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Pricing a Cap under the Affine Term Structure Interest Rate Model

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Abstract

This study seeks to find an explicit price of a cap when the term structure of interest rate is the Pearson-Sun model proposed by Pearson and Sun (1994). This allow company's (investors, risk managers) to hedge against the risk associated with floating interest rate fluctuations going beyond a certain strike rate(the cap rate). The study obtained an expression for the price of the cap which consists of the price of a zero coupon bond written on the Pearson-Sun model plus a cumulative sum of continuously traded caplets, where each caplet is the price of a European call option with strike price as the cap level $\bar{r}$ per unit of the principal amount of the loan, K and written on the interest rate model. The payoff of this contract is $h(r(t),t) = max(r(t),\bar{r})K = (r(t)-\bar{r})^+K$.

Keywords : Interest rate modeling, Pearson-Sun Model, Price of a Cap, Term Structure of Interest Rate, Zero-coupon Bond,

1 INTRODUCTION

Interest rate derivatives have become so important in hedging and managing risk, Investors seek some compensation for the risk they take in the form of returns in investing. The more the returns, the better. But higher returns are associated with higher risk, and since the higher the risk the more uncertain the returns are, and also, investors feel repugnant towards financial risk (downside risk)[1]. In order to realise their goal of securing higher returns or at worst reduce the impact of a loss, investors seek some form of insurance. Interest rate derivatives have become so important in hedging and managing risk, these derivatives are widely used by investors and risk managers for this purpose and hence are becoming increasingly popular.

A derivative security is a financial contract whose value at expiration is determined exactly by the price or prices of the underlying financial assets [2]. That is the payoffs of derivatives are determined by the performance of financial securities called the underlying assets. These underlying assets include bonds, common stock, interest rates etc. Payoffs of derivatives depend on the performance of the underlying assets. When interest rates determine the payoffs of these derivatives, they are called interest rate derivatives. The investor benefits from fluctuations in interest rates while reducing the risk of a loss. Because

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of their usefulness in hedging and managing risk, there has been a tremendous growth in interest rate contingent claims such as caps, swaptions and bond options, and therefore the growth of the derivatives market. This has brought about ever-increasing volume and range of interest rate dependent derivatives[3].

The increasing popularity of these derivatives has made their valuation a major concern [21]. Valuation, basically, involves pricing and prediction of the behavior of these derivatives into the future. A lot of work has been done in developing techniques for pricing these derivatives. Fundamental or basis to pricing interest rate derivatives is the idea that, prices should be the discounted expected values of cash flows [2]. In the process, interest rate models are built. In the valuation of interest rate derivatives, models, to mimic the behavior of interest rates are developed. These are known as interest rate models. An interest rate model is a characterization of the uncertainty of future interest rates, a probabilistic description of future evolution of interest rates based on today's information. There are many models describing the behavior of interest rates. In this research, it seeks to find the explicit price of a cap, a derivative whose payoff depends on the performance of interest rates. It consider pricing it when the evolution of interest rates are defined by the Pearson-Sun model.

An interest rate cap is a contract which protects the buyer from having to pay more than a specified rate, the cap rate. The seller promises to pay the buyer the difference between the floating rate and the cap rate calculated on a principal amount for a period of time. A twelve-month cap, for example, would consist of intervals, say three months, in which floating rates are reset. On these dates, the floating rate is compared with the cap rate and the appropriate decision made on its payoff. These series of options are called caplets. Caplets are the fundamental units of a cap.
The purchased price of the cap is called the premium. The premium is the compensation on the risk taken to buy the option before the expiration. The purchaser of a cap will continue to benefit from the fall in interest rate below the strike price (or rise of the interest rate above the strike price) which makes the cap a popular means of hedging a floating rate loan. A cap is a series of European call option (caplets) which exist for each period the of cap agreement is in existence. The caplets, an European interest rate option sets a maximum interest rate for an interest rate derivative. It is used by investors to hedge against the risk associated with floating interest rate financial products, though investors are more likely to invest in a cap rather than a single caplet. Caplet holders must decide whether to exercise the option or to let it expire when each interest rate payment day is met. The modelling of the interest rate derivative is usually done on a time dependent multi-dimensional lattice built for the underlying risk drivers.
There are many models describing the behavior of interest rates. This performance in the behavior of the interest rate is modelled in order to evaluate or finding the price of assets.

