Decomposition of Continuity and Separation Axioms Via Lower and Upper Approximation

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Abstract
In this paper we study the rough set theory by defined the concepts of rough regularity and rough normality in the topological spaces which we can consider them as results from the general relations on the approximation spaces.

Keywords: Topologized approximation space, rough regularity, rough normality, rough continuity, rough homeomorphism.

1. Introduction
In [3] Pawlak introduced approximation spaces during the early 1980s as part of his research on classifying objects by means their feature. In [1] rough set theory introduced by Pawlak in 1982, as an extension of set theory, mainly in the domain of intelligent systems. In [4,5] M. Jamal and N. Duc rough set theory as a mathematical tool to deal with vagueness and incomplete information data or imprecise by dividing these data into equivalence classes using equivalence relations which result from the same data. This paper study the rough set theory by defined the concepts of rough regularity and rough normality in the topological spaces which are results from the general relations on the approximation spaces.

2. Preliminaries
In [4] Pawlak noted that the approximation space \( \kappa = (X, R) \) with equivalence relation \( R \) defined a uniquely topological space \( (X, \tau_X) \) where \( \tau_X \) is the family of all clopen sets in \( (X, \tau_X) \) and \( X/R \) is a base of \( \tau_X \). Moreover the lower (resp. upper) approximation of any subset \( A \subseteq X \) is exactly the interior (resp. closure) of the subset \( A \) . In this section we shall generalize Pawlak’s concepts to the case of general relations. Hence the approximation space \( \kappa = (X, R) \) with general relation \( R \) defines a uniquely topological space \( (X, \tau_X) \) where \( \tau_X \) is the topology associated to \( \kappa \) (i.e. \( \tau_X \) is the family of all open sets in \( (X, \tau_X) \) and \( X/R = \{xR : x \in X\} \) is a subbase of \( \tau_X \), where \( xR = \{y \in X : yRx\} \)). We give this hypothesis in the following definition.

Definition 2.1 [4]. Let \( \kappa = (X, R) \) be an approximation space with general relation \( R \) and \( \tau_X \) is the topology associated to \( \kappa \). Then the triple \( K = (X, R, \tau_X) \) is called a topologized approximation space.

The following definition introduces the lower and the upper approximations in a topologized approximation space \( K = (X, R, \tau_X) \).

Definition 2.2 [4]. Let \( K = (X, R, \tau_X) \) be a topologized approximation space and \( A \subseteq X \). The lower approximation (resp. upper approximation) of \( A \) is defined by

\[
\overline{R}A = A^- \quad \text{where} \quad A^- = \bigcup \{ G \subseteq X : G \subseteq A \text{ and } G \in \tau \}
\]

(resp. \( \overline{R}A = A^- \quad \text{where} \quad A^- = \bigcap \{ F \subseteq X : A \subseteq F \text{ and } F \in \tau \} \)).

In the following proposition from [4] we introduce some properties of the lower and upper approximations of a set \( A \).

Proposition 2.3 [4]. Let \( K = (X, R, \tau_X) \) be a topologized approximation space. If \( A \) and \( B \) are two subsets of \( X \), then

1) \( \overline{R}A \subseteq A \subseteq \overline{R}A \).
2) \( \overline{R}\emptyset = \emptyset \) and \( \overline{R}X = X \).
3) \( \overline{R}(A \cup B) = \overline{R}A \cup \overline{R}B \).
4) \( \overline{R}(A \cap B) = \overline{R}A \cap \overline{R}B \).
5) If \( A \subseteq B \), then \( \overline{R}A \subseteq \overline{R}B \).
6) If \( A \subseteq B \), then \( \overline{R}A \subseteq \overline{R}B \).
7) \( \overline{R}(A \cup B) = \overline{R}_{\overline{R}A \cup \overline{R}B} \).
8) \( \overline{R}(A \cap B) \subseteq \overline{R}A \cap \overline{R}B \).
9) \( \overline{R}(A)^c = [\overline{R}A]^c \).
10) \( \overline{R}(A)^c = [R(A)]^c \).
11) \( R(RA) = RA \).
12) \( \overline{R} \overline{R}A = \overline{RA} \).

**Definition 2.4.** Let \( K = (X, R, \tau_R) \) be a topologized approximation space and \( x \in X \). A subset \( N \) of \( X \) is said to be rough closed neighborhood of \( x \) iff, there exists a subset \( G \) of \( X \) such that \( x \in G \subseteq \overline{RG} \subseteq N \).

