Mathematical Theory and Modeling
www.iiste.org
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.3, No.6, 2013-Selected from International Conference on Recent Trends in Applied Sciences with Engineering Applications

He-Laplace Method for Special Nonlinear Partial Differential Equations

Hradyesh Kumar Mishra
Department of Mathematics,Jaypee University of Engineering & Technology,Guna-473226(M.P) India,Email: hk.mishra@juet.ac.in

Abstract
In this article, we consider Cauchy problem for the nonlinear parabolic-hyperbolic partial differential equations. These equations are solved by He-Laplace method.. It is shown that, in He-Laplace method, the nonlinear terms of differential equation can be easily handled by the use of He's polynomials and provides better results.
Keywords: Laplace transform method, Homotopy perturbation method, Partial differential equations, He's polynomials.
AMS Subject classification: 35G10; 35G15; 35G25; 35G30; 74S30.

1. Introduction

Nonlinearity exists everywhere and nature is nonlinear in general. The search for a better and easy to use tool for the solution of nonlinear equations that illuminate the nonlinear phenomena of real life problems of science and engineering has recently received a continuing interest. Various methods, therefore, were proposed to find approximate solutions of nonlinear equations. Some of the classical analytic methods are Lyapunov's artificial small parameter method [17], perturbation techniques [6,23,22, 25]. The Laplace decomposition method have been used to solve nonlinear differential equations [1-4, 16, 19, 20, 27]. J.H.He developed the homotopy perturbation method (HPM) [6-13,14-15,21,24,26] by merging the standard homotopy and perturbation for solving various physical problems.Furthermore, the homotopy perturbation method is also combined with the well-known Laplace transformation method [18] which is known as Laplace homotopy perturbation method.

In this paper, the main objective is to solve partial differential equations by using He-Laplace method. It is worth mentioning that He-Laplace method is an elegant combination of the Laplace transformation, the homotopy perturbation method and He's polynomials. The use of He's polynomials in the nonlinear term was first introduced by Ghorbani [5, 23]. This paper contains basic idea of homotopy perturbation method in section 2, He-Laplace method in section 3, examples in section 4 and conclusions in section 5 respectively.

2. Basic idea of homotopy perturbation method

Consider the following nonlinear differential equation

$$A(y) - f(r) = 0\,, \qquad r \in \Omega \tag{1}$$

with the boundary conditions of

$$B\left(y, \frac{\partial y}{\partial n}\right) = 0, \qquad r \in \Gamma, \tag{2}$$

where A, B, $f(r)$ and Γ are a general differential operator, a boundary operator, a known analytic function and the boundary of the domain Ω, respectively.

The operator A can generally be divided into a linear part L and a nonlinear part N. Eq. (1) may therefore be written as:

$$L(y) + N(y) - f(r) = 0, \tag{3}$$

By the homotopy technique, we construct a homotopy $v(r,p): \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(y_0)] + p[A(v) - f(r)] = 0 \tag{4}$$

or

$$H(v,p) = L(v) - L(y_0) + p\,L(y_0) + p[N(v) - f(r)] = 0 \tag{5}$$

where $p \in [0,1]$ is an embedding parameter, while y_0 is an initial approximation of Eq.(1), which satisfies the boundary conditions. Obviously,from Eqs.(4) and (5)
we will have:

$$H(v,0) = L(v) - L(y_0) = 0, \tag{6}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{7}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from y_0 to $y(r)$. In topology, this is called deformation, while $L(v) - L(y_0)$ and $A(v) - f(r)$ are called homotopy. If the embedding parameter p is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of Eqs.(4) and (5) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty \tag{8}$$

Setting $p = 1$ in Eq.(8), we have

$$y = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{9}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The series (9) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$. Moreover, He [6] made the following suggestions:

(1)The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e. $p \to 1$.

(2) The norm of $L^{-1}\left(\frac{\partial N}{\partial v}\right)$ must be smaller than one so that the series converges.