Some researchers have studied and used the Black-Scholes model, Vasicek model Cox, Ingersol and Ross (CIR), Hull and White model coupled with underlying assumption within the various models. The assumption of a log-normal distribution for forward rate in Black-Scholes model. Their model assumed a constant volatility. Though a number of researchers

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have relied upon this approach to price interest rate derivatives, other researchers have tried to incorporate stochastic interest rates into valuation models of options. The Vasicek model is an example of a stochastic interest rate model. Cox, Ingersol and Ross (CIR) model corrects the problems with the Vasicek model allowing interest rates to be negative. The Pearson and Sun (1994) model extends the CIR model in terms of modification by introducing a lower bound on the interest rates and hence, prevents it from getting interest rates below a particular rate.

In order to hedge against the risk associated with floating interest rate fluctuations going beyound a certain strike rate(the cap rate).There is the need to set a cap rate. However the values of derivatives are uncertain due to the randomness in their nature. Explicit pricing of the cap, above which a premium is gained once the floating rate exceed the cap rate is of great relevance. Finding a closed form expression to option prices depend on the distribution of the interest rate process. The study find an expression for the option value under non Gaussian interest rate process using analytical solution.The recent turmoil in financial markets has been partly caused by insufficient attention to rigorous financial modeling. Among the causes of this failing is the relative shortage of researches and trained professionals working on mathematically increasingly technical approaches to the analysis of financial markets.

Abukar and Mohamoud (2003)[7], in pricing interest rate caps and floors, adopted the Black-Scholes approach. They used the implied forward rate at each caplet's maturity as the underlying asset. They also assumed that the underlying forward rates follow a lognormal process. Hence, the price of a cap is the sum of the sequence of caplets, which are essentially, call options. Their model assumed a constant volatility.

2 INTEREST RATE MODELING

2.1 Risk-Neutral Measure

Fundamental to pricing derivatives is risk neutrality. That is, the investor is risk neutral and hence the prices of these derivatives are the discounted expected payoffs at the risk-free rate. But in the real world, prices take into account the uncertainties surrounding the payoffs, the risk. These are the risk-premiums and can not be ignored; they have to be compensated for. The assumption of risk-neutrality in pricing gives rise to risk-neutral probabilities, denoted $\mathbb{Q}$. To price a derivative, the real world probabilities, denoted $\mathbb{P}$, which take risk-premium into consideration must be changed to risk-neutral probabilities, $\mathbb{Q}$.

A risk-neutral measure is a probability measure under which, given information up to a particular time, the price of an asset at that time is the discounted expected future payoffs of the asset at the risk-free rate. Also fundamental to pricing derivatives is the assumption of no-arbitrage. An arbitrage position allows an investor to make profit out of nothing, ie. without taking any risk [14] That is, an arbitrage opportunity is such that with an initial value of zero, it produces a non-negative value at the end of the period.

Changing from one measure to the other is, therefore, important in pricing. Girsanov's

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theorem, stated below, allows us to change from one measure to the other. It also guarantees that there is no arbitrage in the market as we make the change. Furthermore, it ensures that the discounted prices of assets are martingales under the probability measure. Since, the no-arbitrage assumption is fundamental, there is no arbitrage if and only if there exist an equivalent measure [2].

In pricing, we are often required to find an equivalent probability measure $\mathbb{Q}$ under which the underlying price process has the same stochastic return as that of the $\mathbb{P}$ measure, and hence, creating a no-arbitrage condition.
Radon-Nikodym derivative [9] It tells if and how it is possible to change from one probability measure to another, Such changes of probability measure are the cornerstone of the rational pricing of derivatives and are used for converting actual probabilities into those of the risk neutral probabilities.

