Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.10, No.2, 2020

On Asymptotic Linear Arbitrage in Markovian Models of Financial Markets*

Martin Le Doux Mbele Bidima
University of Yaoundé I, Cameroon
E-mail: mbelebidima@gmail.com

January 21, 2020

Abstract

Consider an investor's wealth allocated in a stock with prices modeled as a discrete-time homogeneous Markov process that is not necessarily specified by any stochastic recursion unlike in our previous paper (Mbele Bidima (2014)). Under this modified modeling setting and more stringent (but still verifiable) conditions, using different tools of Branching processes and Large deviations of functions of Markov transitions, we show again existence of asymptotic linear arbitrage with geometrically decaying failure probability in such a market model.

Keyword: Asymptotic linear arbitrage, Markov process, Large deviations.

1 Introduction

A couple of years ago, we introduced in [8] a version of asymptotic arbitrage in discrete-time Markov models of financial markets, which we called asymptotic linear arbitrage (ALA) with geometrically decaying failure probability ($GDFP$). That new concept aimed at discussing existence of trading opportunities in a general discrete-time Markov model that generate risk-less profit at a linear increasing speed in long-term, with a probability of failing to achieve such a linear growth that decays to 0 exponentially fast. (See also [3] and [9] for inspiring and similar works). We recall below the settings and the main result of that paper [8].

In a discrete-time financial market we considered two assets in trading: a risk-less asset (a bank account or a risk-free bond) with fixed interest rate, set to 0 for simplicity, i.e., with prices normalized to $B_t := 1$ for all times $t \in \mathbb{N}$, and a single risky security asset (such

*This article is a personal improved work of an unpublished minor part of my PhD thesis defended ten years ago from Central European University. I still thank my former supervisor there, Miklós Rásonyi.

as stock) whose (discounted) prices process S_t, $t \in \mathbb{N}$, is assumed $\mathbb{R}$-valued and governed by the stochastic difference equation

$$S_{t+1} = S_t + \mu(S_t) + \sigma(S_t)\varepsilon_{t+1},\ t \in \mathbb{N}, \tag{1}$$

and is assumed integrable in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{N}}$, $\mathcal{F}_t := \sigma(S_0, S_1, ..., S_t)$, for all $t \in \mathbb{N}$, the natural filtration of S_t, $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$, with $\sigma > 0$, given measurable functions, and $(\varepsilon_t)_{t\in\mathbb{N}}$ an $\mathbb{R}$-valued sequence of $i.i.d$ random variables. $\mathbb{E}$ and $\mathbb{V}ar$ denoting the expectation and the variance with respect to $\mathbb{P}$.

First, note that (1) could be seen as a time-discretization of a general stochastic differential equation driven by Brownian motion, whose solution is well known to possess the Markov property in the literature (see [6, Theorem 5.6]), and that S_t in the stochastic difference equation (1) is a discrete-time Markov process as it is verifiable from [1, pp. 211-228].

Next, in this model, we considered the so-called (bounded) Markovian strategies i.e., $\mathbb{R}$-valued $\mathbb{F}$-predictable strategies $(\pi_t)_{t\in\mathbb{N}}$ of the form $\pi_t := \pi(S_{t-1})$, for all $t \in \mathbb{N}$, where $\pi : \mathbb{R} \to \mathbb{R}$ i a (bounded) measurable function. Given any such trading opportunity π_t, we assumed that the corresponding (discounted) wealth that an investor allocates in the stock is an $\mathbb{R}$-valued discrete-time stochastic process V_t^π obeying the (self-financing) recursion

$$\begin{cases} V_{t+1}^\pi = V_t^\pi + \pi_{t+1}(S_{t+1} - S_t) \text{ for all time } t \geq 1, \\ V_0^\pi := V_0 \geq 0, \text{ is the investor's initial capital.} \end{cases} \tag{2}$$

Under these modeling settings, we proposed the following definition.

Definition 1.1. *(Definition 2.2 of [8]). We say that a trading opportunity π_t generates asymptotic linear arbitrage (ALA) with geometrically decaying failure probability (GDFP) if from zero initial capital V_0, there are real constants $a > 0$, and $c > 0$ such that,*

$$\mathbb{P}(V_t^\pi \geq at) \geq 1 - e^{-ct},\ \textit{for large enough times } t \geq 1, \tag{3}$$

or equivalently,

$$\mathbb{P}(V_t^\pi < at) < e^{-ct},\ \textit{for large enough times } t \geq 1. \tag{4}$$

Financially speaking, this means the investor's wealth by (3) grows linearly fast in long term with a probability converging to 1 exponentially fast and by (4), failing to achieve such a linear growth profit could be controlled by a probability that decays to 0 geometrically (exponentially) fast.

