Some Fixed Point Theorem for Expansive Type Mapping in Dislocated Metric Space

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Abstract

The purpose of this paper is to present some fixed point theorem in dislocated quasi metric space for expansive type mappings.

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Introduction and Preliminaries:

It is well known that Banach Contraction mappings principle is one of the pivotal results of analysis. Generalizations of this principle have been obtained in several directions .Dass and Gupta [1] generalized Banach's Contraction principle in metric space. Also Rhoades [2] established a partial ordering for various definitions of contractive mappings. In 2005, Zeyada Salunke [4] proved some results on fixed point in dislocated quasimetric spaces. In 2008, Aage and Salunke [4] proved some results on fixed point theorem in dislocated quasimetric spaces. Recently, Isufati [5], proved fixed point theorem for contractive type condition with rational expression in dislocated quasimetric spaces. The following definitions will be needed in the sequel.

Definition 1.1(See [3]). Let X be a nonempty set, and let $d: X \times X \rightarrow [0,\infty)$ be a function, called a distance function. One needs the following conditions:

(M1) d(x, x) = 0,

(M2) d(x, y) = d(y, x) = 0, then x = y

(M3) d(x, y) = d(y, x),

(M4) $d(x, y) \le d(x, z) + d(z, y)$,

(M4) $d(x, y) \le \max\{d(x, z), d(z, y)\}$, for all $x, y, z \in X$.

If d satisfies conditions (M1)-(M4), then it is called a metric on X. If d satisfies conditions (M1), (M2), and (M4), it is called a quasimetric on X. If it satisfies conditions (M2)-(M4) ((M2) and (M4)), it is called a dislocated metric (or simply d-metric) (a dislocated quasimetric (or simply dq-metric)) on X, respectively. If a metric d satisfies the strong triangle inequality (M)['], then it is called an ultrametric.

Definition 1.2 (See [3]). A sequence $\{x_n\}_{n \in \mathbb{N}}$ in dq-metric space (dislocated quasimetric space) (X, d) is called a Cauchy sequence if , for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon$, that is , $\min\{d(x_m, x_n), d(x_n, x_m)\} < \varepsilon$ for all $m, n \ge n_0$.

Definition 1.3 (See [3]). A sequence $\{x_n\}_{n \in N}$ in dq-metric space [d-metric space] is said to be d-converge to $x \in X$ provided that

$$\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0$$
(1.1)

In this case, x is called a dq-limit [d-limit] of $\{x_n\}$ and we write $x_n \to x$.

Definition 1.4 (See [3]). A dq-metric space (X, d) is called complete if every Cauchy sequence in it is a dq-convergent.

Main Results

In this paper, we prove some fixed point theorem for continuous mapping satisfying expansion condition in complete dq-metric space.

Theorem 2.1: Let (X, d) be a complete dislocated metric space and T a continuous mappings satisfying the following condition:

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$$d(Tx, Ty) + \alpha \left[\frac{d(x, Ty) + d(y, Tx)}{1 + d(x, Ty) d(y, Tx)} \right] \ge \beta \frac{d(x, Tx)[1 + d(y, Ty)]}{1 + d(x, y)} + \gamma d(x, y)$$
(2.1)

For all $x, y \in X$, $x \neq y$, where $\alpha, \beta, \gamma \ge 0$ are real constants and $\beta + \gamma > 1 + 2\alpha$, $\gamma > 1 + \alpha$. Then T has a fixed point in X.

Proof: Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n \in N}$ as follows and $Tx_n = x_{n-1}$ for n = 1, 2, 3..... Then, using (2.1) we obtain

$$d(Tx_{n+1}, Tx_{n+2}) + \alpha \left[\frac{d(x_{n+1}, Tx_{n+2}) + d(x_{n+2}, Tx_{n+1})}{1 + d(x_{n+1}, Tx_{n+2}) d(x_{n+2}, Tx_{n+1})} \right] \ge \beta \frac{d(x_{n+1}, Tx_{n+1}) \left[1 + d(x_{n+1}, Tx_{n+2}) \right]}{1 + d(x_{n+1}, x_{n+2})} + \gamma d(x_{n+1}, x_{n+2})$$

$$\Rightarrow d(x_{n}, x_{n+1}) + \alpha \left[\frac{d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_{n})}{1 + d(x_{n+1}, x_{n+1}) \cdot d(x_{n+2}, x_{n})} \right] \ge \beta \frac{d(x_{n+1}, x_{n})[1 + d(x_{n+2}, x_{n+1})]}{1 + d(x_{n+1}, x_{n+2})} + \gamma d(x_{n+1}, x_{n+2})$$

$$\Rightarrow d(x_{n}, x_{n+1}) + \alpha d(x_{n+2}, x_{n}) \ge \beta d(x_{n}, x_{n+1}) + \gamma d(x_{n+1}, x_{n+2})$$

