A Convergent Scheme for Solving Initial Value Problems with Polynomial and Exponential Functions

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ABSTRACT: This paper presents the development of a convergent numerical scheme for the solution of initial value problems of first order ordinary differential equations. The scheme has been derived via the combination of two functions namely, polynomial and exponential functions. The local truncation error $\left(\tau_n(h)\right)$, order of convergence, consistency and stability of the proposed scheme have been analyzed in the present study. The Taylor's series expansion has been used to derive the principal term of $\left(\tau_n(h)\right)$. The Dahlquist's test equation is used to investigate the linear stability region. It is observed that the newly proposed scheme is fourth order convergent, consistent and conditionally stable with the region of linear stability. Three IVPs of different nature have been solved numerically to check the applicability of a new proposed scheme. The absolute error has been calculated at each mesh point of the integration interval. The numerical results show that the scheme is computationally effective, adequate and compares favorably with exact solutions. The aid of MATLAB version: 9.2.0.538062 (R2017a) has been used to carry out all numerical calculations.

Keywords: Local truncation error, Absolute error, stability, consistency, convergence.

MSC: 34A12, 45L05, 65L05, 65L20, 65L70

1. INTRODUCTION

Ordinary differential equations emanate in many context of Mathematics and Social and Natural sciences. The significance of ODEs in fields of science and engineering cannot be denied. Mathematical models emanating from these fields are most often ODEs together with their appropriate boundary and initial conditions. Such physical models report the dynamic aspects of system and represent the real world situation based on the data available in the past and present as detailed in Refs. [1 - 4].

It is a known fact that stiff ODEs constitute the complex and challenging models that cannot be solved analytically. It is difficult to acquire their exact solutions. In such situations, one has to go with numerical approximate solutions of the models achievable by various numerical schemes of different characteristics [5, 6]. Development of new numerical schemes with different characteristics for the solution of IVPs in ODEs has attracted the attention of many numerical analysts in past and recent years as detailed in Refs. [7 - 21].

The main intent of this study is to develop a new numerical scheme of order four via the combination of cubic polynomial and exponential function. Also the local truncation errors, consistency and the conditional stability of the scheme has been thoroughly investigated. Numerical results further confirm the behavior of the scheme when compared with exact solution in order to compute the errors. The rest of the paper is structured as follows; section 2 presents the problem formulation and development of the new numerical scheme. In section 3, the local truncation error of the scheme has been inspected. Also the order of the accuracy of the scheme is obtained. In section 4, analysis properties have been studied to show the efficiency of the new scheme. In Section 5, numerical experiments are carried out. Section 6 presents discussion of results and concluding remarks.

2. PROBLEM FORMULATION AND DERIVATION OF A FOURTH ORDER CONVERGENT SCHEME

2.1 PROBLEM FORMULATION

We consider a first order ordinary differential equation together with initial condition of the form

$$\frac{d}{dx}y(x) = f\left(x, y(x)\right), \qquad y(x_0) = y_0, \; x \in [a, b], \; y \in \mathbb{R} . \qquad \dots (1)$$

Let us suppose that the numerical solution ‘ y_{n+1} ’ estimated at the given interval $[x_n, x_{n+1}]$, $n \geq 0$ to exact solution ‘$y(x_{n+1})$ ’ to (1) be represented by the cubic polynomial and exponential function.

2.2 DERIVATION OF A FOURTH ORDER CONVERGENT SCHEME

Consider a function of the form

$$F(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + Be^{-2x} . \quad \ldots (2)$$

Where A_0 is a constant and A_1, A_2, A_3, B are undetermined constants. The integration interval of [a, b] is defined as

$$a = x_0 \leq x \leq x_n = b . \quad \ldots (3)$$

The step size is defined as

$$h = \frac{b-a}{N} . \quad \ldots (4)$$

The mesh point is defined as

$$x_n = x_0 + nh \quad , \quad n = 1,2,3, \ldots , N \quad \ldots (5)$$

or

$$x_{n+1} = x_0 + (n+1)h \quad . \quad n = 0,1,2,3, \ldots , N-1 \quad \ldots (6)$$

Expanding (2) at the points x_n and x_{n+1} generates

$$F(x_n) = A_0 + A_1x_n + A_2x_n^2 + A_3x_n^3 + Be^{-2x_n} , \quad \ldots (7)$$

and

$$F(x_{n+1}) = A_0 + A_1x_{n+1} + A_2x_{n+1}^2 + A_3x_{n+1}^3 + Be^{-2x_{n+1}} . \quad \ldots (8)$$

Respectively, Differentiating (7) four times and using the fact that

$F'(x_n) = f_n$, $F''(x_n) = f_n^{(1)}$, $F'''(x_n) = f_n^{(2)}$, $F^{iv}(x_n) = f_n^{(3)}$.

