He-Laplace Method for Special Nonlinear Partial Differential Equations

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Abstract
In this article, we consider Cauchy problem for the nonlinear parabolic-hyperbolic partial differential equations. These equations are solved by He-Laplace method. It is shown that, in He-Laplace method, the nonlinear terms of differential equation can be easily handled by the use of He’s polynomials and provides better results.

Keywords: Laplace transform method, Homotopy perturbation method, Partial differential equations, He’s polynomials.

AMS Subject classification: 35G10; 35G15; 35G25; 35G30; 74S30.

1. Introduction
Nonlinearity exists everywhere and nature is nonlinear in general. The search for a better and easy to use tool for the solution of nonlinear equations that illuminate the nonlinear phenomena of real life problems of science and engineering has recently received a continuing interest. Various methods, therefore, were proposed to find approximate solutions of nonlinear equations. Some of the classical analytic methods are Lyapunov’s artificial small parameter method [17], perturbation techniques [6,23,22,25]. The Laplace decomposition method have been used to solve nonlinear differential equations [1-4, 16, 19, 20, 27]. J.H.He developed the homotopy perturbation method (HPM) [6-13,14-15,21,24,26] by merging the standard homotopy and perturbation for solving various physical problems. Furthermore, the homotopy perturbation method is also combined with the well-known Laplace transformation method [18] which is known as Laplace homotopy perturbation method.

In this paper, the main objective is to solve partial differential equations by using He-Laplace method. It is worth mentioning that He-Laplace method is an elegant combination of the Laplace transformation, the homotopy perturbation method and He’s polynomials. The use of He’s polynomials in the nonlinear term was first introduced by Ghorbani [5, 23]. This paper contains basic idea of homotopy perturbation method in section 2, He-Laplace method in section 3, examples in section 4 and conclusions in section 5 respectively.

2. Basic idea of homotopy perturbation method

Consider the following nonlinear differential equation

\[ A(y) - f(r) = 0, \quad r \in \Omega \] (1)

with the boundary conditions of

\[ B\left( y, \frac{\partial y}{\partial n} \right) = 0, \quad r \in \Gamma, \] (2)

where A, B, f(r) and \( \Gamma \) are a general differential operator, a boundary operator, a known analytic function and the boundary of the domain \( \Omega \), respectively.

The operator A can generally be divided into a linear part L and a nonlinear part N. Eq. (1) may therefore be written as:

\[ L(y) + N(y) - f(r) = 0, \] (3)

By the homotopy technique, we construct a homotopy \( v(r, p) : \Omega \times [0,1] \rightarrow R \) which satisfies:

\[ H(v, p) = (1 - p)[L(v) - L(y_0)] + p[A(v) - f(r)] = 0 \] (4)

\[ H(v, p) = L(v) - L(y_0) + pL(y_0) + p[N(v) - f(r)] = 0 \] (5)

where \( p \in [0,1] \) is an embedding parameter, while \( y_0 \) is an initial approximation of Eq.(1), which satisfies the boundary conditions. Obviously, from Eqs.(4) and (5) we will have:
The changing process of \( p \) from zero to unity is just that of \( p r v \), from \( 0 \) to \( 1 \). In topology, this is called deformation, while \( L(v) - L(y_0) \) and \( A(v) - f(r) \) are called homotopy. If the embedding parameter \( p \) is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of Eqs.(4) and (5) can be written as a power series in \( p \):

\[
H(v,0) = L(v) - L(y_0) = 0, \quad H(v,1) = A(v) - f(r) = 0, \tag{6}
\]

\[
\lim_{p \to 0} v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots \to \infty \tag{7}
\]

Setting \( p = 1 \) in Eq.(8), we have

\[
y = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{9}
\]

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The series (9) is convergent for most cases. However, the convergent rate depends on the nonlinear operator \( A(v) \). Moreover, He [6] made the following suggestions:

1. The second derivative of \( N(v) \) with respect to \( v \) must be small because the parameter may be relatively large, i.e. \( p \to 1 \).
2. The norm of \( L^1 \left( \frac{\partial N}{\partial v} \right) \) must be smaller than one so that the series converges.