$$\Rightarrow d(x_{n}, x_{n+1}) + \alpha d(x_{n}, x_{n+1}) + \alpha d(x_{n+1}, x_{n+2}) \ge \beta d(x_{n}, x_{n+1}) + \gamma d(x_{n+1}, x_{n+2})$$

$$\Rightarrow (1 + \alpha - \beta) d(x_{n}, x_{n+1}) \ge (\gamma - \alpha) d(x_{n+1}, x_{n+2})$$
The last inequality gives
$$(1 + \alpha - \beta) d(x_{n}, x_{n+1}) \ge (\gamma - \alpha) d(x_{n+1}, x_{n+2})$$

$$d(x_{n+1}, x_{n+2}) \leq \left(\frac{1+\alpha-\beta}{\gamma-\alpha}\right) d(x_n, x_{n+1})$$

$$\leq k d(x_n, x_{n+1})$$
(2.2)

Where $k = \frac{(1 + \alpha - \beta)}{(\gamma - \alpha)} < 1$. Hence by induction, we obtain

$$d(x_{n+1}, x_{n+2}) \le k^{n+1} d(x_0, x_1)$$

Note that, for $m, n \in N$ such that m > n we have

$$d(x_{m}, x_{n}) \leq d(x_{m}, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n+1}, x_{n})$$

$$\leq \left[k^{m-1} + k^{m-2} + \dots + k^{n}\right] d(x_{0}, x_{1})$$

$$\leq k^{n} (1 + k + k^{2} + \dots + k^{m-n-1}) d(x_{0}, x_{1})$$

$$\leq k^{n} \sum_{r=0}^{\infty} k^{r} d(x_{0}, x_{1})$$

$$= \frac{k^{n}}{1 - k} d(x_{0}, x_{1})$$
(2.3)

Since $0 \le k < 1$, then as $n \to \infty$, $k^n (1-k)^{-1} \to 0$. Hence, $d(x_m, x_n) \to 0$ as $m, n \to \infty$. This forces that $\{x_n\}_{n\in N}$ is a Cauchy sequence in X. But X is a complete dislocated metric space; hence, $\{x_n\}_{n\in N}$ is dconverges. Call the d-limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to \infty$. By continuity of T we have,

$$Tx^{*} = T\left(d - \lim_{n \to \infty} x_{n}\right) = d - \lim_{n \to \infty} Tx_{n} = d - \lim_{n \to \infty} x_{n-1} = x^{*}$$
(2.4)

That is, $Tx^* = x^*$; thus , T has a fixed point in X . Uniqueness

Let y^* be another fixed point of T in X, then $Ty^* = y^*$ and $Tx^* = x^*$.now,

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$$d(Tx^{*}, Ty^{*}) + \alpha \left[\frac{d(x^{*}, Ty^{*}) + d(y^{*}, Tx^{*})}{1 + d(x^{*}, Ty^{*})d(y^{*}, Tx^{*})} \right] \ge \beta \frac{d(x^{*}, Tx^{*})[1 + d(y^{*}, Ty^{*})]}{1 + d(x^{*}, y^{*})} + \gamma d(x^{*}, y^{*})$$
(2.5)
This implies that

$$d(x^{*}, y^{*}) + \alpha \left[\frac{d(x^{*}, y^{*}) + d(y^{*}, x^{*})}{1 + d(x^{*}, y^{*})d(y^{*}, x^{*})} \right] \ge \beta \frac{d(x^{*}, x^{*})[1 + d(y^{*}, y^{*})]}{1 + d(x^{*}, y^{*})} + \gamma d(x^{*}, y^{*})$$

$$\Rightarrow d(x^{*}, y^{*}) + \frac{2\alpha d(x^{*}, y^{*})}{1 + [d(x^{*}, y^{*})]^{2}} \ge \gamma d(x^{*}, y^{*})$$

$$\Rightarrow d(x^{*}, y^{*}) + [d(x^{*}, y^{*})]^{3} + 2\alpha d(x^{*}, y^{*}) \ge \gamma d(x^{*}, y^{*}) + \gamma [d(x^{*}, y^{*})]^{3}$$

$$\Rightarrow (1 + 2\alpha - \gamma)d(x^{*}, y^{*}) \ge (\gamma - 1)[d(x^{*}, y^{*})]^{3}$$

$$d(x^{*}, y^{*}) \le \left(\frac{1 + 2\alpha - \gamma}{\gamma - 1}\right)^{\frac{1}{3}}d(x^{*}, y^{*})$$
(2.6)

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This is true only when $d(x^*, x^*) = 0$. Similarly $d(y^*, x^*) = 0$. Hence $d(x^*, y^*) = d(y^*, x^*) = 0$ and so $x^* = y^*$. Hence, T has a unique fixed point in X.