We obtain

$$A_1 + 2A_2x_n + 3A_3x_n^2 - 2Be^{-2x_n} = f_n , \quad \ldots (9)$$

$$2A_2 + 6A_3x_n + 4Be^{-2x_n} = f_n^{(1)} , \quad \ldots (10)$$

$$6A_3 - 8Be^{-2x_n} = f_n^{(2)}, \quad \ldots (11)$$

$$16Be^{-2x_n} = f_n^{(3)} , \quad \ldots (12)$$

$$B = \frac{f_n^{(3)}}{16e^{-2x_n}} . \quad \ldots (13)$$

Substituting in eq. (11), we obtain

$$6A_3 - \frac{8f_n^{(3)}}{16e^{-2x_n}} \times e^{-2x_n} = f_n^{(2)} ,$$

$$6A_3 - \frac{1}{2}f_n^{(3)} = f_n^{(2)} ,$$

$$6A_3 = f_n^{(2)} + \frac{1}{2}f_n^{(3)} ,$$

$$A_3 = \frac{1}{6}f_n^{(2)} + \frac{1}{12}f_n^{(3)} . \quad \ldots (14)$$

Substituting (13) and (14) into (10), we obtain

$$2A_2 + 6\left(\frac{1}{6}f_n^{(2)} + \frac{1}{12}f_n^{(3)}\right)x_n + 4\left(\frac{f_n^{(3)}}{16e^{-2x_n}}\right) \times e^{-2x_n} = f_n^{(1)} ,$$

$$2A_2 + f_n^{(2)}x_n + \frac{1}{2}f_n^{(3)}x_n + \frac{f_n^{(3)}}{4} = f_n^{(1)} ,$$

$$2A_2 + \left(f_n^{(2)} + \frac{1}{2}f_n^{(3)}\right)x_n = \left(f_n^{(1)} - \frac{1}{4}f_n^3\right) ,$$

$$A_2 = \left(\frac{1}{2}f_n^{(1)} - \frac{1}{8}f_n^3\right) - \left(\frac{1}{2}f_n^{(2)} + \frac{1}{8}f_n^{(3)}\right)x_n \quad . \quad \ldots (15)$$

Substituting (13), (14) and (15) into (9), we obtain

$$A_1 = f_n - 2A_2x_n - 3A_3x_n^2 + 2Be^{-2x_n} ,$$

$$A_1 = f_n - 2\left\{\left(\frac{1}{2}f_n^{(1)} - \frac{1}{8}f_n^3\right) - \left(\frac{1}{2}f_n^{(2)} + \frac{1}{4}f_n^{(3)}\right)x_n\right\}x_n - 3\left\{\frac{1}{6}f_n^{(2)} + \frac{1}{12}f_n^{(3)}\right\}x_n^2 + 2\left\{\frac{f_n^{(3)}}{16e^{-2x_n}}\right\}e^{-2x_n} ,$$

$$A_1 = f_n - f_n^{(1)}x_n + \frac{1}{4}f_n^{(3)}x_n + f_n^{(2)}x_n^2 + \frac{1}{2}f_n^{(3)}x_n^2 - \frac{1}{2}f_n^{(2)}x_n^2 - \frac{1}{4}f_n^{(3)}x_n^2 + \quad \frac{1}{8}f_n^{(3)} ,$$

$$A_1 = \left(f_n + \frac{1}{8}f_n^{(3)}\right) - \left(f_n^{(1)} - \frac{1}{4}f_n^{(3)}\right)x_n + \left(\frac{1}{2}f_n^{(2)} + \frac{1}{4}f_n^{(3)}\right)x_n^2 . \quad \ldots (16)$$

Subtracting (7) from (8) gives the following:

$$F(x_{n+1}) - F(x_n) = (A_0 + A_1x_{n+1} + A_2x_{n+1}^2 + A_3x_{n+1}^3 + Be^{-2x_{n+1}}) - (A_0 + A_1x_n + A_2x_n^2 + A_3x_n^3 + Be^{-2x_n}) ,$$

$$F(x_{n+1})-F(x_n)=A_1(x_{n+1}-x_n)+A_2(x_{n+1}^2-x_n^2)+A_3(x_{n+1}^3-x_n^3)+B(e^{-2x_{n+1}}-e^{-2x_n}) \quad \dots (17)$$

Since;

$$x_n=x_0+nh\,, \quad \dots (18)$$

and

$$x_{n+1}=x_0+(n+1)h\,. \quad \dots (19)$$

Therefore,

$$x_{n+1}-x_n=h\,, \quad \dots (20)$$

$$x_{n+1}^2-x_n^2=\{x_0+(n+1)h\}^2-\{x_0+nh\}^2\,,$$

$$x_{n+1}^2-x_n^2=x_0^2+2x_0nh+2x_0h+(n^2+2n+1)h^2-x_0^2-2x_0nh-n^2h^2\,,$$

$$x_{n+1}^2-x_n^2=2x_0nh+2x_0h+n^2h^2+2nh^2+h^2-2x_0nh-n^2h^2\,,$$

$$x_{n+1}^2-x_n^2=2x_0h+(2n+1)h^2\,, \quad \dots (21)$$

and

$$x_{n+1}^3-x_n^3=\{x_0+(n+1)h\}^3-\{x_0+nh\}^3\,,$$

$$x_{n+1}^3-x_n^3=\{x_0^3+(n^3+1+3n^2+3n)h^3+3x_0^2(n+1)h+3x_0(n^2+2n+1)h^2\}-\{x_0^3+n^3h^3+3x_0^2nh+3x_0n^2h^2\}\,,$$