**Theorem 2.5.** Let \( K = (X, R, \tau_R) \) be a topologized approximation space and \( G \subseteq X \), then \( G = \overline{RG} \) iff, \( G \) a rough closed neighborhood of each of its points.

**Proof.** Let \( G = \overline{RG} \) and \( x \in G \), then \( x \in \overline{RG} \subseteq G \) and \( G \subseteq G \), therefore \( G \) a rough closed neighborhood of each of its points.

**Conversely.** Let \( G \) be a rough closed neighborhood of each of its points. Let \( x \in G \). Then there exists \( N_x \subseteq G \) such that \( x \in \overline{R}N_x \subseteq G \). Therefore \( G = \bigcup_{x \in G} \overline{R}N_x = \overline{R} \bigcup_{x \in G} N_x = \overline{RG} \), because \( \forall x \in G \exists N_x \subseteq G \), so \( \bigcup_{x \in G} N_x = G \). Hence \( G = \overline{RG} \).

**Definition 2.6.** Let \( K = (X, R, \tau_R) \) be a topologized approximation space and \( Y \subseteq X \). Then \( Q = (Y, R, \sigma_Q) \) where \( \sigma_Q = \{ G \cap Y | G \in \tau_R \} \) is a topologized approximation space of \( Y \), called the relative topologized approximation space for \( Y \). The fact that a subset of \( X \) is being given this topologized approximation space is signified by referring to it as a subspace of \( X \).

**Theorem 2.7.** Let \( Q = (Y, R, \sigma_Q) \) be a subspace of a topologized approximation space \( K = (X, R, \tau_R) \), then:

i) If \( H \subseteq Y \) then \( \overline{R}_Y H = H \) iff \( H = G \cap Y \) where \( \overline{R}_Y G = G \).
ii) If \( F \subseteq Y \) then \( \overline{R}_Y F = F \) iff \( F = B \cap Y \) where \( \overline{R}_Y B = B \).
iii) if \( A \subseteq Y \), then \( \overline{R}_Y A = Y \cap \overline{R}_X A \).

**Proof.** By definitions of subspace of a topologized approximation space and upper approximation, the proof is obvious.

**Theorem 2.8.** Let \( Q = (Y, R, \sigma_Q) \) be a subspace of a topologized approximation space \( K = (X, R, \tau_R) \). Then for \( A, B \subseteq Y \) we have:

i) If \( \overline{R}_Y A = A \) and \( \overline{R}_Y Y = Y \) then \( \overline{R}_X A = A \).
ii) If \( \overline{R}_Y B = B \) and \( \overline{R}_Y Y = Y \) then \( \overline{R}_X B = B \).

**Proof.** By Theorem 2.7 the proof is obvious.

**Definition 2.9.** Let \( K = (X, R_1, \tau_R) \) and \( Q = (Y, R_2, \sigma_Q) \) be two topologized approximation spaces. Then a mapping \( f: K \rightarrow Q \) is said to be rough continuous at a point \( x \) of \( X \) iff, for each subset \( V \) contains \( f(x) \) in \( Y \), there exists a subset \( U \) contains \( x \) in \( X \) such that \( f(\overline{R}_1 U) \subseteq \overline{R}_2 V \). The mapping \( f \) is said to a rough continuous iff it is rough continuous at every point of \( X \).

**Theorem 2.10.** Let \( K = (X, R_1, \tau_R) \) and \( Q = (Y, R_2, \sigma_Q) \) be two topologized approximation spaces and \( f: K \rightarrow Q \) be a mapping, then the following statements are equivalent:

i) \( f \) is rough continuous.
ii) For each subset \( F \) of \( Y \), \( \overline{R}_2 f^{-1}(\overline{R}_2 F) = f^{-1}(\overline{R}_2 F) \).
iii) For each subset \( E \) of \( X \), \( f(\overline{R}_2 E) \subseteq \overline{R}_2 f(E) \).
iv) For each subset \( B \) of \( Y \), \( \overline{R}_2 f^{-1}(B) \subseteq f^{-1}(\overline{R}_2 B) \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( F \) be a set in \( Y \). We are going to prove that \( \overline{R}_2 f^{-1}(\overline{R}_2 F) = f^{-1}(\overline{R}_2 F) \). For this purpose, let \( p \) be a point in \( f^{-1}(\overline{R}_2 F) \). Then \( f(p) \) is a point in \( \overline{R}_2 F \). Since \( f \) is rough continuous at the point \( p \), there exits a subset \( U \) of \( X \) such that \( p \in U \) and \( f(\overline{R}_1 U) \subseteq \overline{R}_2 [\overline{R}_2 F] \), then by (12) of Proposition 2.3 we have \( f(\overline{R}_1 U) \subseteq \overline{R}_2 F \). This implies that \( \overline{R}_1 U \subseteq f^{-1}(\overline{R}_2 F) \). By Theorem 2.5, it follows that for each subset \( F \) of \( Y \), \( \overline{R}_1 f^{-1}(\overline{R}_2 F) = f^{-1}(\overline{R}_2 F) \).
(ii) $\iff$ (iii) (1) of Proposition 2.3 implies $f(E) \subseteq \overline{R}_2 f(E)$. Then $E \subseteq f^{-1}(\overline{R}_2 f(E))$, thus again by (5) of Proposition 2.3 we have $\overline{R}_1 E \subseteq \overline{R}_1 f^{-1}(\overline{R}_2 f(E))$. Then from (ii), we have $\overline{R}_1 f^{-1}(\overline{R}_2 f(E)) = f^{-1}(\overline{R}_2 f(E))$, therefore $\overline{R}_1 E \subseteq f^{-1}(\overline{R}_2 f(E))$. Hence $f(\overline{R}_1 E) \subseteq \overline{R}_2 f(E)$.

(iii) $\iff$ (iv). Let $B$ be a subset of $Y$. Then by (iii), we have $f \left( \overline{R}_1 f^{-1}(B) \right) \subseteq \overline{R}_2 f \left( f^{-1}(B) \right) \subseteq \overline{R}_2 B$, therefore $f \left( \overline{R}_1 f^{-1}(B) \right) \subseteq \overline{R}_2 B$. Hence $\overline{R}_1 f^{-1}(B) \subseteq f^{-1}(\overline{R}_2 B)$.

(iv) $\iff$ (i). Let $p$ be a point of $X$ and let $V$ be a subset of $Y$ such that $f(p) \in V$. Our hypothesis (iv) and (12) of Proposition 2.3 lead to $\overline{R}_1 f^{-1}(\overline{R}_2 V) \subseteq f^{-1}(\overline{R}_2 \overline{R}_2 V) = f^{-1}(\overline{R}_2 V)$. So
\[
\overline{R}_1 f^{-1}(\overline{R}_2 V) \subseteq f^{-1}(\overline{R}_2 V).
\]

On the other hand (1) of Proposition 2.3 implies
\[
f^{-1}(\overline{R}_2 V) \subseteq \overline{R}_1 f^{-1}(\overline{R}_2 V).
\]

From (1) and (2) we obtain
\[
\overline{R}_1 f^{-1}(\overline{R}_2 V) = f^{-1}(\overline{R}_2 V).
\]

Then $p \in \overline{R}_1 f^{-1}(\overline{R}_2 V)$ and $f \left( \overline{R}_1 f^{-1}(\overline{R}_2 V) \right) \subseteq \overline{R}_2 V$, therefore $f$ is a rough continuous at $p$. Hence $f$ is a rough continuous.

**Definition 2.11.** Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_q)$ be two topologized approximation spaces. Then a mapping $f: K \rightarrow Q$ is said to be a rough closed mapping if $f(\overline{R}_1 F) = \overline{R}_2 f(\overline{R}_1 F)$ for each subset $F$ of $X$.

**Definition 2.12.** Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_q)$ be two topologized approximation spaces. Then a mapping $f: K \rightarrow Q$ is said to be a rough open mapping if $f(\overline{R}_1 G) = \overline{R}_2 f(\overline{R}_1 G)$ for each subset $G$ of $X$.

**Definition 2.13.** Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_q)$ be two topologized approximation spaces. Then a mapping $f: K \rightarrow Q$ is said to be a rough homeomorphism if:

i) $f$ is bijective.
ii) $f$ is rough continuous.
iii) $f^{-1}$ is rough continuous.

In this case, we say $X$ and $Y$ are rough homeomorphic.

**Theorem 2.14.** Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_q)$ be two topologized approximation spaces and $f: K \rightarrow Q$ be an onto mapping, then $f$ is rough continuous if $f$ is rough open.