Theorem 2.2: Let (X, d) be a complete dislocated metric space and T a sujective mapping satisfying the condition (2.1) for all $x, y \in X, x \neq y$, where $\alpha, \beta, \gamma \ge 0$ are real constants and $\beta + \gamma > 1 + 2\alpha$, $\gamma > 1 + \alpha$. Then, T has a fixed point in X.

Proof : Choose $x_0 \in X$ to be arbitrary , and define the iterative sequence $\{x_n\}_{n \in N}$ as follows : $Tx_n = x_{n-1}$ for $n = 1, 2, 3, \dots$. Then, using (2.1), we obtain, sequence $\{x_n\}_{n \in N}$ is a Cauchy sequence in X.But X is a complete dislocated metric space; hence $\{x_n\}_{n\in N}$ is a d-converges. Call the d-limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to \infty$.

Existence of fixed point

Since T is a surjective map, so there exists a point y in X, such that x = Ty. Consider $d(\mathbf{r} \cdot \mathbf{r}) - d(T_{\mathbf{r}} \cdot T_{\mathbf{r}})$

$$\begin{aligned} a(x_{n}, x) &= d(Ix_{n+1}, Iy) \\ &\geq -\alpha \bigg[\frac{d(x_{n+1}, Ty) + d(y, Tx_{n+1})}{1 + d(x_{n+1}, Ty) \cdot d(y, Tx_{n+1})} \bigg] + \beta \frac{d(x_{n+1}, Tx_{n+1})[1 + d(y, Ty)]}{1 + d(x_{n+1}, y)} + \gamma d(x_{n+1}, y) \end{aligned}$$
Taking $n \to \infty$, we get
$$(2.7)$$

Taking $n \to \infty$, we get

$$d(x,x) \ge -\alpha \left[\frac{d(x,x) + d(y,x)}{1 + d(x,x)d(y,x)} \right] + \beta \frac{d(x,x)[1 + d(y,x)]}{1 + d(x,y)} + \gamma d(x,y)$$

$$0 \ge -\alpha d(x,y) + \gamma d(x,y)$$

$$\Rightarrow (\gamma - \alpha)d(x,y) \le 0$$

$$\Rightarrow d(x,y) = 0 \text{ as } \gamma > \alpha$$
Similarly, $d(y,x) = 0$ Hence, $d(x,y) = d(y,x) = 0$
(2.8)

Similarly, d(y,x) = 0. Hence d(x,y) = d(y,x) = 0This implies x = y and so Tx = x, that is x is fixed point of T.

Uniqueness

Let
$$y^*$$
 be another fixed point of T in X , then $Ty^* = y^*$ and $Tx^* = x^*$. Now,
 $d(Tx^*, Ty^*) + \alpha \left[\frac{d(x^*, Ty^*) + d(y^*, Tx^*)}{1 + d(x^*, Ty^*) d(y^*, Tx^*)} \right] \ge \beta \frac{d(x^*, Tx^*)[1 + d(y^*, Ty^*)]}{1 + d(x^*, y^*)} + \gamma d(x^*, y^*)$

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This implies that

$$d(x^{*}, y^{*}) + \alpha \left[\frac{d(x^{*}, y^{*}) + d(y^{*}, x^{*})}{1 + d(x^{*}, y^{*}) d(y^{*}, x^{*})} \right] \ge \beta \frac{d(x^{*}, x^{*})[1 + d(y^{*}, y^{*})]}{1 + d(x^{*}, y^{*})]} + \gamma d(x^{*}, y^{*})$$

$$\Rightarrow d(x^{*}, y^{*}) + \frac{2\alpha d(x^{*}, y^{*})}{1 + [d(x^{*}, y^{*})]^{2}} \ge \gamma d(x^{*}, y^{*})$$

$$\Rightarrow d(x^{*}, y^{*}) + [d(x^{*}, y^{*})]^{3} + 2\alpha d(x^{*}, y^{*}) \ge \gamma d(x^{*}, y^{*}) + \gamma [d(x^{*}, y^{*})]^{2}$$

$$\Rightarrow (1 - 2\alpha - \gamma) d(x^{*}, y^{*}) \ge (\gamma - 1)[d(x^{*}, y^{*})]^{3}$$

$$\Rightarrow d(x^{*}, y^{*}) \le \left(\frac{1 + 2\alpha - \gamma}{\gamma - 1}\right)^{\frac{1}{3}} d(x^{*}, y^{*})$$
This is true only when $d(x^{*}, y^{*}) = 0$. Similarly $d(y^{*}, x^{*}) = 0$. Hence $d(x^{*}, y^{*}) = d(y^{*}, x^{*}) = 0$ and

This is true only when d(x, y) = 0. Similarly, d(y, x) = 0. Hence d(x, y) = d(y, x) = 0 and so $x^* = y^*$. Hence T has a unique fixed point in X.

The proof is completed.

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