$$x_{n+1}^3-x_n^3=\{x_0^3+n^3h^3+h^3+3n^2h^3+3nh^3+3x_0^2nh+3x_0^2h+3x_0n^2h^2+6x_0nh^2+3x_0h^2-x_0^3-n^3h^3-3x_0^2h-3x_0n^2h^2\}\,,$$

$$x_{n+1}^3-x_n^3=3x_0^2h+3x_0(1+2n)h^2+(3n^2+3n+1)h^3\,, \quad \dots (22)$$

Setting $x_0=0$ in eq (18), (19), (21) and (22) yields

$$x_n=nh\,, \quad \dots (23)$$

$$x_{n+1}=(n+1)h\,, \quad \dots (24)$$

$$x_{n+1}^2-x_n^2=(2n+1)h^2\,, \quad \dots (25)$$

$$x_{n+1}^3-x_n^3=(3n^2+3n+1)h^3\,, \quad \dots (26)$$

Substituting (23), (24), (25) and (26) in (17), we obtain

$$F(x_{n+1})-F(x_n)=A_1h+A_2h^2(2n+1)+A_3h^3(3n^2+3n+1)+B(e^{-2(n+1)h}-e^{-2nh})\,,$$

$$F(x_{n+1})-F(x_n)=A_1h+A_2h^2(2n+1)+A_3h^3(3n^2+3n+1)+Be^{-2nh}(e^{-2h}-1) \quad \dots (27)$$

Substitute the values of A_1, A_2, A_3 and B in eq. (27) yields

$$F(x_{n+1})-F(x_n)=\left[\left(f_n+\tfrac{1}{8}f_n^{(3)}\right)-\left(f_n^{(1)}-\tfrac{1}{4}f_n^{(3)}\right)x_n+\left(\tfrac{1}{2}f_n^{(2)}+\tfrac{1}{4}f_n^{(3)}\right)x_n^2\right]h+\left[\left(\tfrac{1}{2}f_n^{(1)}-\tfrac{1}{8}f_n^{(3)}\right)-\left(\tfrac{1}{2}f_n^{(2)}+\tfrac{1}{4}f_n^{(3)}\right)x_n\right]h^2(2n+1)+\left[\tfrac{1}{6}f_n^{(2)}+\tfrac{1}{12}f_n^{(3)}\right]h^3(3n^2+3n+1)+\frac{f_n^{(3)}}{16e^{-2nh}}\times e^{-2nh}(e^{-2h}-1) \quad \dots (28)$$

Substitute $x_n=nh$ into eq. (28), we obtain

$$F(x_{n+1})-F(x_n)=\left[\left(f_n+\tfrac{1}{8}f_n^{(3)}\right)-\left(f_n^{(1)}-\tfrac{1}{4}f_n^{(3)}\right)nh+\left(\tfrac{1}{2}f_n^{(2)}+\tfrac{1}{4}f_n^{(3)}\right)n^2h^2\right]h+\left[\left(\tfrac{1}{2}f_n^{(1)}-\tfrac{1}{8}f_n^{(3)}\right)-\left(\tfrac{1}{2}f_n^{(2)}+\tfrac{1}{4}f_n^{(3)}\right)nh\right]h^2(2n+1)+\left[\tfrac{1}{6}f_n^{(2)}+\tfrac{1}{12}f_n^{(3)}\right]h^3(3n^2+3n+1)+\frac{f_n^{(3)}}{16e^{-2nh}}\times e^{-2nh}(e^{-2h}-1)\,,$$

$$F(x_{n+1})-F(x_n)=\left(f_n+\tfrac{1}{8}f_n^{(3)}\right)h+\left(\tfrac{1}{2}f_n^{(1)}-\tfrac{1}{8}f_n^{(3)}\right)h^2(2n+1)-\left(f_n^{(1)}-\tfrac{1}{4}f_n^{(3)}\right)nh^2+\left(\tfrac{1}{2}f_n^{(2)}+\tfrac{1}{4}f_n^{(3)}\right)n^2h^3-\left(\tfrac{1}{2}f_n^{(2)}+\tfrac{1}{4}f_n^{(3)}\right)n(2n+1)h^3+\left(\tfrac{1}{6}f_n^{(2)}+\tfrac{1}{12}f_n^{(3)}\right)h^3(3n^2+3n+1)+\frac{f_n^{(3)}}{16}(e^{-2h}-1)\,,$$