Under the mean-reverting condition (6) and other verifiable assumptions (A_1), (A_2), (A_3) set respectively on the stock prices process S_t, on the stock prices driving sequence ε_t's, on the drift and volatility functions μ and σ and on the so-called stock market price of risk stated in [8] (but which are not necessary needed to be recalled here), we proved the following main result,

Theorem 1.2. (Theorem 2.4 of [8]). *There is an explicitly constructed Markovian strategy π_t^0 that realizes ALA with GDFP in the investor's wealth process* (2) *with the stock prices model* (1).

To show this result, we began with proving a large set of preliminary results using advanced Markov chains theory in [10], Non-linear Functional analysis and the Spectral theory of multiplicative regular Markov processes in [7], until we finally applied the Gärtner-Ellis Large deviations theorem in [2, Theorem 2.3.6].

In our present paper, we state and prove a variant of that Theorem 2.4 of [8] in Theorem 2.12 of the section below under modified modeling settings with a different set of conditions. A practical example is worked out to show the relevance of this existence theorem. And we end with concluding remarks in the last section.

2 Modeling Setup and the Main Theorem

On contrary to the conditions set in [8], we consider the following.

Assumption 2.1. *1) The stock prices process S_t is not necessarily specified by a stochastic difference equation like in* (1)*,*

2) But still, S_t is a general discrete-time homogeneous Markov process with state space a compact interval I instead of the entire real line $\mathbb{R}$ as in (1).

Remark 2.2. 1) The assumption on an interval state space of S_t could be considered realistic using empirical justifications as follows: after long time historical observations of a stock prices on a given market, one may record the smallest value, say a, taken by the stock and it is possible to chose/predict a higher enough value b that the stock may not hit even after long-time trading. This is true since stock prices never hit infinite values in none market.

2) And the assumption that the interval $[a, b]$ can be taken as the state space I of the stock price process S_t is a mathematical condition that we need to use since closed bounded intervals in $\mathbb{R}$ are compact.

Next, let λ and λ_2 denote the Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^2$ respectively, and let $\mathcal{B}(I)$ be the Borel σ-algebra on I. For $x \in I$ and $A \in \mathcal{B}(I)$, let $P(x, A) := \mathbb{P}(S_{t+1} \in A \,|\, S_t = x)$, $t \geq 0$, be the one-step transition probability kernel of the Markov process S_t. And denote $P^t(x, A) := \mathbb{P}(S_t \in A \,|\, S_0 = x)$, $t \geq 1$, the corresponding t-step transition probability kernel. We assume the following stringent (but still verifiable) conditions in our present modeling settings.

Assumption 2.3. *1) The one-step kernel $P(x,\cdot)$, for all $x \in I$, has a positive density $p(x,\cdot) : I \to \mathbb{R}_+$, with respect to the Lebesgue measure λ.*

2) That kernel density $p(x,\cdot)$ is uniformly positive and bounded i.e., there exist constants $c, d \in \mathbb{R}$ such that $0 < c < p(x,y) \leq d < \infty$, for all $x, y \in I$.

3) Markovian strategies π_t in such a general Markovian model are (uniformly) bounded i.e., the functions π are bounded on I.

For any such Markovian strategy, we still consider the wealth process V_t^π of an economic agent as specified in the stochastic difference equation (2). Note that we may write V_t^π as

$$V_t^\pi = V_0 + \sum_{n=1}^{t} Z_n^\pi, \text{ for all time } t \geq 1, \text{ where } Z_n^\pi := \pi(S_{n-1})(S_n - S_{n-1})$$

is the wealth increment by time $n \leq t$, for all time $t \geq 1$.

2.1 Preliminary Results from Markov Chains Theory

Before stating the main theorems of this article, we prove first the following set of results.

Proposition 2.4. *1) The t-step transition probability kernel $P^t(x,\cdot)$ has density, say $p^t(x,\cdot) : I \to \mathbb{R}_+$, with respect to the Lebesgue measure λ.*

2) For all $t \geq 1$, the law of S_t has density, say $p_t : I \to \mathbb{R}_+$, with respect to λ.