So, the Girsanov's theorem allows us to change from one measure to the other. But, since our interest rate model is already under an equivalent measure $\mathbb{Q}$, for the rest of the study, we will be using $\mathbb{Q}$ as defined in the Girsanov's theorem and $\tilde{\mathscr{B}}(t)$ to represent a Brownian motion under this measure.

Given the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, if $r(t)$ is the interest rate process driven by the Brownian motion $\tilde{\mathscr{B}}(t)$, and $G(t)$ is any $\mathscr{F}(t)$-adapted contingent claim payable at time t, then its value at that time is given by

$$G(t) = \mathbb{E}_{\mathbb{Q}}\left(\exp\left[-\int_t^T r(s)ds\right] G(T) \mid \mathscr{F}(t)\right).$$

2.2 Bonds

Zero-coupon bond are the basis in pricing interest rate contingent assets. Bonds are debt securities issued by governments and corporate entities to raise money. The holder of the bond (the lender) gives an initial amount to the issuer. In return, the entity periodically pays a predetermined interest rate, called the coupon rate, to the holder for the duration of the bond, and finally pays the principal at the maturity date, called the face value or par value. Bonds are sold either at premium or discount. At premium it is sold at a price greater than its face value and at discount, at a price below its face value. A person would buy a bond at premium if it has a coupon rate greater than the existing interest rate. A bond is traded at discount if its coupon rate is less than the existing interest rate. Also, bonds exist either with or without coupons, variable or fixed coupons etc. In this study,we will be using bonds without coupons. A zero-coupon bond, also called a T-bond, is a bond without regular payments or coupons and only have a one-time payment at maturity. A zero-coupon bond with maturity date T, also called a T-bond, is a contract that guarantees the holder a cash payment of one unit on the date T. The price at time t of a bond with maturity date T is denoted $p(t,T)$. A unit amount is due immediately, hence we have $p(t,t) = 1$ for all t. Also, the assumption of no arbitrage in the market gives us the condition $p(t,T) \leq 1$ for all T [18].
In interest rate theory, zero-coupon bonds are fundamental. They are often used as the basic quantities from which all rates can be recovered. Also, the prices of zero coupon bonds can be defined in terms of a given type of interest rates[19].

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2.3 Interest Rate Models

We begin by assuming that the short rate process $r(t)$ solves a stochastic differential equation under $\mathbb{Q}$ and is modelled by the SDE of the form

$$dr(t) = \beta(t, r(t))dt + \gamma(t, r(t))d\bar{\mathscr{B}}(t) \tag{2.3.1}$$

where $\beta(t, r(t))$ is the drift and $\gamma(t, r(t))$ the volatility of the process and $\bar{\mathscr{B}}$ is under an equivalent martingale measure $\mathbb{Q}$.
These models are usually called short-term models because $r(t)$ is the interest rate for short-term borrowing. The interest rate process is being driven by a single Brownian motion resulting in a single SDE, hence, these are known as one-factor models. There are some interest rate models, such as Black's pricing model, which do not take into account the evolution of interest rates with time. A class that take this relationship into account are the term structure models.

2.4 Affine Term-Structure Models

A term structure is a function which maps the time to maturity of a discount bond to its current price or yield to maturity [3]. The expectation of the participants of the bond market is reflected in the term structure. In this study, we will be using the one-factor affine term structure model. Hence, we focus our attention on Vasicek, Cox, Ingersoll, and Ross (CIR) and the Pearson-Sun models.

2.4.1 Vasicek Model

With the following drift and volatility

$$\begin{aligned}\beta(t, r) &= -\alpha r(t) + \alpha\mu = \alpha(\mu - r(t)),\\ \gamma(t, r) &= \sqrt{\sigma^2} = \sigma\end{aligned}$$

we have the model proposed by Vasicek (1977). The SDE for modelling the interest rate $r(t)$ is given by

$$dr(t) = \alpha(\mu - r(t))dt + \sigma d\bar{\mathscr{B}}(t) \tag{2.4.1}$$

where α is the speed of the reversion to the constant mean, μ is the long-term level of interest rate (the long-term mean) and σ is the volatility of the interest rate process.