3. He-Laplace method

Consider the following Cauchy problem for the nonlinear parabolic-hyperbolic differential equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{\partial^2}{\partial t^2} - \Delta\right)y = F(y), \tag{10}$$

with initial conditions

$$\frac{\partial^k y}{\partial t^k}(0, X) = \phi_k(X),\ X = (x_{1,} x_{2,} \ldots\ldots\ldots, x_i), k = 0,1,2,3\ldots\ldots \tag{11}$$

where the nonlinear term is represented by $F(y)$, and Δ is the Laplace operator in R^n. we rewrite the eqn(10) as follows:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)y = F(y),$$

or

$$\frac{\partial^3 y}{\partial t^3} - \frac{\partial^3 y}{\partial t \partial x^2} - \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\partial^4 y}{\partial x^4} = F(y)\ , \tag{12}$$

Applying the laplace transform of both sides of (12), we have

$$L\left[\frac{\partial^3 y}{\partial t^3}\right] = L\left[\frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^4 y}{\partial x^4}\right] + L[F(y)] \tag{13}$$

$$s^3 L[y(x,t)] - s^2 y(x,0) - s\, y'(x,0) - y''(x,0) = L\left[\frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^4 y}{\partial x^4}\right] + L[F(y)] \tag{14}$$

Using initial conditions (11) in (14), we have

$$s^3 L[y(x,t)] - s^2 \phi_0(x) - s\,\phi_1(x) - \phi_2(x) = L\left[\frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^4 y}{\partial x^4}\right] + L[F(y)] \tag{15}$$

$$L[y(x,t)] = \frac{\phi_0(x)}{s} + \frac{\phi_1(x)}{s^2} + \frac{\phi_2(x)}{s^3} + \frac{1}{s^3} L\left[\frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^4 y}{\partial x^4}\right] + \frac{1}{s^3} L[F(y)] \quad (16)$$

Taking inverse Laplace transform, we have

$$y(x,t) = \phi_0(x) + t\,\phi_1(x) + \frac{t^2}{2}\phi_2(x) + L^{-1}\left(\frac{1}{s^3} L\left[\frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^4 y}{\partial x^4}\right] + \frac{1}{s^3} L[F(y)]\right) \quad (17)$$

Now, we apply homotopy perturbation method[18],

$$y(x,t) = \sum_{n=0}^{\infty} p^n y_n(x,t) \quad (18)$$

Where the term y_n are to recursively calculated and the nonlinear term $F(y)$ can be decomposed as

$$F(y) = \sum_{n=0}^{\infty} p^n H_n(y) \quad (19)$$

Here, He's polynomials H_n are given by

$$H_n(y_0, y_1, y_2, \ldots\ldots\ldots\ldots, y_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[F\left(\sum_{i=0}^{\infty} p^i y_i\right)\right]_{p=0}, \quad n = 0,1,2,3,\ldots\ldots\ldots\ldots$$

Substituting Eqs.(18) and (19) in (17), we get

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = \phi_0(x) + t\,\phi_1(x) + \frac{t^2}{2}\phi_2(x) + p\left(L^{-1}\left\{\frac{1}{s^3} L\left[\sum_{n=0}^{\infty} p^n y_n(x,t)\right] + \frac{1}{s^3} L\left[\sum_{n=0}^{\infty} p^n H_n(y)\right]\right\}\right) \quad (20)$$

Comparing the coefficient of like powers of p, we obtained $y_0(x,t), y_1(x,t), y_2(x,t)$.................... Adding all these values, we obtain $y(x,t)$.

4. Examples

Example 4.1. Consider the following differential equation [22]:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)y = -\left(\frac{1}{3}\frac{\partial^2 y}{\partial x^2}\right)^2 + \left(\frac{1}{6}\frac{\partial^2 y}{\partial t^2}\right)^3 - 16y \quad (21)$$

with the following condition:

$$y(x,0) = -x^4, \quad \frac{\partial y}{\partial t}(x,0) = 0, \quad \frac{\partial^2 y}{\partial t^2}(x,0) = 0. \quad (22)$$

The exact solution of above problem is given by

$y(x,t) = -x^4 + 4t^3$.