Proof. We prove 1) by induction. Note first that $p^t(x,\cdot)$ is just a notation, not power t^{th} of the density $p(x,\cdot)$ of $P(x,\cdot)$. Next, by induction, if $t = 1$ and $x \in I$, then we have $P^1(x,\cdot) = P(x,\cdot)$ which has density $p(x,\cdot)$ by hypothesis. Set $p^1(x,\cdot) := p(x,\cdot)$, so $P^1(x,\cdot)$ has density $p^1(x,\cdot)$. Suppose for $t > 1$ that $P^t(x,\cdot)$ has density, say $p^t(x,\cdot)$, then by Chapman-Kolmogorov's Theorem (see for instance [10, Theorem 3.4.2]), we have for all $A \in \mathcal{B}(I)$ that,

$$P^{t+1}(x,A) = \int_I P(x,dy)P^t(y,A) = \int_I P(x,dy)\int_A p^t(y,u)\lambda(du),$$

by induction hypothesis.

So if $\lambda(A) = 0$, then $P^{t+1}(x,A) = 0$, which means P^{t+1} is dominated by the Lebesgue measure λ. Hence by Radon-Nikodym Theorem, P^{t+1} also has a density $p^{t+1}(x,\cdot)$. We therefore conclude that for all $t \geq 1$, $P^t(x,\cdot)$ has a density $p^t(x,\cdot)$.

Statement 2) is derived from 1) as follows: for all $t \geq 1$, and all $A \in \mathcal{B}(I)$ we have, $\mathbb{P}(S_t \in A) = P^t(S_0, A) = \int_A p^t(S_0,y)\lambda(dy)$. Hence $p_t(y) := p^t(S_0,y)$, $y \in I$, is the density of S_t, as required. □

Proposition 2.5. *The Markov chain X_t is ψ-irreducible and aperiodic.*

Proof. First, by definition of irreducibility and Proposition 4.2.1 both stated in [10, p. 91], we have to show that if $A \in \mathcal{B}(I)$ with $\lambda(A) > 0$, then there is an integer $t \geq 1$ such that $P^t(x, A) > 0$ for all $x \in I$. Indeed, set $t := 1$, then we have

$$\begin{aligned} P(x, A) &= \int_A p(x, y)\lambda(dy) \\ &\geq \int_A c\lambda(dy) \text{ by Assumption 2.3, 2)} \\ &= c\lambda(A). \end{aligned} \tag{5}$$

Since $\lambda(A) > 0$ and $c > 0$, it follows that $P(x, A) > 0$ and hence by Proposition 4.2.2 of [10, p. 91] the Markov process S_t is ψ-irreducible.

For the aperiodicity property, from the last equation in (5) above, setting $\nu_1 := c\lambda$, we obtain by Definition (5.14) in [10, p. 111] that the whole compact state space I is a ν_1-small set for the discrete-time Markov process S_t. It follows that $1 \in E_I := \{t \geq 1 :$ I is ν_t-small with $\nu_t = \delta_t \nu_1$, for some constant $\delta_t > 0\}$, which implies that $d := g.c.d(E_I)$, i.e., the greatest common divisor of the set E_I, is exactly 1. Moreover since $\lambda(I) > 0$ i.e., $I \in \mathcal{B}^+(I) := \{A \in \mathcal{B}(I) : \lambda(A) > 0\}$, we conclude by Theorem 5.4.4 and the definition of aperiodicity in [10, pp. 122-123] that the Markov process S_t is aperiodic, ending the proof. □

Next, we state and prove the following technical lemma.

Lemma 2.6. *There is a unique invariant (probability) measure φ for the Markov process S_t, having a (stationary) positive density $\gamma : I \to \mathbb{R}_+$, with respect to the Lebesgue measure λ, such that the following limit holds,*

$$\lim_{t\to\infty} \mathbb{P}(S_t \in A) = \varphi(A) = \int_A \gamma(x)\lambda(dx), \text{ for all } A \in \mathcal{B}(I). \tag{6}$$