$$F(x_{n+1})-F(x_n)=\left(f_n+\tfrac{1}{8}f_n^{(3)}\right)h+\left[\left(\tfrac{1}{2}f_n^{(1)}-\tfrac{1}{8}f_n^{(3)}\right)(2n+1)-\left(f_n^{(1)}-\tfrac{1}{4}f_n^{(3)}\right)n\right]h^2+\left[\left(\tfrac{1}{2}f_n^{(2)}+\tfrac{1}{4}f_n^{(3)}\right)n^2-\left(\tfrac{1}{2}f_n^{(2)}+\tfrac{1}{4}f_n^{(3)}\right)n(2n+1)+\left(\tfrac{1}{6}f_n^{(2)}+\tfrac{1}{12}f_n^{(3)}\right)(3n^2+3n+1)\right]h^3+\frac{f_n^{(3)}}{16}(e^{-2h}-1)\,. \quad \dots (29)$$

Since one-step numerical scheme will be developed, let

$$y_{n+1}-y_n=F(x_{n+1})-F(x_n)$$

$$y_{n+1}=y_n+\left(F(x_{n+1})-F(x_n)\right)\,. \quad \dots (30)$$

Substituting (29) into (30), we have

$$y_{n+1}=y_n++\frac{f_n^{(3)}}{16}(e^{-2h}-1)+\left(f_n+\tfrac{1}{8}f_n^{(3)}\right)h+\left[\left(\tfrac{1}{2}f_n^{(1)}-\tfrac{1}{8}f_n^{(3)}\right)(2n+1)-\left(f_n^{(1)}-\tfrac{1}{4}f_n^{(3)}\right)n\right]h^2+\left[\left(\tfrac{1}{2}f_n^{(2)}+\tfrac{1}{4}f_n^{(3)}\right)n^2-\left(\tfrac{1}{2}f_n^{(2)}+\tfrac{1}{4}f_n^{(3)}\right)n(2n+1)+\left(\tfrac{1}{6}f_n^{(2)}+\tfrac{1}{12}f_n^{(3)}\right)(3n^2+3n+1)\right]h^3\,.$$

… (31)

The simplified form of (31) is

$$y_{n+1} = y_n + \frac{1}{16}(e^{-2h} - 1)f_n^{(3)} + h\left(f_n + \frac{1}{8}f_n^{(3)}\right) + \frac{h^2}{2}\left(f_n^{(1)} - \frac{1}{4}f_n^{(3)}\right) + \frac{h^3}{2}\left(\frac{1}{3}f_n^{(2)} + \frac{1}{6}f_n^{(3)}\right). \quad \dots (32)$$

Equation (32) is the newly developed fourth order one-step numerical scheme.

3. ORDER OF ACCURACY OF THE NEW NUMERICAL SCHEME

According to Ref. [15], local truncation error measures the order of accuracy for any numerical scheme. Consider the Taylor's series expansion of the form

$$y(x_n + h) = y(x_n) + hf + \frac{1}{2!}h^2 f^{(1)} + \frac{1}{3!}h^3 f^{(2)} + \frac{1}{4!}h^4 f^{(3)} + \frac{1}{5!}h^5 f^{(iv)} + O(h^6). \quad \dots (33)$$

From the definition of a local truncation error for explicit one step scheme, we write that:

Local Truncation Error $= \tau_{n+1} = y(x_n + h) - y_{n+1}$. … (34)

Substituting (32) and (33) into (34), we obtain

$$\tau_{n+1} = \left[y(x_n) + hf + \frac{1}{2!}h^2 f^{(1)} + \frac{1}{3!}h^3 f^{(2)} + \frac{1}{4!}h^4 f^{(3)} + \frac{1}{5!}h^5 f^{(iv)}\right] - \left[y_n + \frac{1}{16}(e^{-2h} - 1)f_n^{(3)} + h\left(f_n + \frac{1}{8}f_n^{(3)}\right) + \frac{h^2}{2}\left(f_n^{(1)} - \frac{1}{4}f_n^{(3)}\right) + \frac{h^3}{2}\left(\frac{1}{3}f_n^{(2)} + \frac{1}{6}f_n^{(3)}\right)\right] + O(h^6). \quad \dots (35)$$

Replacing the term e^{-2h} from eq. (35) by its Maclaurin's series, we obtain

$$\tau_{n+1} = \left[y(x_n) + hf + \frac{1}{2!}h^2 f^{(1)} + \frac{1}{3!}h^3 f^{(2)} + \frac{1}{4!}h^4 f^{(3)} + \frac{1}{5!}h^5 f^{(iv)}\right] - \left[y_n + \frac{1}{16}\left(1 - 2h + \frac{(-2h)^2}{2!} + \frac{(-2h)^3}{3!} + \frac{(-2h)^4}{4!} + \frac{(-2h)^5}{5!} + \cdots - 1\right)f_n^{(3)} + h\left(f_n + \frac{1}{8}f_n^{(3)}\right) + \frac{h^2}{2}\left(f_n^{(1)} - \frac{1}{4}f_n^{(3)}\right) + \frac{h^3}{2}\left(\frac{1}{3}f_n^{(2)} + \frac{1}{6}f_n^{(3)}\right)\right] + O(h^6). \quad \dots (36)$$