**Proof.** Assume that $f$ is rough closed and $F$ is a subset of $X$, with $f(X - \overline{R}_1 F) = \overline{R}_2 f(X - \overline{R}_1 F)$, by (9) and (10) of Proposition 2.3, we have
\[
f(\overline{R}_1 F) = Y - \overline{R}_2 f(X - \overline{R}_1 F) = Y - \overline{R}_2 \left( f(X) - f(\overline{R}_1 F) \right) = Y - \overline{R}_2 \left( Y - f(\overline{R}_1 F) \right) = \overline{R}_2 \left( f(\overline{R}_1 F) \right)^c = \overline{R}_2 \left( f(\overline{R}_1 F) \right)^c = \overline{R}_2 f(\overline{R}_1 F)
\]

Conversely. Similarly to the first part.

**Theorem 2.15.** Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_q)$ be two topologized approximation spaces and $f: K \rightarrow Q$ be a bijective mapping, then $f$ is rough homeomorphism if $f$ is rough continuous and rough closed.

**Proof.** Assume that $f$ is a bijective and $f$ is a rough homeomorphism, then by Definition 2.12, we have $f$ is rough continuous. To prove $f$ is a rough closed, let $g$ be the inverse mapping of $f$, therefore $g = f^{-1}$ and $f = g^{-1}$, since $f$ is bijective, therefore $g$ is bijective. Let $F$ be a subset of $X$, then by Definition 2.12, $g$ is a rough continuous, therefore $g^{-1}(\overline{R}_1 F) = \overline{R}_2 g^{-1}(\overline{R}_1 F)$, since $g = g^{-1}$, then $f(\overline{R}_1 F) = \overline{R}_2 f(\overline{R}_1 F)$, hence $f$ is a rough closed.

Conversely. Assume that $f$ is a bijective, rough continuous and rough closed. To prove $f$ is a rough homeomorphism, we must show that $f^{-1}$ is a rough continuous. Let $g$ be the inverse mapping of $f$, therefore $g = f^{-1}$ and $f = g^{-1}$. Let $F$ be a subset of $X$, since $f$ is a rough closed, then $f(\overline{R}_1 F) = \overline{R}_2 f(\overline{R}_1 F)$, thus $g^{-1}(\overline{R}_1 F) = \overline{R}_2 g^{-1}(\overline{R}_1 F)$, therefore $g$ is a rough continuous. So $f^{-1}$ is a rough continuous. Hence $f$ is a rough homeomorphism.

**Theorem 2.16.** Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_q)$ be two topologized approximation spaces and $f: K \rightarrow Q$ be a bijective mapping, then $f$ is rough homeomorphism if $f$ is rough continuous and rough open.

**Proof.** From Theorem 2.14 and Theorem 2.15 the proof is obvious.

**Definition 2.17.** A rough property of a topologized approximation space $K = (X, R, \tau_k)$ is said to be a rough hereditary if, every subspace of the topologized approximation space $K$ has that rough property.
Definition 2.18. A rough property of a topologized approximation space $K = (X,R,τ_k)$ is said to be a topologized approximation rough property if, each rough homeomorphic space of $K$ has that rough property whenever $K$ has that rough property.

3. Rough Regular Spaces

We define rough regular space and introduce several theorems about rough regularity in topological spaces which are results from the general relations on the approximation spaces.

Definition 3.1. Let $K = (X,R,τ_k)$ be a topologized approximation space. Then $K$ is said to be a rough regular space if, for every subset $F$ of $X$ and $x \notin \overline{R}F$, there exist two subsets $A$ and $B$ of $X$ such that $x \in RA$, $\overline{RB} \subseteq \overline{RB}$ and $RA \cap RB = \emptyset$.

Definition 3.2 [4]. Let $K = (X,R,τ_k)$ be a topologized approximation space. Then $K$ is said to be a rough $k_1$ space (briefly $k_1$-space), if for every two distinct points $x, y \in X$, there exist two subsets $A$ and $B$ of $X$ such that $x \in RA, y \notin RA$ and $y \in RB, x \notin RB$.

Definition 3.3 [4]. Let $K = (X,R,τ_k)$ be a topologized approximation space. Then $K$ is said to be a rough $k_2$ space (briefly $k_2$-space), if for every two distinct points $x, y \in X$, there exist two subsets $A$ and $B$ of $X$ such that $x \in RA, y \notin RB$ and $RA \cap RB = \emptyset$.

Theorem 3.4 [4]. Let $K = (X,R,τ_k)$ be a topologized approximation space. Then $K$ is a $k_1$-space if and only if $\{x\} = \overline{R}\{x\}$ for every $x \in X$.