Proof. We proved in Proposition 2.5 that the whole compact state space I is ν_1-small for the Markov process S_t, hence by Theorem 16.0.2 of [10], S_t is uniformly ergodic, hence ergodic applying Theorem 16.0.1 of the same reference. It follows by Definition (13.8) of ergodicity on p. 319 of this reference that there is a unique invariant measure φ for the Markov process S_t such that $\|P^t(x, \cdot) - \varphi\| \to 0$ as $t \to \infty$ for all $x \in I$. Which means by Definition (13.7) of [10, p. 319] again that we have, in particular for the initial constant $x := S_0 \in I$,

$$\sup_{f:|f|\leq 1} |P^t(S_0, f) - \varphi(f)| \to 0 \text{ as } t \to \infty,$$

where f runs over the set of real measurable functions on I, $\varphi(f) := \int_I f(y)\varphi(dy)$ and $P^t(S_0, f) := \int_I f(y)P^t(S_0, dy)$. In other words, we have

$$\sup_{f:|f|\leq 1} \left| \int_I f(y)P^t(S_0, dy) - \int_I f(y)\varphi(dy) \right| \to 0 \text{ as } t \to \infty.$$

Setting $f := \mathbf{1}_A$ for any $A \in \mathcal{B}(I)$, we obtain that $|P^t(S_0, A) - \varphi(A)| \to 0$ as $t \to \infty$. Since $P^t(S_0, A) = \mathbb{P}(S_t \in A)$, it follows that $\mathbb{P}(S_t \in A) \to \varphi(A)$, as $t \to \infty$.

Finally we show that φ has a positive density. Since φ is invariant, using its defining property (10.1) in [10, p. 237], we obtain the following, for all $A \in \mathcal{B}(I)$,

$$\begin{aligned}\varphi(A) = \int_I P(x, A)\varphi(dx) &= \int_I \int_A p(x, y)\lambda(dy)\varphi(dx) \\ &= \int_A \Big(\int_I p(x, y)\varphi(dx) \Big) \lambda(dy),\end{aligned}$$

by Fubini Theorem. Setting $\gamma(y) := \int_I p(x, y)\varphi(dx)$, for all $y \in I$, which by Assumption 2.3. 2) is positive, we conclude with the later equality that φ has positive density γ, showing the lemma, as we required. □

This lemma implies the key result below.

Proposition 2.7. *For any Markovian strategy π_t in the wealth model (2), there exists $z_\pi \in \mathbb{R}$ such that the sequence of expected wealth increments $\mathbb{E}(Z_t^\pi)$ converges to z_π. We call this number z_π, the asymptotic expectation of the wealth increment Z_t^π.*

Proof. By Proposition 2.4 above, for all time t, S_t has density p_t. So for all $A, B \in \mathcal{B}(I)$, we have for $t \geq 1$,

$$\begin{aligned}\mathbb{P}(S_{t-1} \in A, S_t \in B) &= \int_A \mathbb{P}(S_t \in B | S_{t-1} = x) p_{t-1}(x)\lambda(dx) \\ &= \int_A \int_B p(x, y)\lambda(dy) p_{t-1}(x)\lambda(dx) \\ &= \int_A \int_B p(x, y) p_{t-1}(x)\lambda_2(dx, dy).\end{aligned}$$

This means, for $t \geq 1$, that (S_{t-1}, S_t) has density $p(x, y)p_{t-1}(x)$, for $x, y \in I$. Next by Lemma 2.6 above, since $\pi(x)(y - x)p(x, y)$ is bounded on I^2 (and is measurable), then we get,

$$\begin{aligned}\mathbb{E}(Z_t^\pi) &= \int_{I^2} \pi(x)(y - x)p(x, y)p_{t-1}(x)\lambda_2(dx, dy) \\ &\to \int_{I^2} \pi(x)(y - x)p(x, y)\gamma(x)\lambda_2(dx, dy), \text{ as } t \to \infty.\end{aligned}$$

The later integral is finite since $p(x, \cdot)$ and γ are densities, and $\pi(x)(y - x)$ is bounded on I^2 for all x, y. It is now enough to take $z_\pi := \int_{I^2} \pi(x)(y - x)p(x, y)\gamma(x)\lambda_2(dx, dy)$, to terminate the proof. □

2.2 Existence Theorem on Asymptotic Linear Arbitrage

First, we state and prove the following key lemmas.