This means that interest rates are being pulled to the long-term mean, μ at a speed α. This is known as mean reversion . The drift $\alpha(\mu - r(t))$ ensures that this phenomenon works. When the state of $r(t)$ is above μ, the drift of the process is negative and positive when $r(t)$ is below μ, reverting the process to the long-term mean at a speed rate α. When $\alpha = 0$, interest rates follow a random walk and do not revert to a long-term mean. When α is small, the interest rates tend to the long-term mean slowly and hence have high persistence. For a large α, the mean reversion is very quick.

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The solution to equation 2.4.1 is given by

$$dr + \alpha r dt = \alpha\mu dt + \sigma d\tilde{\mathscr{B}}(t),$$

Multiply through by integration factor $e^{\alpha t}$

$$\begin{aligned} e^{\alpha t} dr + e^{\alpha t}\alpha r dt &= \alpha\mu e^{\alpha t} dt + \sigma e^{\alpha t} d\tilde{\mathscr{B}}(t) \\ d\left(e^{\alpha t} r(t)\right) &= \alpha\mu e^{\alpha t} dt + \sigma e^{\alpha t} d\tilde{\mathscr{B}}(t) \end{aligned}$$

taking integral of both sides

$$\begin{aligned} \int_0^t e^{\alpha t} r(s) ds &= \int_0^t \alpha\mu e^{\alpha s} ds + \int_0^t \sigma e^{\alpha s} d\tilde{\mathscr{B}}(s) \\ e^{\alpha t} r(t) - e^{\alpha t} r(0) &= \mu(e^{\alpha t} - e^{\alpha 0}) + \sigma \int_0^t e^{\alpha s} d\tilde{\mathscr{B}}(s) \\ e^{\alpha t} r(t) &= r(0) + \mu(e^{\alpha t} - 1) + \sigma \int_0^t e^{\alpha s} d\tilde{\mathscr{B}}(s) \end{aligned}$$

Multiply both sides by $e^{-\alpha t}$

$$\begin{aligned} r(t) &= r(0)e^{-\alpha t} + \mu(1 - e^{\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} d\tilde{\mathscr{B}}(s) \\ r(t) &= r(0)e^{-\alpha t} + \mu - \mu e^{\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} d\tilde{\mathscr{B}}(s) \\ r(t) &= \mu + e^{-\alpha t}(r(0) - \mu) + \sigma \int_0^t e^{-\alpha(t-s)} d\tilde{\mathscr{B}}(s) \end{aligned}$$

hence, we have

$$r(t) = \mu + \exp^{-\alpha t}(r(0) - \mu) + \sigma \int_0^t \exp(-\alpha(t-s)) d\tilde{\mathscr{B}}(s).$$

Also, comparing $\beta(t,r)$ and $\gamma^2(t,r)$ of the vasicek model to their general affine form for $\beta(t,r)$ and $\gamma^2(t,r)$ $(\kappa(t) = \alpha\mu, \quad \delta(t) = \sigma^2, \quad \alpha = \alpha, \quad \eta(t) = 0)$ then here is the following boundary value problems as ;

$$\begin{aligned} B_t(t,T) - \alpha B(t,T) &= -1, \\ B(T;T) &= 0 \end{aligned}$$

and

$$\begin{aligned} A_t(t,T) &= \alpha\mu B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T;T) &= 0 \end{aligned}$$

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The price of a zero-coupon bond is given by, $p(t,T) = \exp(A(t,T) - B(t,T)r(t)$ where

$$B(t,T) = \frac{1-\exp(-\alpha(T-t)}{\alpha},$$

$$A(t,T) = [B(t,T)-(T-t)]\left(\mu - \frac{\sigma^2}{2\sigma^2}\right) - \frac{\sigma^2}{4\alpha}B(t,T)^2$$

and

$$r(t) = \mu - \exp^{-\alpha t}(r(0)-\mu) + \sigma\int_0^t \exp(-\alpha(t-s))d\tilde{\mathscr{B}}(s).$$

The distribution of the interest rate is Gaussian in the Vasicek model and hence, there is a positive non-negative probability of a negative interest rate. The probability of a negative interest rate is not desirable. Also, it assumes a constant volatility of interest rates. These are some challenges with Vasicek model.