Solving further, eq. (36), we obtain

$$\tau_{n+1} = y(x_n) + hf + \frac{1}{2}h^2 f^{(1)} + \frac{1}{6}h^3 f^{(2)} + \frac{1}{24}h^4 f^{(3)} + \frac{1}{120}h^5 f^{(iv)} - y_n - hf_n - \frac{h^2}{2}f_n^{(1)} - \frac{h^3}{6}f_n^{(2)} - \frac{h^4}{24}f_n^{(3)} + \frac{h^5}{60}f_n^{(3)} + O(h^6). \quad \dots (37)$$

By means of the localizing assumption, the terms up to h^4 have been eliminated and eq. (37) has been reduced into the following expression:

$$\tau_{n+1} = \frac{1}{60}h^5\left(\frac{1}{2}f^{(iv)} - f_n^{(3)}\right) + O(h^6). \quad \dots (38)$$

Thus, the leading term of the local truncation error involves h^5, which confirms the fourth order accuracy of the new proposed numerical scheme, given by eq. (32).Hence, the new proposed scheme has the convergence of fourth order.

4. ANALYSIS OF THE PROPERTIES OF THE PROPOSED SCHEME

This section presents the consistency, stability and convergence properties of the proposed numerical scheme (32) as follows.

4.1 CONSISTENCY. According to Ref. [6], consistency essentially requires that: for a numerical scheme to be consistent, it is important for the truncation errors to be zero when the step size h gets smaller and ultimately converges to zero. Alternatively, the consistency of a one-step numerical scheme can be measured as follows:

$$\lim_{h\to 0}\left(\frac{LTE}{h}\right) = \lim_{h\to 0}\left(\frac{\tau_{n+1}}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{60}h^5\left(\frac{1}{2}f^{(iv)} - f_n^{(3)}\right) + O(h^6)}{h}\right) = 0. \quad \dots (39)$$

From above criterion, it is easy to observe that the proposed numerical scheme has consistency characteristics.

4.2 STABILITY. According to Ref. [6], numerical schemes are stated to be numerically stable if they are able to damping out the small fluctuations achieved with inside the enter data. Some factors that affect stability include the differential equation that's being solved, or the numerical method that's being used, or the step size h used in numerical calculations. To illustrate the idea of stability analysis of the proposed numerical scheme, consider the following Dahlquist's test problem:

$$y'(x) = \mu y(x), \ y(0) = 1, \ \mu < 0 \quad \dots (40)$$

where μ is a complex constant.

The exact solution of eq. (40) is given by

$$y(x) = e^{\mu x}, \mu < 0 \quad \dots (41)$$

For an integration interval $[x_n, x_{n+1}]$ where $h = x_{n+1} - x_n$; the exact solution at the point $x = x_{n+1}$ is acquired as

$$y^{Exact}(x_{n+1}) = e^{\mu x_{n+1}} = e^{\mu(x_n + h)} = e^{\mu x_n}e^{\mu h} = y^{Exact}(x_n)e^{\mu h}. \quad \dots (42)$$

When implemented the proposed numerical scheme (30) on this test problem; it produce

$y_{n+1} = \varphi y_n$ Where $\varphi = 1 + \mu h + \frac{(\mu h)^2}{2!} + \frac{(\mu h)^3}{3!} + \frac{(\mu h)^4}{4!}$ … (43)

Comparing the Equations (42) and (43), it can see clearly that the Eq. (43) is the fifth term of the series expansion of the function $e^{\mu h}$ in the exact solution. The error amplification factor given by Eq. (43) can be restrained by $|\varphi| < 1$ so that the errors may not magnify. Thus, the region of stability of the proposed one-step numerical scheme satisfies

$$\left|1 + \mu h + \frac{(\mu h)^2}{2!} + \frac{(\mu h)^3}{3!} + \frac{(\mu h)^4}{4!}\right| < 1 \,. \quad \ldots (44)$$

We substitute $z = \mu h$, then Equation (44) provides

$$\left|1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!}\right| < 1 \,. \quad \ldots (45)$$

Using Equation (45), the stability region for the proposed one-step numerical scheme is plotted in Figure 1. Hence, the proposed numerical scheme (32) is observe to be conditionally stable with the region of linear stability given below.

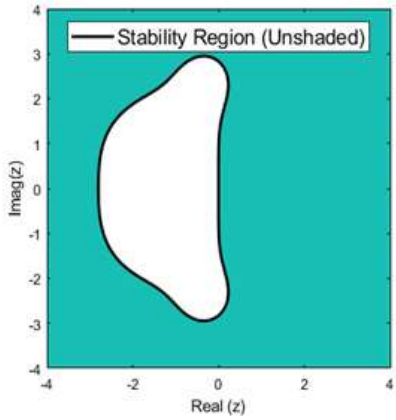

Fig. 1. The stability region (Un-shaded) for the proposed scheme (32).