Lemma 2.8. *Let π_t be any Markovian strategy in the wealth model (2) such that $\lambda(\{x \in I : \pi(x) \neq 0\}) > 0$. Then, there is a positive analytic function $\beta : \mathbb{R} \to \mathbb{R}_+$, such that the average investor's wealth sequence $(V_t^\pi - V_0)/t$ satisfies a large deviations principle with good convex rate function Λ^* i.e., the convex conjugate function of Λ defined as $\Lambda(\theta) := \log\big(\beta(\theta)\big)$ for all $\theta \in \mathbb{R}$.*

Proof. For $\theta \in \mathbb{R}$, consider the so-called scaled kernels $K_\theta(x, y) := e^{\theta\alpha(x,y)}p(x, y)$, where $\alpha(x, y) := \pi(x)(y - x)$, for all $x, y \in I$. Since, by Assumption 2.3. 3), the process $\alpha(S_{n-1}, S_n)$ is bounded for all n, it follows again by Assumption 2.3. 3) and 1) that K_θ satisfies the conditions of Theorem 10.1 in [4, p. 67], for all θ. So K_θ has a positive eigenvalue, say β i.e., as in definitions (10.3) and (10.4) of [4, p. 67], there are two functions $f, g \neq 0$ on I (the left and right eigenfunctions associated to β) such that $\beta(\theta)f(y) = \int_I f(x)K_\theta(x, y)\lambda(dx)$ and $\beta(\theta)g(x) = \int_I K_\theta(x, y)g(y)\lambda(dy)$, for all $x, y \in I$. It hence follows by Theorem 1 of [5] that $\lim_{t\to\infty} \big(\mathbb{E}\big(e^{\theta(V_t^\pi - V_0)}\big)\big)^{1/t} = \beta(\theta)$, and that β is analytic in θ. This implies by continuity of Logarithm that $\frac{1}{t}\log\mathbb{E}\big(e^{\theta(V_t^\pi - V_0)}\big) \to \log(\beta(\theta))$ as $t \to \infty$.

Define $\Lambda(\theta) := \log\big(\beta(\theta)\big)$, for all $\theta \in \mathbb{R}$. From the basic properties $\mathbb{E}(X) = M_X'(0)$ and $\mathbb{V}ar(X) = M_X''(0) - (M_X'(0))^2$ for any random variable with finite expectation and moment generating function $M_X(t) := \mathbb{E}e^{tX}$, $t \in \mathbb{R}$, the following corresponding equality holds for the so-called asymptotic variance of $(V_t^\pi - V_0)$,

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{V}ar\left(V_t^\pi - V_0\right) = \beta''(0) - z_\pi^2 = \Lambda''(0).$$

Suppose that the asymptotic variance $\lim_{t\to\infty} \frac{1}{t}\mathbb{V}ar\left(V_t^\pi - V_0\right)$ is nonzero i.e., $\Lambda''(0) > 0$, then the analytic function Λ is a nonlinear, hence it satisfies the conditions of Gärtner-Ellis Theorem 2.3.6 in [2] i.e., $(V_t^\pi - V_0)/t$ satisfies a large deviations principle in $\mathbb{R}$ with good convex rate function Λ^*.

If on contrary $\Lambda''(0)$ was zero, then, since as in Proposition 2.7 we have

$$\begin{aligned}\Lambda''(0) &= \int_{I^2} \pi^2(x)(y-x)^2 p(x,y)\gamma(x)\lambda_2(dx,dy)\\ &- \left(\int_{I^2} \pi(x)(y-x)p(x,y)\gamma(x)\lambda_2(dx,dy)\right)^2.\end{aligned}$$

$\Lambda''(0) = 0$ would imply $\pi(x)(y - x)$ is constant λ_2-*a.e.* on I^2, which happens only if $\pi(x) = 0$ λ-*a.e.* on I, a case we excluded in the statement of this lemma. □

Next,

Lemma 2.9. *For any Markovian strategy π_t in the investor's wealth model (2), the corresponding asymptotic expectation z_π is the unique minimizer of the convex rate function Λ^*. Moreover, we have $\Lambda^*(x) > 0$ for all $x \neq z_\pi$.*

Proof. In the proof of Proposition 2.7, we obtained $\lim_{t\to\infty}\big(\mathbb{E}\big(e^{\theta(V_t^\pi - V_0)}\big)\big)^{1/t} = \beta(\theta)$. Setting $\theta := 0$, then we get $\beta(0) = 1$. Thus, $\Lambda(0) = \log(\beta(0)) = 0$. So, for all $x \in \mathbb{R}$, we have $\Lambda^*(x) \geq 0 \times x - \Lambda(0) = 0$. Hence in particular we have $\Lambda^*(z_\pi) \geq 0$. Conversely, let us also show that $\Lambda^*(z_\pi) \leq 0$ and conclude that $\Lambda^*(z_\pi) = 0 \leq \Lambda^*(x)$ for all $x \in \mathbb{R}$. Indeed, for all $\theta \in \mathbb{R}$, we have,