2.4.2 The Cox, Ingersoll, and Ross (CIR) model

The Cox, Ingersoll, and Ross (CIR) model seeks to solve these problems. A class of interest rate models which seek to solve this negativity problem are the square-root processes. They solved these problems by introducing a square root of the interest rate into the model. Square-root process is popular in financial modelling because of its positivity property.The CIR model proposed by Cox et al. (1985) is an example of a square-root process. Cox, Ingersoll, and Ross (CIR) model With the following drift and volatility

$$\beta(t,r) = \alpha(\mu - r(t)),$$
$$\gamma(t,r) = \sqrt{\sigma^2 r(t)}$$

Here is the CIR interest rate model given by

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}d\tilde{\mathscr{B}}(t) \qquad (2.4.2)$$

The CIR model has the same drift as the Vasicek model but differs in the volatility of the interest rate process.
The interpretation of the terms are the same as in the Vasicek model. The only difference is, this model has a volatility of $\sigma\sqrt{r(t)}$ which is dependent on time. In fact, the volatility increases as the interest rate increase, as compared to the Vasicek model which has a constant volatility σ.

The solution to equation 2.4.2 is given by

$$r(t) = \mu + \exp^{-\alpha t}(r(0)-\mu) + \sigma\int_0^t \sqrt{r(s)}\exp(-\alpha(t-s))d\tilde{\mathscr{B}}(s).$$

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Also, we have the following boundary value problems

$$A(t,T) = \frac{2\alpha\mu}{\sigma^2}\log\left(\frac{2\rho\exp[\frac{\rho+\alpha}{2}(T-t)]}{(\rho+\alpha)(\exp[\rho(T-t)]-1)+2\rho}\right) \quad (2.4.3)$$

$$B(t,T) = \frac{2(\exp[\rho(T-t)]-1)}{(\rho+\alpha)(\exp[\rho(T-t)]-1)+2\rho} \quad (2.4.4)$$

and

$$\rho = \sqrt{\alpha^2+2\sigma^2}$$

The price of a zero coupon bond is given by $p(t,T) = \exp(A(t,T) - B(t,T)r(t)$ with the r(t), A(t,T) and B(t,T) under the CIR model.

Even though the CIR model has this positivity property, it has the possibility of the interest rate getting very close to and becoming zero. When this happens, the rates loose their randomness.

2.4.3 Pearson and Sun model

Pearson and Sun (1994) model extends the CIR model by introducing a lower bound on the interest rates and hence, prevents it from getting below a particular rate. Pearson-Sun model again changing the parameters of the drift and volatility of the interest rate process gives us another affine term structure model, the Pearson-Sun model. With the following drift and volatility

$$\beta(t,r) = \alpha(\mu - r(t)),$$
$$\gamma(t,r) = \sqrt{\sigma^2(r(t)-\upsilon)}$$

Here, the Pearson-Sun interest rate model given by

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{(r(t)-\upsilon)}d\tilde{\mathscr{B}}(t) \quad (2.4.5)$$

The solution to equation 2.4.5 is given by

$$r(t) = \mu + \exp^{-\alpha t}(r(0)-\mu) + \sigma\int_0^t \sqrt{(r(t)-\upsilon)}\exp(-\alpha(t-s))d\tilde{\mathscr{B}}(s).$$

This model introduces a lower bound on the interest rate as υ.
let $X(t) = r(t) - \upsilon \Rightarrow r(t) = X(t) + \upsilon$
we have

$$dr(t) = \alpha(\mu - (X(t)+\upsilon))dt + \sigma\sqrt{(X(t))}d\tilde{\mathscr{B}}(t)$$

and the solution to this is given by

$$r(t) = \mu + \exp^{-\alpha t}((X(0)+\upsilon)-\mu) + \sigma\int_0^t \sqrt{X(s)}\exp(-\alpha(t-s))d\tilde{\mathscr{B}}(s).$$