By applying the aforesaid method subject to the initial conditions, we have

$$y(x,t) = -x^4 + L^{-1}\left(\begin{array}{l} \frac{1}{s^3} L\left[\frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^4 y}{\partial x^4} - 16y\right] \\ + \frac{1}{s^3} L\left[-\frac{1}{9}\left(\frac{\partial^2 y}{\partial x^2}\right)^2 + \frac{1}{216}\left(\frac{\partial^2 y}{\partial t^2}\right)^3\right] \end{array}\right) \quad (23)$$

Now, we apply the homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = -x^4 + p\left(L^{-1}\left[\frac{1}{s^3}\left(L\left[\sum_{n=0}^{\infty} p^n y_n\right]\right) + \frac{1}{s^3}\left(L\left[\sum_{n=0}^{\infty} p^n H_n(y)\right]\right)\right]\right) \quad (24)$$

Where, $H_n(y)$ are He,s polynomials.

Comparing the coefficient of like powers of p, we have

$$p^0: \quad y_0(x,t) = -x^4$$

$$p^1: \quad y_1(x,t) = 4t^3$$

Hence, the solution of $y(x,t)$ is given by

$$y(x,t) = y_0 + y_1 + ...$$
$$= -x^4 + 4t^3$$

Which is the exact solution of the problem.

Example 4.2. Consider the following nonlinear PDE [22]:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)y = y\frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial x} \qquad (25)$$

with the following condition:

$$y(x,0) = \cos x, \quad \frac{\partial y}{\partial t}(x,0) = -\sin x, \quad \frac{\partial^2 y}{\partial t^2}(x,0) = -\cos x. \qquad (26)$$

The exact solution of above problem is given by

$y(x,t) = \cos(x+t)$.

By applying the aforesaid method subject to the initial conditions, we have

$$y(x,t) = \cos x - t\sin x - \frac{t^2}{2!}\cos x + L^{-1}\left(\begin{array}{l}\frac{1}{s^3}L\left[\frac{\partial^3 y}{\partial t\partial x^2} + \frac{\partial^4 y}{\partial x^2\partial t^2} - \frac{\partial^4 y}{\partial x^4}\right] \\ +\frac{1}{s^3}L\left[y\frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial x}\right]\end{array}\right) \qquad (27)$$

Now, we apply the homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = \cos x - t\sin x - \frac{t^2}{2!}\cos x + p\left(L^{-1}\left[\frac{1}{s^3}\left(L\left[\sum_{n=0}^{\infty} p^n y_n\right]\right) + \frac{1}{s^3}\left(L\left[\sum_{n=0}^{\infty} p^n H_n(y)\right]\right)\right]\right) \qquad (28)$$

Where, $H_n(y)$ are He,s polynomials.

Comparing the coefficient of like powers of p, we have

$$p^0: \quad y_0(x,t) = \cos x - t\sin x - \frac{t^2}{2!}\cos x$$

$$p^1: \quad y_1(x,t) = \frac{t^3}{3!}\sin x + \frac{t^4}{4!}\cos x + \frac{t^4}{4!}\sin x + \frac{t^5}{5!}\cos x + \frac{t^4}{4!}\sin^2 x - \frac{t^4}{4!}\cos^2 x + \frac{t^5}{5!}\sin 2x + \frac{3}{6!}t^6\cos^2 x.$$

Hence, the solution of $y(x,t)$ is given by

$$y(x,t) = y_0 + y_1 + y_2 + ...$$
$$= \cos x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - ....\infty\right) - \sin x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - ....\infty\right)$$
$$= \cos x\cos t - \sin x\sin t$$
$$= \cos(x+t)$$

Which is the exact solution of the problem.

5. Conclusions

In this work, we used He-Laplace method for solving nonlinear partial differential parabolic-hyperbolic equations. The results have been approved the efficiency of this method for solving these problems. It is worth mentioning that the He-Laplace method is capable of reducing the volume of the computational work and gives high accuracy in the numerical results.