$$\begin{aligned}
\theta z_\pi - \Lambda(\theta) &= \theta z_\pi + \lim_{t\to\infty}\frac{1}{t}\Big(-\log\mathbb{E}\big(e^{\theta\sum_{n=1}^t Z_n^\pi}\big)\Big) \\
&\leq \theta z_\pi + \lim_{t\to\infty}\frac{1}{t}\big(\mathbb{E}(-\theta\sum_{n=1}^t Z_n^\pi)\big) \text{ by Jensen-inequality} \\
&= \theta z_\pi - \theta\lim_{t\to\infty}\frac{1}{t}\big(\sum_{n=1}^t \mathbb{E}(Z_n^\pi)\big) \\
&= \theta z_\pi - \theta z_\pi \text{ since } \lim_{n\to\infty}\mathbb{E}(Z_n^\pi) = z_\pi \\
&= \theta(z_\pi - z_\pi) \\
&= 0.
\end{aligned}$$

Taking the supremum over all $\theta \in \mathbb{R}$ we get that $\Lambda^*(z_\pi) \leq 0$. Hence, we have proved that $\Lambda^*(z_\pi) = 0 \leq \Lambda^*(x)$ for all $x \in \mathbb{R}$. This implies that z_π is a global minimun for Λ^*.

On the other hand, β is analytic hence differentiable on $\mathbb{R}$; and since $\beta(\theta) > 0$ for all $\theta \in \mathbb{R}$, it follows that $\Lambda = \log\beta$ is also differentiable on $\mathbb{R}$. Which implies that its convex conjugate Λ^* is strictly convex on its effective domain $\mathbb{R}$. We conclude by strict convexity that z_π is the unique minimum for Λ^*.

Moreover, let $x_0 \neq z_\pi$ such that $\Lambda^*(x_0) \leq 0$, then $\Lambda^*(x_0) \leq \Lambda^*(x)$ for all $x \in \mathbb{R}$. This means, x_0 is a different global minimum for Λ^*, contradicting the unicity of z_π. This completes the proof, as we required. □

These two lemmas allow us to establish the following last key result, required to state our main theorem.

Proposition 2.10. *For every Markovian strategy π_t in the investor's wealth model (2) such that $\lambda(\{x \in I : \pi(x) \neq 0\}) > 0$, and for any arbitrarily small $\epsilon > 0$, the wealth process V_t^π satisfies the following estimate,*

$$\mathbb{P}\big(V_t^\pi \geq V_0 + (z_\pi - \epsilon)t\big) \geq 1 - e^{-t\Lambda^*(z_\pi - \epsilon)}, \textit{ for large time } t. \tag{7}$$

Proof. Lemma 2.8 above entitled that $(V_t^\pi - V_0)/t$ satisfies a large deviations principle with good rate function Λ^*. Hence taking only the large deviations upper bound in Definition

(1.2.4) of [2, p. 5], we obtain for any arbitrary small $\epsilon > 0$ that,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\frac{V_t^\pi - V_0}{t} < z_\pi - \epsilon\Big) \leq - \inf_{x\in(-\infty, z_\pi-\epsilon]} \Lambda^*(x).$$

In the proof of the preceding lemma, we obtained that Λ^* is strictly convex, so it is non-increasing on $(-\infty, z_\pi]$. It follows from this lemma that,

$$\inf_{x\in(-\infty, z_\pi-\epsilon]} \Lambda^*(x) = \Lambda^*(z_\pi - \epsilon) > 0.$$

Hence, $\mathbb{P}\big(V_t^\pi \geq V_0 + (z_\pi - \epsilon)t\big) \geq 1 - e^{-t\Lambda^*(z_\pi-\epsilon)}$ for large time t. As we required. □

Remark 2.11. *In this result, one may not get in* (7) *a straightforward linear growth of the wealth V_t^π in the long-run, if $z_\pi = 0$ for all strategies π_t. So we resort to Martingale Theory to get a more precise formulation of our main theorem as below.*

Theorem 2.12. *In the investor's wealth model* (2),

1) If there is a Markovian strategy π_t with $\lambda(\{x \in I : \pi(x) \neq 0\}) > 0$, such that $z_\pi \neq 0$, then π_t is an ALA with GDFP in the model.

2) There is no Markovian strategy π_t such that $z_\pi \neq 0$ if and only if, for λ-almost all $x \in I$, the Markov process S_t starting from $S_0 = x$, with transition density $p(x,\cdot)$, is a martingale with respect to the natural filtration $\mathcal{F}_t$. However,

3) Under Assumption 2.3. 2), S_t cannot be a martingale for almost all $S_0 = x$. Hence under the condition of Proposition 2.10, there is always ALA with GDFP.