4.3 CONVERGENCE. According to Ref. [7], the necessary and sufficient conditions for a one-step numerical scheme to be convergent are consistency and stability. Since the proposed numerical scheme satisfied these conditions effectively, we can conclude that the new proposed numerical one-step scheme is convergent.

5. IMPLEMENTATION OF THE NUMERICAL SCHEME (32)

This last section presents some numerical experiments to test the performance of the proposed numerical scheme (32).

5.1. NUMERICAL EXPERIMENTS

It is always necessary to signify the implementability, compatibility and accuracy of the newly developed one-step numerical scheme for solving IVPs arising from dynamic systems. To do so, the scheme was written in an algorithm form, transformed into computer codes with the help of MATLAB programming language and employed with couple of IVPs on a digital computer.

We consider the following numerical experiments.

5.1.1. Experiment 1

$\frac{dy}{dx} = x + y\,;\ \ y\,(0) = 1; \quad h = 0.1; \ \ 0 \le x \le 1$

Analytical Solution:

$y(x) = 2e^x - x - 1$

The comparative analysis of the results are shown in Table 1 below

Table 1. The comparative analysis of the results produced via the new scheme ($'y_n'$) in the context of the exact solution ($'y(x_n)'$) in the interval of integration $x \in [0,1]$ with large constant step-size $h = 0.1$

n	h	x_n	y_n	$y(x_n)$	$A.E$
1	0.1	0.0000	1.0000	1.0000	0.0000
2	0.1	0.1000	1.1103	1.1103	0.0000
3	0.1	0.2000	1.2428	1.2428	0.0000
4	0.1	0.3000	1.3997	1.3997	0.0000
5	0.1	0.4000	1.5836	1.5836	0.0000
6	0.1	0.5000	1.7974	1.7974	0.0000
7	0.1	0.6000	2.0442	2.0442	0.0000
8	0.1	0.7000	2.3275	2.3275	0.0000
9	0.1	0.8000	2.6511	2.6511	0.0000
10	0.1	0.9000	3.0192	3.0192	0.0000
11	0.1	1.0000	3.4366	3.4366	0.0000

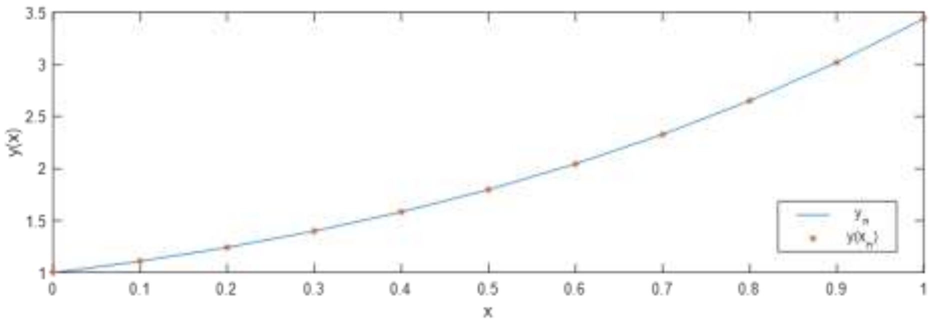

Fig.2. Comparison of exact solution with proposed scheme (32) for the numerical experiment 1.

5.1.2. Experiment 2

$\frac{dy}{dx} = 1 + y^2$; $y(0) = 1$; $h = 0.01$; $0 \le x \le 0.5$

Analytical Solution:

$y(x) = tan\left(x + \frac{\pi}{4}\right)$

The comparative analysis of the results are shown in Table 2 below

Table 2. The comparative analysis of the results produced via the new scheme ('y_n') in the context of the exact solution ('$y(x_n)$') in the interval of integration $x \in [0,0.5]$ with small constant step-size $h = 0.01$

n	h	x_n	y_n	$y(x_n)$	$A.E$
0	0.01	0.0000	1.0000	1.0000	0.0000
5	0.01	0.0500	1.1053	1.1054	0.0000
10	0.01	0.1000	1.2230	1.2230	0.0000
15	0.01	0.1500	1.3560	1.3561	0.0001
20	0.01	0.2000	1.5084	1.5085	0.0001
25	0.01	0.2500	1.6856	1.6858	0.0002
30	0.01	0.3000	1.8956	1.8958	0.0002
35	0.01	0.3500	2.1494	2.1497	0.0003
40	0.01	0.4000	2.4643	2.4650	0.0007
45	0.01	0.4500	2.8677	2.8689	0.0011
50	0.01	0.5000	3.4062	3.4082	0.0021