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But the price of the zero coupon bond given some information $\mathscr{F}(t)$ adapted contingent claim payable at time t, with interest rate driven by the Brownian motion $\bar{\mathscr{B}}(t)$ under the equivalent martingale measure $\mathbb{Q}$ is defined as

$$p(t,T)=\mathbb{E}_{\mathbb{Q}}\left[\exp\left[\left(-\int_t^T r(s)d(s)\right)\right] \mid \mathscr{F}(t)\right]$$

But here, $r(s)=X(s)+\upsilon$, hence we have

$$\begin{aligned}
p(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[\exp[\left(-\int_t^T (\upsilon+X(s))ds\right)] \mid \mathscr{F}(t)\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_t^T \upsilon ds]\exp[\left(-\int_t^T X(s)ds\right)] \mid \mathscr{F}(t)\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp[-\upsilon(T-t)]\exp[\left(-\int_t^T X(s)ds\right)] \mid \mathscr{F}(t)\right]\\
&= \exp[-\upsilon(T-t)]\mathbb{E}_{\mathbb{Q}}\left[\exp[\left(-\int_t^T X(s)ds\right)] \mid \mathscr{F}(t)\right]\\
&= \exp[-\upsilon(T-t)]\exp[A(t,T)-B(t,T)X(t)]\\
&= \exp[-\upsilon(T-t)]\exp[A(t,T)-B(t,T)X(t)-\upsilon B(t,T)+\upsilon B(t,T)]\\
&= \exp[-\upsilon(T-t)]\exp[A(t,T)-B(t,T)(X(t)+\upsilon)+\upsilon B(t,T)]
\end{aligned}$$

By substituting $r(t)=X(t)+\upsilon$, gives

$$\begin{aligned}
p(t,T) &= \exp[-\upsilon(T-t)]\exp[A(t,T)-B(t,T)r(t)+\upsilon B(t,T)]\\
&= \exp[A(t,T)-\upsilon(T-t)+\upsilon B(t,T)-B(t,T)r(t)]
\end{aligned}$$

Hence the price of the zero coupon bond under person sun model given as

$$p(t,T)=\exp[A(t,T)-\upsilon(T-t)+\upsilon B(t,T)-B(t,T)r(t)]$$

where $A_p(t,T)=A(t,T)-\upsilon(T-t)+\upsilon B(t,T)$.

$A(t,T)$ and $B(t,T)$ are given as the same for equation 2.4.3 and equation 2.4.4 respectively in the CIR model where we replace μ by $\mu-\upsilon$.

The boundary value problems are as follows;

$$A(t,T)=\frac{2\alpha(\mu-\upsilon)}{\sigma^2}\log\left(\frac{2\rho\exp[\frac{\rho+\alpha}{2}(T-t)]}{(\rho+\alpha)(\exp[\rho(T-t)]-1)+2\rho}\right) \tag{2.4.6}$$

$$B(t,T)=\frac{2(\exp[\rho(T-t)]-1)}{(\rho+\alpha)(\exp[\rho(T-t)]-1)+2\rho} \tag{2.4.7}$$

and

$$\rho=\sqrt{\alpha^2+2\sigma^2}$$

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3 PRICE OF A CAP UNDER THE PEARSON-SUN MODEL

The Pearson-Sun model under an equivalent martingale measure $\mathbb{Q}$ is given by

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{(r(t) - \upsilon)}d\tilde{\mathscr{B}}(t) \tag{3.0.1}$$

where μ is the long-term level of interest rate (the long-term mean), α is the speed of reversion to the constant mean μ and $\sigma\sqrt{(r(t) - \upsilon)}$ is the volatility of the interest rate process and $\tilde{\mathscr{B}}(t)$ is a Brownian motion under a risk-neutral probability measure $\mathbb{Q}$.
The solution to equation 3.0.1 is given by

$$r(t) = \mu + \exp(-\alpha(t))[r(0) - \mu] + \sigma\int_0^t \sqrt{(r(s) - \upsilon)}\exp(-\alpha(t-s))d\tilde{\mathscr{B}}(s). \tag{3.0.2}$$

A cap (a loan at variable interest rate), essentially, is a portfolio of elementary units called caplets. We, therefore, define a cap with a cap rate $\hat{r}$, as a series of caplets each with a payoff $(r(t) - \hat{r})^+$ on the continuous reset dates $t \in (0,T)$. That is, we define the payoff of a cap as a sequence of payments on dates $t \in (0,T)$ with a terminal payoff on T.