Proof. 1) Let π_t be a Markovian strategy such that $z_\pi \neq 0$. Then if $z_\pi > 0$, we choose ϵ small enough such that $z_\pi - \epsilon > 0$, hence we get by inequality (7) an asymptotic linear arbitrage with geometrically decaying failure probability. Similarly if $z_\pi < 0$, we choose the "opposite" strategy $-\pi$ for which $z_{-\pi} = -z_\pi$ which is strictly positive. So, with a similar choice of ϵ, one also gets an ALA with $GDFP$.

2) Let π_t be any Markovian strategy. If S_t is a martingale with respect to $\mathcal{F}_t$ for λ-$a.e.$ starting point x, then for all time t, $\mathbb{E}(S_t|\mathcal{F}_{t-1}) = S_{t-1}$. This holds whatever the law of S_{t-1} is. By a property of Conditional Expectation, we get $\mathbb{E}\big(\pi(S_{t-1})(S_t - S_{t-1})|S_{t-1}\big) = 0$. Hence $\mathbb{E}(Z_t) = 0$ for all time t, implying that $z_\pi = 0$.

Conversely, suppose that for some $A \in \mathcal{B}(I)$ with $\lambda(A) > 0$ and for all $x \in A$ we have for example,

$$\mathbb{E}(S_1 - S_0|S_0 = x) = \int_I p(x,y)(y-x)\lambda(dy) > 0.$$

Then consider the Markovian strategy $\pi(x) := \mathbf{1}_A(x)$ for all $x \in I$. From the proof of Proposition 2.7, we have

$$\begin{aligned} z_\pi &= \int_{I^2} \pi(x)(y-x)p(x,y)\gamma(x)\lambda_2(dx,dy) \\ &= \int_A \int_I (y-x)p(x,y)\lambda(dy)\gamma(x)\lambda(dx) > 0. \end{aligned}$$

Since $\int_I (y-x)p(x,y)\lambda(dy) > 0$, $\lambda(A) > 0$ and γ is positive on I, it follows that $z_\pi > 0$.

3) Finally, without loss of generality, we may suppose that the state space is $I = [0,1]$. If S_t was a martingale for almost all $S_0 = x$ then there would be a sequence $x_n \to 1$ such that

$$\mathbb{E}[S_1|S_0 = x_n] = x_n \to 1, \text{ as } n \to \infty. \tag{8}$$

On the other hand, under Assumption 2.3. 2), let $M > 1$ be an upper bound for $p(x,\cdot)$ on I, then

$$\mathbb{E}[S_1|S_0 = x_n] = \int_{[0,1]} yp(x_n, y)dy \leq \int_{[1-1/M,1]} yMdy < 1,$$

contradicting (8). We may hence conclude, as required. □

We construct below an example of a Markovian model (not necessarily governed by any stochastic difference equation like the one in [8]) and for which the stringent conditions in Assumption 2.3. 1) and 2) are verified.

Example 2.13. *(Markovian model with uniformly distributed-like transitions). Consider a discrete-time process S_t valued in a compact interval $I := [a,b]$ where $0 < a < b$. Suppose for every $x \in I$, and for any time $t \in \mathbb{N}$, the conditional probability law of $S_{t+1}|S_t = x$ is given by the density $p(x,\cdot)$ defined for all $y \in I$ by,*

$$p(x,y) := \begin{cases} \frac{1}{b-a} & \text{if } x = a \text{ or } x = b, \\ \frac{1}{2(x-a)} & \text{if } a < x < b \text{ and } a \leq y \leq x, \\ \frac{1}{2(b-x)} & \text{if } a < x < b \text{ and } x < y \leq b. \end{cases}$$

It follows from Subsection 3.4.1 of [10, p. 68] that the one-step transition probability kernel $P(x,\cdot)$ so defined, generates a time-homogeneous Markov process, which is indeed S_t by construction, and has density $p(x,\cdot)$ with respect to the Lebesgue measure λ.