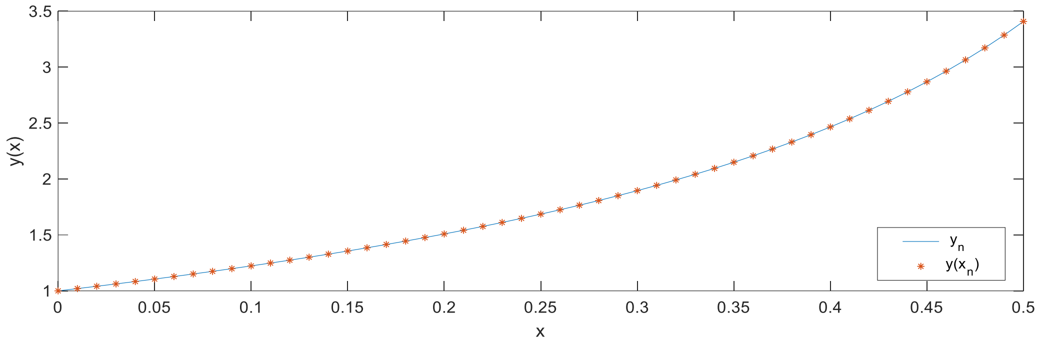

Fig.3. Comparison of exact solution with proposed scheme (32) for the numerical experiment 2.

5.1.3. Experiment 3

$\frac{dy}{dx} = xy^3$; $y(0) = 1$; $h = 0.01$; $0 \leq x \leq 0.5$

Analytical Solution:

$y(x) = \frac{1}{\sqrt{1-x^2}}$

The comparative analysis of the results are shown in Table 3 below

Table 3. The comparative analysis of the results produced via the new scheme ('y_n') in the context of the exact solution ('$y(x_n)$') in the interval of integration $x \in [0,0.5]$ with small constant step-size $h = 0.01$

n	h	x_n	y_n	$y(x_n)$	$A.E$
0	0.01	0.0000	1.0000	1.0000	0.0000
5	0.01	0.0500	1.0013	1.0031	0.0000
10	0.01	0.1000	1.0050	1.0050	0.0000
15	0.01	0.1500	1.0114	1.0114	0.0000
20	0.01	0.2000	1.0206	1.0206	0.0000
25	0.01	0.2500	1.0328	1.0328	0.0000
30	0.01	0.3000	1.0483	1.0483	0.0000
35	0.01	0.3500	1.0675	1.0675	0.0000
40	0.01	0.4000	1.0911	1.0911	0.0000
45	0.01	0.4500	1.1198	1.1198	0.0000
50	0.01	0.5000	1.1547	1.1547	0.0000

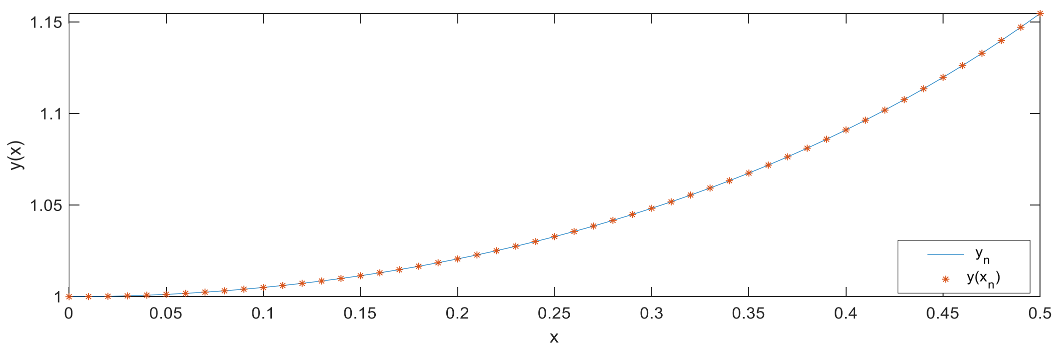

Fig.4. Comparison of exact solution with proposed scheme (32) for the numerical experiment 3

6. DISCUSSION OF RESULTS AND CONCLUDING REMARKS

This section presents the brief description of resultss achieved via new proposed one-step numerical scheme (32).

6.1 DISCUSSION OF RESULTS

The initial value problems in ordinary differential equations arising from scientific computations can be solved effectively by using new proposed numerical scheme (32). Three numerical examples have been solved to verify the performance of the new proposed numerical scheme in terms of the accuracy in the context of the exact solution and also the absolute relative errors computed at the final mesh point of the integration interval under consideration as shown in Table 1-3. The mesh points (‘$\boldsymbol{x_n}$’), numerical solution (‘$\boldsymbol{y_n}$’), the exact solution (‘$\boldsymbol{y(x_n)}$’) and the absolute errors (‘$\boldsymbol{A.E}$’) are displayed in third, fourth, fifth and sixth columns respectively. The new scheme has the behavior of decreasing errors with small step-size. Table 1-3 clearly indicated that the results of the new scheme and the exact solution increase over time. From figure 2-4, we discovered a similar behavior in exact and numerical solutions, which indicated that there is no disruption in the simulation curve and new approximate solution is in good agreement with the exact solution.