Let us define the sequence of payoffs per unit of the principal amount of the loan as a function $h(r(t),t) = (r(t) - \hat{r})^+$ and the terminal payoff $g(r(T),T) = 1$ at maturity.

Given a sequence of payoffs of cap and a terminal payoff, the goal is to find the value of the cap at any time $t < T$. The price of the cap is the sum of all the discounted sequence of payoffs, including the discounted terminal payoff. For this case, given the payoffs at time $t \in (0,T)$ as $h(r(t),t)$ and a terminal payoff as $g(r(T),T)$, here, the value of a cap at time t, maturing at time T, given as

$$F(r(t),t) = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T \varphi_{t,s}h(r(s),s)ds + \varphi_{t,T}g(r(T),T) \mid \mathscr{F}(T)\right] \qquad 0 \le t \le T \tag{3.0.3}$$

where $\varphi_{t,s} = \exp(-\int_t^s r(u)du)$, where $\varphi_{t,s}$ is a discounting factor (see definition 4.1.8)

Now seek to simplify the right-hand side of equation 3.0.3 Since the terminal condition is $g(r(T),T) = 1$, substituting into equation 3.0.3 gives us

$$F(r(t),t) = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T \varphi_{t,s}h(r(s),s)ds + \varphi_{t,T} \mid \mathscr{F}(T)\right] \qquad 0 \le t \le s \le T \tag{3.0.4}$$

Given that the expectation of a sum is the same as the sum of the expectation and using Fubini's theorem **??**, rewrite and simplify equation 3.0.4 as

$$F(r(t),t) = \mathbb{E}_{\mathbb{Q}}[\int_t^T \varphi_{t,s}h(r(s),s)ds \mid \mathscr{F}(T)] + \mathbb{E}_{\mathbb{Q}}[\varphi_{t,T} \mid \mathscr{F}(T)]$$

$$F(r(t),t) = \int_t^T \mathbb{E}_{\mathbb{Q}}[\varphi_{t,s}h(r(s),s) \mid \mathscr{F}(T)]ds + \mathbb{E}_{\mathbb{Q}}[\varphi_{t,T} \mid \mathscr{F}(T)]$$

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Let

$$f(r(t),t) = \mathbb{E}_{\mathbb{Q}}[\varphi_{t,s}h(r(s),s) \mid \mathscr{F}(T)] \tag{3.0.5}$$

This gives

$$F(r(t),t) = \int_t^T f(r(s),s)ds + \mathbb{E}_{\mathbb{Q}}[\varphi_{t,T} \mid \mathscr{F}(T)]. \tag{3.0.6}$$

But $\mathbb{E}_{\mathbb{Q}}[\varphi_{t,T}]$ is the price of a zero coupon bond at time t maturing at time T. Let's make the transformation $r(s) = \upsilon + X(s) \Rightarrow X(s) = r(s) - \upsilon$, so that the Pearson-Sun model given in equation 3.0.1 becomes

$$dr(t) = \alpha[\mu - \upsilon - (X(t)]dt + \sigma\sqrt{(X(t))}d\mathscr{B}(t) \tag{3.0.7}$$

which is in the form of the CIR interest rate model. This gives

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\varphi_{t,T} \mid \mathscr{F}(T)] &= \mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_t^T r(s)ds] \mid \mathscr{F}(T)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T [\upsilon + X(s)]ds\right) \mid \mathscr{F}(T)\right]
\end{aligned}$$

$$\mathbb{E}_{\mathbb{Q}}[\varphi_{t,T}] = \exp[A_C(t,T) - B(t,T)r(t)] \tag{3.0.8}$$