References

[1]Adomian, G., Solution of physical problems by decomposition, Computers and Mathematics with Applications,2(1994)145-154.
[2]Dehghan,M., Weighted finite difference techniques for the one dimensional advection-diffusion equation, Applied Mathematics and Computation, 147(2004)307-319.
[3]Ganji D.D., Sadighi, A., Application of He's homotopy perturbation method to nonlinear coupled systems of reaction-diffusion equations, International Journal of Nonlinear Sciences and Numerical Simulation 7(2006)411-418.
[4]Ganji, ,D.D. The application of He's homotopy perturbation method to nonlinear equation arising in heat transfer, Physics Letter A 335(2006)337-341.
[5]Ghorbani, ,A., Beyond Adomian's polynomials: He's polynomials, Chaos, Solitons, Fractals, 39(2009)1486-1492.
[6]He J.H., Homotopy perturbation technique, Computer methods in Applied Mechanics and Engineering 178 (1999)257-262.
[7] He, J.H., Homotopy perturbation method: A new nonlinear analytical technique, Applied Mathematics and Computation, 135(2003)73-79.
[8]He,J.H., Homotopy, Comparion of homotopy perturbation method and homotopy analysis method, Applied Mathematics and Computation, 156(2004)527-539.
[9]He,J.H. Homotopy, The homotopy perturbation method for nonlinear oscillators with discontinuities, Applied Mathematics and Computation, 151(2004)287-292.
[10] He,J.H., Recent developments of the homotopy perturbation method, Topological methods in Nonlinear Analysis 31(2008)205-209.
[11]He,J.H., New Interpretation of homotopy perturbation method, International Journal of Modern Physics, 20(2006)2561-2568.
[12]He,J.H.,A coupling method of homotopy technique and a perturbation technique for nonlinear problems, International Journal of Nonlinear Mechanics, 35(2000)37-43.
[13] He,J.H., Variational iteration method for autonomous ordinary differential systems, Applied Mathematics and Computation, 114(2000)115-123.
[14]Hesameddini,E., Latifizadeh,H., An optimal choice of initial solutions in the homotopy perturbation method, International Journal of Nonlinear Sciences and Numerical Simulation 10(2009)1389-1398.
[15]Hesameddini, E., Latifizadeh, H., A new vision of the He's homotopy perturbation method, International Journal of Nonlinear Sciences and Numerical Simulation 10(2009)1415-1424.
[16] Khan,Y., Austin,F., Application of the Laplace decomposition method to nonlinear homogeneous and non-homogeneous advection equations, Zeitschrift fuer Naturforschung, 65 a (2010)1-5.
[17]Lyapunov,A.M.,The General Problem of the Stability of Motion,Taylor & Francis, London,UK,1992,English translation.
[18]Madani, M., Fathizadeh, M., Homotopy perturbation algorithm using Laplace transformation, Nonlinear Science Letters A1 (2010)263-267.
[19]H.K.Mishra, A comparative study of Variational Iteration method and He-Laplace method, Applied Mathematics, 3 (2012)1193-1201.
[20]Mohyud-Din, S.T.,Yildirim, A., Homotopy perturbation method for advection problems, Nonlinear Science Letter A 1(2010)307-312.
[21]Rafei,M., Ganji,D.D., Explicit solutions of helmhotz equation and fifth-order KdV equation using homotopy perturbation method, International Journal of Nonlinear Sciences and Numerical Simulation 7(2006)321-328.
[22]A.Roozi,E.Alibeiki,S.S.Hosseini,S.M.Shafiof,M.Ebrahimi,Homotopy perturbation method for special nonlinear partial differential equations, Journal of King Saud University science,23(2011)99-103.
[23]Saberi-Nadjafi, J., Ghorbani, A., He's homotopy perturbation method: an effective tool for solving nonlinear integral and integro-differential equations, Computers and Mathematics with Applications, 58(2009)1345-1351.
[24]Siddiqui,A.M., Mohmood,R., Ghori,Q.K.., Thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method, International Journal of Nonlinear Sciences and Numerical Simulation 7(2006)7-14.
[25]Sweilam, N.H., Khadar, M.M., Exact solutions of some coupled nonlinear partial differential equations using the homotopy perturbation method, Computers and Mathematics with Applications, 58(2009)2134-2141.
[26]Xu, L., He's homotopy perturbation method for a boundary layer equation in unbounded domain, Computers and Mathematics with Applications, 54(2007)1067-1070.
[27]Yusufoglu, E., Numerical solution of Duffing equation by the Laplace decomposition algorithm, Applied Mathematics and Computation, 177(2006)572-580.