6.2 CONCLUDING REMARKS

In this paper, we have developed a new fourth order numerical scheme for the solution of IVPs in ordinary differential equations via the combination of cubic polynomial and exponential function. We have examined the truncation error, convergence, consistency and stability of the fourth order numerical scheme (32).The numerical results in Figures 2-4 depict that the fourth order numerical scheme is accurate and converges quicker to the exact solution. It is also observed that the proposed numerical scheme is consistent with conditional stability as shown in Figure 1. Hence, from the practical viewpoint, the suggested fourth order numerical scheme is effective, robust, accurate and very near to the exact solution. Finally, all the numerical calculations were carried out with the help of MATLAB version: 9.2.0.538062 (R2017a).

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REFERENCES

[1] Ansari, M. Y., Shaikh, A. A., & Qureshi, S. (2018). Error bounds for a numerical scheme with Reduced Slope evaluations. *J. Appl. Environ. Biol. Sci*, *8*(7), 67-76.

[2] Bird, J. (2017). Higher Engineering Mathematics. Taylor and Francis Group, London.

[3] Butcher, J. C. (2016). *Numerical methods for ordinary differential equations*. John Wiley & Sons.

[4] Davis, M. E. (2013). *Numerical methods and modeling for chemical engineers*. Courier Corporation.

[5] Qureshi, S., & Emmanuel, F. S. (2018). Convergence of a numerical technique via interpolating Function to approximate physical dynamical systems. *Journal of Advanced Physics*, *7*(3), 446-450.

[6] Fadugba, S. E., & Idowu, J. O. (2019). Analysis of the properties of a third order Convergence Numerical method derived via transcendental function of exponential form. *International Journal of Applied Mathematics and Theoretical Physics*, *5*(4), 97-103.

[7] Ogunrinde, R. B., & Fadugba, S. E. (2012). Development of a new scheme for the Solution of Initial value problems in ordinary differential equations. *International Organization of Scientific Research Journal of Mathematics (IOSRJM)*, *2*, 24-29.

[8] Fadugba, S. E., & Okunlola, J. T. (2017). Performance measure of a new one-step Numerical Technique via interpolating function for the solution of initial value problem of first order Differential equation. *World Scientific News*, *90*, 77-87.

[9] Fadugba, S. E., & Olaosebikan, T. E. (2018). Comparative study of a class of one-step methods For the numerical solution of some initial value problems in ordinary differential Equations. *Research Journal of Mathematics and Computer Science*, *2*(9), 1-11.

[10] QURESHI, S., SHAIKH, A. A., & CHANDIO, M. S. (2019). A New iterative Integrator for Cauchy Problems. *Sindh University Research Journal-SURJ (Science Series)*, *45*(3).

[11] Qureshi, S., & Ramos, H. (2018). L-stable explicit nonlinear method with constant and Variable Step-size formulation for solving initial value problems. *International Journal of Nonlinear Sciences and Numerical Simulation*, *19*(7-8), 741-751.

[12] Rabiei, F. A. R. A. N. A. K., & Ismail, F. (2012). Fifth-order Improved Runge-Kutta method for Solving ordinary differential equation. *Australian Journal of Basic and Applied Sciences*, *6*(3), 97- 105.

[13] Rabiei, F. A. R. A. N. A. K., Ismail, F., & Suleiman, M. (2013). Improved Runge-Kutta methods For solving ordinary differential equations. *Sains Malaysiana*, *42*(11),1679-1687.

[14] Emmanuel, F. S., & Qureshi, S. (2020). Convergent numerical method using transcendental Function of

exponential type to solve continuous dynamical systems. *Punjab University Journal Of Mathematics*, *51*(10).

[15] Ogunrinde, R. B., Olayemi, K. S., Isah, I. O., & Salawu, A. S. (2020). A Numerical Solver for First Order Initial Value Problems of Ordinary Differential Equation via the Combination of Chebyshev Polynomial and Exponential Function. *Journal of Physical Sciences*, *2*(1), 17-32.

[16] SO, A., & Ibijola, E. A. (2015). A new numerical method for solving first order differential Equations. *American journal of applied mathematics and statistics*, *3*(4),156-160.

[17] Ochoche, A. (2008). Improving the improved modified Euler method for better performance on autonomous initial value problems. *Leonardo Journal of Sciences*, *12*, 57-66.

[18] Kama, P., & Ibijola, E. A. (2001). On a new one-step method for numerical solution of initial-value problems in ordinary differential equations. *International journal of computer mathematics*, *77*(3), 457-467.

[19] Zill, D. G. (2012). *A first course in differential equations with modeling applications*.Cengage Learning.

[20] Jain, M. K. (2003). *Numerical methods for scientific and engineering computation*.New Age International.

[21] Wang, S. (2014). Lax Equivalence Theorem. *Student's Book Numerical Functional Analysis*, 15.