Furthermore, it is easy to check that for all $x \in I$, $p(x,\cdot)$ verifies Assumption 2.3. 1) and 2). Hence according to Theorem 2.12, with such a Markovian model of stock prices S_t, it is possible to find a (bounded) Markovian strategy that generates asymptotic linear arbitrage with GDFP in the investor's wealth model (2). ■

3 Conclusion

To conclude, we draw a comparative remark of Theorem 2.12 to Theorem 2.4 of [8]. Note first that, as known from [1, pp. 211-228], a discrete-time stochastic process X_t is a Markov process if and only if the process evolves in time according to a stochastic recursion of the form,

$$X_{t+1} = f(X_t, \varepsilon_{t+1}), \text{ for } t \geq 0, \tag{9}$$

where ε_t is a "driving" sequence of $i.i.d.$ random variables independent from X_0 and f is some measurable function. In [8], we proved existence of ALA with $GDFP$ (Theorem 2.4) with the investor's wealth in (2) in the more common modeling setting that the stock prices process S_t in (1) satisfies the stochastic recursion (9) and takes values in the entire real line $\mathbb{R}$. To achieve this, we used the Markovian structure provided by the difference equation (1) by imposing structural conditions $(A_1), (A_2), (A_3)$ and (6) stated in [8, pp. 92-93]. And we checked the result in Example 2.18 by considering a discrete-time Ornstein-Uhlenbeck process where ε_t are standard normally distributed, hence with density supported in the entire $\mathbb{R}$. In the present paper, we proved existence of ALA with $GDFP$ in Theorem 2.12 with the same investor's wealth process in a modified modeling case where the stock prices process S_t is not specified by any stochastic recursion of the type (9), but is just assumed to be any general discrete-time homogeneous Markov process taking values this time in a compact interval I. We notice that, even if any discrete-time Markov process admits a stochastic recursion of the form (9) as established from [1, pp. 211-228], by avoiding the use of the Markovian structure of the stochastic recursion as in [8], we had to find and impose different stringent but still realistic conditions (Assumption 2.1. 2) and Assumption 2.3. 1) and 2)) on the Markov process S_t in order to prove the new corresponding existence theorem. By verifying this result in Example 2.13 above, we found that we necessarily had to choose the (kernel) density of the one-step transition probability kernel that is supported in the compact interval I and no longer in the entire real line $\mathbb{R}$.

Finally, we notice that Theorem 2.4 of [8] and Theorem 2.12 of this paper differ also from each other not only on the mathematical tools used to derive each, but also on the fact that the Markovian strategy realizing asymptotic linear arbitrage in the former is explicitly and uniquely constructed while one may find several Markovian strategies π_t satisfying the required conditions $\lambda(\{x \in I : \pi(x) \neq 0\}) > 0$ and $z_\pi \neq 0$ in the later. Consequently, for an arbitrageur trader, the modeling setting of the present article offers more opportunities to make risk-less profit in long-term in the sense of our proposed Definition 2.2 in [8].

Conflict of interest

The author declares no conflict of interest regarding this paper.

References

[1] Bhattacharya, R.N. & Waymire, E.C. (1992), "Stochastic Processes with Applications", John Wiley & Sons Inc., USA.

[2] Dembo, A. & Zeitouni, O. (1998), "Large Deviations Techniques and Applications", Second Edition, Springer, USA.

[3] Föllmer, H. & Schachermayer W. (2008), "Asymptotic Arbitrage and Large Deviations", *Math. Financ. Econ.*, **1**, 213-249.

[4] Harris, T.E. (1963), "The Theory of Branching Processes", Springer, Berlin.

[5] Kim, G. & David, H.T. (1979), "Large Deviations of Functions of Markovian Transitions and Mathematical Programming Duality", *Ann. Probab.*, **7**, 874-881.

[6] Klebaner, F.C. (2005), "Introduction to Stochastic Calculus with Applications", Second Edition, Imperial College Press, London.

[7] Kontoyiannis, I. & Meyn, S. (2005), "Large Deviations Asymptotics and the Spectral Theory of Multiplicative Regular Markov Processes", *Electronic. Journ. Probab.*, **10**, 61-123.

[8] Mbele Bidima, M.L.D. (2014), "Asymptotic Linear Arbitrage and Utility-based Asymptotic Linear Arbitrage in Mean-reverting Financial Markets", *Int. J. Appl. Math.*, **27**, 89-107.

[9] Mbele Bidima, M.L.D. & Rásonyi, M. (2015), "Asymptotic Exponential Arbitrage and Utility-based Asymptotic Arbitrage in Markovian Models of Financial Markets", *Acta Appl. Math.*, *Springer*, **138**, 1-15.

[10] Meyn, S. & Tweedie, R. (2009), "Markov Chains and Stochastic Stability", Second Edition, Cambridge University Press, Cambridge.