Definition 3.5. Let $K = (X,R,τ_k)$ be a topologized approximation space. Then $K$ is said to be a rough $k_3$ space (briefly $k_3$-space) if, it is both rough regular space and $k_1$-space.

Theorem 3.6. Every $k_3$-space is $k_2$-space.

Proof. Let $K = (X,R,τ_k)$ be a $k_3$-space (i.e. $K$ is a rough regular $k_1$-space). Let $x, y \in X$ such that $x \neq y$, then by Theorem 3.4, we have $\{x\} = \overline{R}\{x\}$. Since $y \notin \{x\} = \overline{R}\{x\}$ and $K$ is rough regular space, then there exist two subsets $A$ and $B$ of $X$ such that $x \in RA, y \notin RB$ and $y \in RB, x \notin RB$.

Hence $K$ is $k_3$-space.

Theorem 3.7. Rough regularity is rough hereditary property.

Proof. Let $K = (X,R,τ_k)$ be a rough regular space and let $Y$ be a subset of $X$.

Let $H$ be a subset of $Y$ and $y \notin \overline{R}H$, $\forall y \in Y$. By Theorem 2.7, we have $\overline{R}H = \overline{R}H \subseteq Y$ and $\overline{R}H$, so we get that $y \notin \overline{R}H$. Now, $K$ rough regular then there exist two subsets $A$ and $B$ of $X$ such that $y \in RA, \overline{R}H \subseteq \overline{R}B$ and $\overline{R}A \cap \overline{R}B = \emptyset$. Therefore $y \in RA \cap \overline{R}H \subseteq \overline{R}B \cap \overline{R}H$, and $\overline{R}H \subseteq \overline{R}B \cap \overline{R}H$. Also by Theorem 2.7, we have $(\overline{R}A \cap Y$) and $(\overline{R}B \cap Y)$ are subsets of $Y$ such that $\overline{R}(\overline{R}A \cap Y) = (\overline{R}A \cap Y)$ and $\overline{R}\overline{R}(\overline{R}B \cap Y) = (\overline{R}B \cap Y)$. Then by (4) of Proposition 2.3, we have $\overline{R}(\overline{R}A \cap Y) \cap \overline{R}\overline{R}(\overline{R}B \cap Y) = \overline{R}(\overline{R}A \cap \overline{R}B \cap Y) = \overline{R}(\overline{R}(\overline{R}A \cap \overline{R}B) \cap Y) = \overline{R}(\overline{R}A \cap \overline{R}B) \cap Y = \overline{R}(\emptyset \cap Y) = \overline{R}\emptyset = \emptyset$. Therefore $Q$ is rough regular space.

Hence rough regularity is rough hereditary property.

Theorem 3.8. Rough regularity is a topologized approximation rough property.

Proof. Let $K = (X,R,τ_k)$ be a rough regular space and let $Q = (Y,R,τ_q)$ be a rough homeomorphic image of $K = (X,R,τ_k)$ under a map $f$. Let $F$ be a subset of $Y$ and $y$ be a point of $Y$ which is not in $\overline{R}F$. Since $f$ is bijective function. There exits $x \in X$ such that $f(x) = y$. Now $f$ being rough continuous, $f^{-1}(\overline{R}F)$ is a subset of $X$ such that $f^{-1}(\overline{R}F) = \overline{R}(f^{-1}(\overline{R}F))$. Since $y \notin \overline{R}F$ then $x = f^{-1}(y) \notin f^{-1}(\overline{R}F) = \overline{R}(f^{-1}(\overline{R}F))$, thus $x \in \overline{R}(f^{-1}(\overline{R}F))$. Since $K$ is rough regular, then there exist two subsets $M, N$ of $X$ such that $x \in M, N$. Therefore $f(x) \in f(M)$. Since $f$ is rough homeomorphism, then $f$ is rough open, therefore $f(M) = f(M, N)$ and $f(N) = f(N)$. Moreover by Proposition 2.3, $f(M, N) \cap f(N) = f(M) \cap f(N) = f(M)$. Hence $K$ is rough regular space. Hence rough regularity is a topologized approximation rough property.