3. **He-Laplace method**

Consider the following Cauchy problem for the nonlinear parabolic-hyperbolic differential equation:

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{\partial^2}{\partial t^2} - \Delta \right) y = F(y), \tag{10}
\]

with initial conditions

\[
\frac{\partial^k y}{\partial t^k} (0, X) = \phi_k(X), \quad X = (x_1, x_2, \ldots, x_i), k = 0, 1, 2, 3, \ldots \tag{11}
\]

where the nonlinear term is represented by \( F(y) \), and \( \Delta \) is the Laplace operator in \( R^n \). We rewrite the eqn(10) as follows:

\[
\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) y = F(y),
\]

or

\[
\frac{\partial^3 y}{\partial t^3} - \frac{\partial^3 y}{\partial x^3} - \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\partial^4 y}{\partial x^4} = F(y). \tag{12}
\]

Applying the laplace transform of both sides of (12), we have

\[
L \left[ \frac{\partial^3 y}{\partial t^3} \right] = L \left[ \frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^3 \partial t^2} - \frac{\partial^4 y}{\partial x^4} \right] + L[F(y)] \tag{13}
\]

\[
s^3 L[y(x,t)] - s^2 y(x,0) - s y'(x,0) - y''(x,0) = L \left[ \frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^3 \partial t^2} - \frac{\partial^4 y}{\partial x^4} \right] + L[F(y)] \tag{14}
\]

Using initial conditions (11) in (14), we have

\[
s^3 L[y(x,t)] - s^2 \phi_0(x) - s \phi_1(x) - \phi_2(x) = L \left[ \frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^3 \partial t^2} - \frac{\partial^4 y}{\partial x^4} \right] + L[F(y)] \tag{15}
\]
Now, we apply the homotopy perturbation method, we have

\[ L[y(x,t)] = \frac{\phi_0(x)}{s} + \frac{\phi_1(x)}{s^2} + \frac{\phi_2(x)}{s^3} + \frac{1}{s^3} \left[ \frac{\partial^4 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t} - \frac{\partial^4 y}{\partial x^4} \right] + \frac{1}{s^3} L[F(y)] \]  

(16)

Taking inverse Laplace transform, we have

\[ y(x,t) = \phi_0(x) + t \phi_1(x) + \frac{t^2}{2} \phi_2(x) + L^{-1} \left[ \frac{1}{s^3} L \left[ \frac{\partial^4 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t} - \frac{\partial^4 y}{\partial x^4} \right] + \frac{1}{s^3} L[F(y)] \right] \]  

(17)

Now, we apply homotopy perturbation method[18],

\[ y(x,t) = \sum_{n=0}^{\infty} p^n y_n(x,t) \]  

(18)

Where the term \( y_n \) are to recursively calculated and the nonlinear term \( F(y) \) can be decomposed as

\[ F(y) = \sum_{n=0}^{\infty} p^n H_n(y) \]  

(19)

Here, He’s polynomials \( H_n \) are given by

\[ H_n(y_0, y_1, y_2, \ldots \ldots \ldots, y_n) = \frac{\partial^n}{\partial p^n} \left[ F \left( \sum_{i=0}^{\infty} p^i y_i \right) \right] \quad , \quad n = 0, 1, 2, 3, \ldots \ldots \ldots \]

Substituting Eqs.(18) and (19) in (17), we get

\[ \sum_{n=0}^{\infty} p^n y_n(x,t) = \phi_0(x) + t \phi_1(x) + \frac{t^2}{2} \phi_2(x) + p \left[ L^{-1} \left\{ \frac{1}{s^3} L \left[ \sum_{n=0}^{\infty} p^n y_n(x,t) \right] \right\} + \frac{1}{s^3} L \left[ \sum_{n=0}^{\infty} p^n H_n(y) \right] \right] \]  

(20)

Comparing the coefficient of like powers of \( p \), we obtained \( y_0(x,t), y_1(x,t), y_2(x,t) \) \ldots \ldots \ldots \ldots \ldots. \) Adding all these values, we obtain \( y(x,t) \).

4. Examples

Example 4.1. Consider the following differential equation [22]:

\[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right)^2 y = \left( \frac{1}{3} \frac{\partial^2 y}{\partial x^2} \right)^2 + \left( \frac{1}{6} \frac{\partial^2 y}{\partial t^2} \right)^3 - 16y \]  

(21)

with the following condition:

\[ y(x,0) = -x^4, \quad \frac{\partial y}{\partial t}(x,0) = 0, \quad \frac{\partial^2 y}{\partial t^2}(x,0) = 0. \]  

(22)

The exact solution of above problem is given by \( y(x,t) = -x^4 + 4t^3 \).

