx, a, t) \right\} \right\} \zeta(t) dt
\]

for all \( x, y, a \in X, k \in (0,2), t > 0 \) and \( \emptyset \in \varphi \). Then \( A, B, S, T \) and \( A \) have a unique common fixed point in \( X \).

Proof: If we get \( A = B = I \) (the identity mapping) in theorem 3.1, then it is easy to check that the pairs \((I, S)\) and \((I, T)\) are compatible of type (II) and the identity mapping \( I \) is continuous. Hence, by theorem 3.1, \( T \) and \( S \) have a unique common fixed point in \( X \).

Theorem 3.7: Let \((X, M, *)\) be a complete fuzzy-2 metric space with \( t * t = t \) for all \( t \in [0,1] \). Let \( S, T \) and two sequences \( \{A_i\}, \{B_j\} \) for all \( i, j \in N \) be mappings from \( X \) into itself such that

(i) There exists \( i_0, j_0 \in N \) such that \( A_{i_0}(X) \subseteq T(X), B_{j_0}(X) \subseteq S(X) \),

(ii) There exists a constant \( k \in (0, \frac{1}{2}) \) such that

\[
\int_0^\infty M(x, y, a, kt) \zeta(t) dt \geq \int_0^\infty \left\{ \min \left\{ M(Sx, Ty, a, t), M(Ax, Sy, a, t), M(At, Py, a, t), M(At, Aa, t) \right\} \right\} \zeta(t) dt
\]

for all \( x, y, a \in X, k \in (0,2), t > 0 \) and \( \emptyset \in \varphi \). If the mappings \( A^n, S^n \) satisfy any one of the following conditions:

(iii) The pairs \((A^n, S^n)\) is compatible of type (II) \( A^n \) and \( S^n \) is continuous,

(iv) The pairs \((A^n, S^n)\) is compatible of type (I) and \( S^n \) is continuous,

(v) \( A^n = SA^n \) and \( SA^n = S^nA \), then \( A \) and \( S \) have a unique common fixed point in \( X \).

Proof: If we set \( A = B = I \) and \( S = T = S^n \) in theorem 3.1, then \( A^n \) and \( S^n \) have a unique fixed point in \( X \).

That is, there exists \( z \in X \) such that \( A^nz = S^nz = z \). Therefore, \( A^nAz = A^nAz = S^nz = S^nz \) which implies that \( Az \) is a fixed point of \( A^n \) and \( S^n \) and hence \( A \) is a fixed point of \( S^n \).

Similarly, we have \( S = z = 0 \).

Theorem 3.7: Let \((X, M, *)\) be a complete fuzzy-2 metric space with \( t * t = t \) for all \( t \in [0,1] \). Let \( S, T \) and two sequences \( \{A_i\}, \{B_j\} \) for all \( i, j \in N \) be mappings from \( X \) into itself such that

(i) There exists \( i_0, j_0 \in N \) such that \( A_{i_0}(X) \subseteq T(X), B_{j_0}(X) \subseteq S(X) \),

(ii) There exists a constant \( k \in (0, \frac{1}{2}) \) such that

\[
\int_0^\infty M(x, y, a, kt) \zeta(t) dt \geq \int_0^\infty \left\{ \min \left\{ M(Sx, Ty, a, t), M(Ax, Sy, a, t), M(At, Py, a, t), M(At, Aa, t) \right\} \right\} \zeta(t) dt
\]

for all \( x, y, a \in X, k \in (0,2), t > 0 \) and \( \emptyset \in \varphi \). If the mappings \( A_{i_0}, B_{j_0} \) satisfy any one of the following conditions:

(iii) The pairs \((A_{i_0}, S)\) and \((B_{j_0}, T)\) are compatible of type (II) and \( A_{i_0} \) or \( B_{j_0} \) is continuous,

(iv) The pairs \((A_{i_0}, S)\) and \((B_{j_0}, T)\) are compatible of type (I) and \( S \) or \( T \) is continuous, then \( A_{i_0}B_{j_0}, S \) and \( T \) have a unique common fixed point in \( X \) for \( i, j = 1, 2, 3, \ldots \).

Proof: By theorem 3.1, the mappings \( S, T \) and \( A_{i_0}B_{j_0} \) for some \( i_0, j_0 \in N \) have a unique common fixed point in \( X \).

That is, there exists a unique point \( z \in X \) such that \( Sz = Tz = B_{j_0}z = z \). Suppose that there exists \( i \in N \) such that \( i \neq i_0 \). Then we have
\[
\int_0^\infty M(\alpha z, \beta z, \gamma, \delta t) \, dt = \int_0^\infty M(\alpha z, \beta z, \gamma, \delta t) \, \zeta(t) \, dt \\
\geq \int_0^\infty \left\{ \min \left\{ M(\alpha z, \beta z, \gamma, \delta t), M(\alpha z, \beta z, \gamma, \delta t) \right\} \right\} \zeta(t) \, dt
\]

For \( \alpha = 1 \), we have

\[
\int_0^\infty M(\alpha z, \beta z, \gamma, \delta t) \, dt \geq \int_0^\infty \left\{ \min \left\{ M(\alpha z, \beta z, \gamma, \delta t), M(\alpha z, \beta z, \gamma, \delta t) \right\} \right\} \zeta(t) \, dt
\]

Which is a contradiction. Hence, for all \( i \in N \), it follows that \( A_i z = z \). Similarly, for \( j \in N \), we have \( B_j z = z \). Therefore, for all \( i, j \in N \), we have \( A_i z = B_j z = S z = T z = z \).

Acknowledgement:
One of the Author (Dr RKB) is thankful to MPCOST Bhopal for the project No.2556.

References:
[32] Pathak HK, Cho YJ, Kang SM, Lee BS. Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming. Le Matematiche 1995; L:15–33.
[34] Pathak HK, Mishra N, Kalinde AK. Common fixed point theorems with applications to nonlinear integral equation. Demonstratio Math 1999; 32:517–564.
[47] Turkoglu D, Altun I, Cho YJ. Compatible maps and compatible maps of types (a) and (b) in intuitionistic fuzzy metric spaces, in press.
This academic article was published by The International Institute for Science, Technology and Education (IISTE). The IISTE is a pioneer in the Open Access Publishing service based in the U.S. and Europe. The aim of the institute is Accelerating Global Knowledge Sharing.

More information about the publisher can be found in the IISTE’s homepage:
http://www.iiste.org

CALL FOR PAPERS

The IISTE is currently hosting more than 30 peer-reviewed academic journals and collaborating with academic institutions around the world. There’s no deadline for submission. **Prospective authors of IISTE journals can find the submission instruction on the following page:** http://www.iiste.org/Journals/

The IISTE editorial team promises to the review and publish all the qualified submissions in a fast manner. All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Printed version of the journals is also available upon request of readers and authors.

**IISTE Knowledge Sharing Partners**

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digital Library, NewJour, Google Scholar