Theorem 3.9. Let $K = (X,R,τ_k)$ be a topologized approximation space. Then the following statements are equivalent:

i) $K$ is rough regular space.

ii) For every subset $F$ of $X$ and $x \notin \overline{RF}$, there exist two subsets $A$ and $B$ of $X$ such that $x \in RA, \overline{RF} \subseteq RB$ and $\overline{RA} \cap RB = \emptyset$.

iii) For every subset $F$ of $X$ and $x \notin \overline{RF}$, there is a subset $A$ of $X$ such that $x \in RA$ and $\overline{RA} \cap \overline{RF}$.
iv) For every subset $G$ of $X$ and $x \in RG$, there is a subset $B$ of $X$ such that $x \in RB \subseteq \overline{RB} \subseteq RG$.

v) For every subset $F$ of $X$, we have:
$$RF = \overline{RB} : B \text{ subset of } X \text{ and } RF \subseteq RB \}.$$

Proof. (i) (ii). Let $x \in X$ and $F$ be a subset of $X$ such that $x \notin RF$, since $K$ is rough regular, there exist two subsets $A$ and $B$ of $X$ such that $x \in RA, RF \subseteq RB$ and $RA \cap RB = \emptyset$. Then by Proposition 2.3, we have $RA \subseteq \overline{RB}$, so $\overline{RA} \subseteq \overline{RB}$. Hence $\overline{RA} \cap RB = \emptyset$.

(iii) (iv). Let $x \in X$ and $F$ be a subset of $X$ such that $x \notin RF$. By (ii), there exist two subsets $A$ and $B$ of $X$ such that $x \in RA, RF \subseteq RB$ and $RA \cap RB = \emptyset$. Since $RF \subseteq RB$. Then $\overline{RA} \cap RF = \emptyset$.

(iv) (v). Let $y \in X$ and $F$ be a subset of $X$ such that $y \notin RF$. Then by (10) of Proposition 2.3, we have $y \in RF^c$. By (iv), there is a subset $B$ of $X$ such that $y \notin RB \subseteq \overline{RB} \subseteq RF^c$. Then $RF \subseteq \overline{RB} \subseteq RF^c$ and $y \notin RF^c$. Let $RW = \overline{RB}$, then $RF \subseteq RW$ since $RW \subseteq RF^c$ then by (5) of Proposition 2.3, we have $\overline{RW} \subseteq \overline{RB}$, thus $\overline{RW} \subseteq RF^c$, therefore $y \in \overline{RW}$. This implies that that $y \in \overline{RF} : W$ subset of $X$ and $RF \subseteq RW$. Then $\overline{RF} \subseteq \overline{RW} : W$ subset of $X$ and $RF \subseteq RW$. Hence $\overline{RF} \subseteq \overline{RW} : W$ subset of $X$ and $RF \subseteq RW$.

(v) (i). Let $x \in X$ and $F$ be a subset of $X$ such that $x \notin RF$. By (v), there is a subset $B$ of $X$ such that $\overline{RF} \subseteq RB$ and $x \notin \overline{RB}$. Put $RA = \overline{RB}$. Then $x \in RA$. Moreover, $RB \cap RA = \emptyset$. Hence $K$ is rough regular space.

4. Rough Normal Spaces

We define rough normal spaces and introduce several theorems about rough normality in topological spaces which are results from the general relations on the approximation spaces.

Definition 4.1. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then $K$ is said to be a rough normal space if, for every two subsets $G$ and $H$ of $X$ such that $\overline{G} \cap RH = \emptyset$, there exist two subsets $A$ and $B$ of $X$ such that $RG \subseteq RA, RH \subseteq RB$, and $RA \cap RB = \emptyset$.

Definition 4.2. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then $K$ is said to be a rough $k_4$ space (briefly $k_4$-space) if, it is both rough normal and $k_1$-space.

Theorem 4.3. Every $k_4$-space is $k_3$-space.

Proof. Let $K = (X, R, \tau_k)$ be a $k_4$-space (i.e. $K$ is a rough normal $k_1$-space). Let $F$ be a subset of $X$ and $x \in X$ such that $x \notin RF$, then by Theorem 3.4, we have $\{x\} = \overline{RF} \subseteq RF$. Since $K$ is rough normal space, then there exist two subsets $A$ and $B$ of $X$ such that $\overline{RF} \subseteq RA, RH \subseteq RB$, and $RA \cap RB = \emptyset$. Thus $x \in RA, RF \subseteq RB$ and $RA \cap RB = \emptyset$. Therefore $K$ is rough regular space, since $K$ is $k_1$-space. So $K$ is $k_3$-space.

Theorem 4.4. If $K = (X, R, \tau_k)$ is a rough normal space and $Y$ is a subset of $X$ such that $Y = \overline{XY}$. Then $Q = (Y, R, \sigma_q)$ is rough normal.