By applying the aforesaid method subject to the initial conditions, we have

\[ y(x,t) = -x^4 + L^{-1} \left\{ \frac{1}{s^3} L \left[ \frac{\partial^4 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t} - \frac{\partial^4 y}{\partial x^4} \right] + \frac{1}{s^3} L \left[ -\frac{1}{9} \left( \frac{\partial^2 y}{\partial x^2} \right)^2 + \frac{1}{216} \left( \frac{\partial^2 y}{\partial t^2} \right)^3 \right] \right\} \]  

(23)

Now, we apply the homotopy perturbation method, we have

\[ \sum_{n=0}^{\infty} p^n y_n(x,t) = -x^4 + p \left[ L^{-1} \left\{ \frac{1}{s^3} L \left[ \sum_{n=0}^{\infty} p^n y_n \right] \right\} + \frac{1}{s^3} \left( L \left[ \sum_{n=0}^{\infty} p^n H_n(y) \right] \right) \right] \]  

(24)

Where, \( H_n(y) \) are He’s polynomials.

Comparing the coefficient of like powers of \( p \), we have

\[ 115 \]
\( p^0 : \quad y_0(x,t) = -x^4 \)
\( p^1 : \quad y_1(x,t) = 4t^3 \)

Hence, the solution of \( y(x,t) \) is given by
\[
y(x,t) = y_0 + y_1 + \ldots \]
\[= -x^4 + 4t^3 \]

Which is the exact solution of the problem.

**Example 4.2.** Consider the following nonlinear PDE [22]:
\[
\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) y = y \frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2} \frac{\partial y}{\partial x} \quad (25)
\]

with the following condition:
\[
y(x,0) = \cos x, \quad \frac{\partial y}{\partial t}(x,0) = -\sin x, \quad \frac{\partial^2 y}{\partial t^2}(x,0) = -\cos x. \quad (26)
\]

The exact solution of above problem is given by
\[
y(x,t) = \cos(x + t). \quad (27)
\]

By applying the aforesaid method subject to the initial conditions, we have
\[
y(x,t) = \cos x - t \sin x - \frac{t^2}{2!} \cos x + L^{-1} \left( \frac{1}{s^3} L \left[ \frac{\partial^3 y}{\partial t^3} + \frac{\partial^4 y}{\partial t^4} - \frac{\partial^5 y}{\partial t^5} \right] \right)
\]
\[
+ \frac{1}{s^2} L \left[ y \frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2} \frac{\partial y}{\partial x} \right] \quad (28)
\]

Now, we apply the homotopy perturbation method, we have
\[
\sum_{n=0}^{\infty} p^n y_n(x,t) = \cos x - t \sin x - \frac{t^2}{2!} \cos x + p \left( L^{-1} \left[ \frac{1}{s^3} L \left( \sum_{n=0}^{\infty} p^n y_n \right) \right] + \frac{1}{s^2} L \left[ \sum_{n=0}^{\infty} p^n H_n(y) \right] \right)
\]

Where, \( H_n(y) \) are He,s polynomials.

Comparing the coefficient of like powers of \( p \), we have
\[
p^0 : \quad y_0(x,t) = \cos x - t \sin x - \frac{t^2}{2!} \cos x
\]
\[
p^1 : \quad y_1(x,t) = \frac{t^3}{3!} \sin x + \frac{t^4}{4!} \cos x + \frac{t^4}{4!} \sin x + \frac{t^5}{5!} \cos x + \frac{t^4}{4!} \sin x - \frac{t^4}{4!} \cos x + \frac{t^5}{5!} \sin 2x + \frac{3}{6!} t^6 \cos^2 x .
\]

Hence, the solution of \( y(x,t) \) is given by
\[
y(x,t) = y_0 + y_1 + y_2 + \ldots \]
\[= \cos x \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots \right) - t \sin x \left( \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots \right)
\]
\[= \cos(x + t)
\]

Which is the exact solution of the problem.

5. Conclusions

In this work, we used He-Laplace method for solving nonlinear partial differential parabolic-hyperbolic equations. The results have been approved the efficiency of this method for solving these problems. It is worth mentioning that the He-Laplace method is capable of reducing the volume of the computational work and gives high accuracy in the numerical results.

References
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