Proof. Assume that $A$ and $B$ are subsets of $Y$ such that $\overline{YA} \cap \overline{YB} = \emptyset$. Since $Y = \overline{XY}$, then by Theorem 2.8, we have $\overline{YA} = \overline{YB}$ and $\overline{YB} = \overline{YA}$. Therefore $\overline{YA} \cap \overline{YB} = \emptyset$. Since $K$ rough normal space, then there exist two subsets $G$ and $H$ of $X$ such that $\overline{YA} \subseteq R_g A, \overline{YB} \subseteq R_g B$ and $R_g A \cap R_g B = \emptyset$. Therefore $\overline{R_g A} \subseteq R_g G \cap \overline{YB} \subseteq R_g H \cap \overline{YB}$. Also by Theorem 2.7, we have $R_g G \cap \overline{YB} \subseteq R_g H \cap \overline{YB}$ are subsets of $Y$ such that $R_g (R_g G \cap \overline{YB}) = R_g (R_g H \cap \overline{YB})$. So by (4) and (2) of Proposition 2.3, we have $\overline{R_g (R_g G \cap \overline{YB})} = \overline{R_g (R_g H \cap \overline{YB})} = \overline{R_g (R_g G \cap \overline{YB})} = \overline{R_g (R_g H \cap \overline{YB})} = \emptyset$. Hence $Q$ is rough normal space.

Theorem 4.5. Rough normality is a topologized approximation rough property.

Proof. Let $K = (X, R, \tau_k)$ be a rough normal and let $Q = (Y, R, \sigma_q)$ be a rough homeomorphic image of $K = (X, R, \tau_k)$ under a map $f$. Let $G, H$ be two subsets of $Y$ such that $\overline{RG} \cap \overline{RH} = \emptyset$. Since $f$ is rough continuous, then $f^{-1}(\overline{RG}) = \overline{f^{-1}(RG)}$ and $f^{-1}(\overline{RH}) = \overline{f^{-1}(RH)}$. Then by (v) of Theorem 2.10 and (12) Proposition 2.3, we have $\overline{f^{-1}(\overline{RG})} \subseteq f^{-1}(\overline{RG}) = f^{-1}(\overline{RG})$ and $\overline{f^{-1}(\overline{RG})} \subseteq f^{-1}(\overline{RG}) = f^{-1}(\overline{RG})$. Therefore $f^{-1}(\overline{RG})$ and $f^{-1}(\overline{RG})$ are rough normal.
\( f^{-1}(\overline{R}_2 H) \), therefore \( \overline{R}_1 f^{-1}(\overline{R}_2 G) \cap \overline{R}_1 f^{-1}(\overline{R}_2 H) \subseteq f^{-1}(\overline{R}_2 G) \cap f^{-1}(\overline{R}_2 H) = f^{-1}(\overline{R}_2 G \cap \overline{R}_2 H) = f^{-1}(\emptyset) = \emptyset \), hence \( \overline{R}_1 f^{-1}(\overline{R}_2 G) \cap \overline{R}_1 f^{-1}(\overline{R}_2 H) = \emptyset \). Since \( K \) is rough normal, then there exist two subsets \( M, N \) of \( X \) such that \\
\[ \overline{R}_1 f^{-1}(\overline{R}_2 G) \subseteq \overline{R}_1 M, \quad \overline{R}_1 f^{-1}(\overline{R}_2 H) \subseteq \overline{R}_1 N \] \quad \text{and} \quad \overline{R}_1 M \cap \overline{R}_1 N = \emptyset. \] Therefore \\
\[ f(f^{-1}(\overline{R}_2 G)) = f(\overline{R}_1 f^{-1}(\overline{R}_2 G)) \subseteq f(\overline{R}_1 M) \] and \\
\[ f(f^{-1}(\overline{R}_2 H)) = f(\overline{R}_1 f^{-1}(\overline{R}_2 H)) \subseteq f(\overline{R}_1 N), \] thus \( \overline{R}_2 G \subseteq f(\overline{R}_1 M) \) and \( \overline{R}_2 H \subseteq f(\overline{R}_1 N) \). Since \( f \) is rough homeomorphism, then \( f(f(\overline{R}_1 M)) = \overline{R}_2 f(\overline{R}_1 M) \) and \( f(f(\overline{R}_1 N)) = \overline{R}_2 f(\overline{R}_1 N) \). Moreover by (2) and (4) of Proposition 2.3, we have \( \overline{R}_2 f(\overline{R}_1 M) \cap \overline{R}_2 f(\overline{R}_1 N) = \overline{R}_2 f(\overline{R}_1 M \cap \overline{R}_1 N) = \overline{R}_2 f(\emptyset) = \overline{R}_2 \emptyset = \emptyset \). Therefore \( Q \) is rough normal space. Hence rough normality is a topologized approximation rough property.

**Theorem 4.6.** Let \( K = (X, R, \tau_R) \) be a topologized approximation space. Then the following statements are equivalent:

i) \( K \) is rough normal space.

ii) For every two subsets \( G \) and \( H \) of \( X \) such that \( \overline{R}G \cap \overline{R}H = \emptyset \), there are two subsets \( A \) and \( B \) of \( X \) such that \( \overline{R}G \subseteq \overline{R}A, \overline{R}H \subseteq \overline{R}B \) and \( \overline{R}A \cap \overline{R}B = \emptyset \).

iii) For every two subsets \( G \) and \( H \) of \( X \) such that \( \overline{R}G \cap \overline{R}H = \emptyset \), there is a subset \( A \) of \( X \) such that \( \overline{R}G \subseteq \overline{R}A \) and \( \overline{R}A \cap \overline{R}H = \emptyset \).

iv) For every two subsets \( G \) and \( H \) of \( X \) such that \( \overline{R}G \subseteq \overline{R}H \), there is a subset \( B \) of \( X \) such that \( \overline{R}G \subseteq \overline{R}B \subseteq \overline{R}B \cap \overline{R}H \).

**Proof:** (i) \( \implies \) (ii). Assume that \( G \) and \( H \) are subsets of \( X \) such that \( \overline{R}G \cap \overline{R}H = \emptyset \), since \( K \) is rough normal, there exist two subsets \( A \) and \( B \) of \( X \) such that \( \overline{R}G \subseteq \overline{R}A, \overline{R}H \subseteq \overline{R}B \) and \( \overline{R}A \cap \overline{R}B = \emptyset \). Then by (9), (5), (12) of Proposition 2.3, we have \( \overline{R}A \subseteq \overline{R}Bc \) and \( \overline{R}A \subseteq \overline{R}Bc = \overline{R}B \). Hence \( \overline{R}A \cap \overline{R}B = \emptyset \).

(ii) \( \implies \) (iii). Assume that \( G \) and \( H \) are subsets of \( X \) such that \( \overline{R}G \cap \overline{R}H = \emptyset \). By (ii), there exist two subsets \( A \) and \( B \) of \( X \) such that \( \overline{R}G \subseteq \overline{R}A, \overline{R}H \subseteq \overline{R}B \) and \( \overline{R}A \cap \overline{R}B = \emptyset \). Since \( \overline{R}H \subseteq \overline{R}B \), then \( \overline{R}A \cap \overline{R}H = \emptyset \).

(iii) \( \implies \) (iv). Assume that \( G \) and \( H \) are subsets of \( X \) such that \( \overline{R}G \subseteq \overline{R}H \). Then \( \overline{R}G \cap \overline{R}Hc = \emptyset \). By (iii), there is a subset \( A \) of \( X \) such that \( \overline{R}Hc \subseteq \overline{R}A \) and \( \overline{R}A \cap \overline{R}G = \emptyset \). Then by (10) of Proposition 2.3, we have \( \overline{R}G \subseteq \overline{R}A \subseteq \overline{R}A \subseteq \overline{R}Hc \). Hence \( \overline{R}Hc \cap \overline{R}G = \emptyset \).

(iv) \( \implies \) (i). Let \( G \) and \( H \) be subsets of \( X \) such that \( \overline{R}G \cap \overline{R}Hc = \emptyset \) and satisfies (iv). Then \( \overline{R}G \subseteq \overline{R}Hc \). Now by hypothesis (iv) there exist a subset \( M \) of \( X \) such that \( \overline{R}G \subseteq \overline{R}M \) and \( \overline{R}RM \subseteq \overline{R}Hc \), then \( \overline{R}H \subseteq \overline{R}RMc \), also \( \overline{R}M \cap \overline{R}RMc = \emptyset \). But by (1) of Proposition 2.3 \( \overline{R}RMc = \overline{R}Mc \). Thus \( \overline{R}M \cap \overline{R}RMc = \emptyset \). Hence \( K \) is rough normal space.

**References**