thus where $r(s) = X(s) + \upsilon$, and solving for $\mathbb{E}_{\mathbb{Q}}[\varphi_{t,T}]$ gives,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\varphi_{t,T}] &= \mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_t^T \upsilon ds]\exp[\left(-\int_t^T X(s)ds\right)] \mid \mathscr{F}(T)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp[-\upsilon(T-t)]\exp[\left(-\int_t^T X(s)ds\right)] \mid \mathscr{F}(T)\right] \\
&= \exp[-\upsilon(T-t)]\mathbb{E}_{\mathbb{Q}}\left[\exp[\left(-\int_t^T X(s)ds\right)] \mid \mathscr{F}(T)\right] \\
&= \exp[-\upsilon(T-t)]\exp[A(t,T) - B(t,T)X(t)] \\
&= \exp[-\upsilon(T-t)]\exp[A(t,T) - B(t,T)X(t) - \upsilon B(t,T) + \upsilon B(t,T)] \\
&= \exp[-\upsilon(T-t)]\exp[A(t,T) - B(t,T)(X(t) + \upsilon) + \upsilon B(t,T)]
\end{aligned}$$

By substituting $r(t) = X(t) + \upsilon$, gives

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\varphi_{t,T}] &= \exp[-\upsilon(T-t)]\exp[A(t,T) - B(t,T)r(t) + \upsilon B(t,T)] \\
&= \exp[A(t,T) - \upsilon(T-t) + \upsilon B(t,T) - B(t,T)r(t)]
\end{aligned}$$

Hence the price of the zero coupon bond under person sun model given as

$$\mathbb{E}_{\mathbb{Q}}[\varphi_{t,T}] = \exp[A(t,T) - \upsilon(T-t) + \upsilon B(t,T) - B(t,T)r(t)]$$

$$\mathbb{E}_{\mathbb{Q}}[\varphi_{t,T}] = \exp[A_C(t,T) - B(t,T)r(t)]$$

where $A_C(t,T) = A(t,T) - \upsilon(T-t) + \upsilon B(t,T)$. $A(t,T)$ and $B(t,T)$ are given by equations 3.0.9 and 3.0.10 below respectively.

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$$A(t,T) = \frac{2\alpha(\mu-\upsilon)}{\sigma^2}\log\left(\frac{2\rho\exp[\frac{(\rho+\alpha)}{2}(T-t)]}{(\rho+\alpha)(\exp[\rho(T-t)]-1)+2\rho}\right) \tag{3.0.9}$$

$$B(t,T) = \frac{2(\exp[\rho(T-t)]-1)}{(\rho+\alpha)(\exp[\rho(T-t)]-1)+2\rho} \tag{3.0.10}$$

and

$$\rho = \sqrt{\alpha^2+2\sigma^2}$$

Solve equation 3.0.5 by Substituting $h(r(t),t) = (r(t)-\hat{r})^+$ into equation 3.0.5, we have

$$f(r(t),s) = \mathbb{E}_{\mathbb{Q}}[\varphi_{t,s}(r(t)-\hat{r})^+] \tag{3.0.11}$$

Equation 3.0.11 is the price of a caplet at some time t with a reset date s. This, can also be observed as a European call option maturing at time s with strike price $\hat{r}$, with $r(t)$ as the underlying. The problem is that $r(t)$ is not a Gaussian, if not we could have used Black-Scholes approach to price the caplets. This leaves us with the choice of using the PDE approach to solving this problem.

The terminal condition of the caplet is defined when $t = s$ in equation 3.0.5. That is

$$\begin{aligned} f(r(s),s) &= \mathbb{E}_{\mathbb{Q}}[\varphi_{s,s}h(r(s),s) \mid \mathscr{F}(T)] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_s^s r(u)du]h(r(s),s)\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[h(r(s),s)\right] \end{aligned}$$

But $h(r(s),s) = (r(s)-\hat{r})^+$, giving

$$f(r(s),s) = \mathbb{E}_{\mathbb{Q}}\left[(r(s)-\hat{r})^+\right] \tag{3.0.12}$$

From equation 3.0.5, the price of the caplet at time (t) with reset date (s)

$$\begin{aligned} f(r(t),s) &= \mathbb{E}_{\mathbb{Q}}[\varphi_{t,s}h(r(s),s)] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_t^s r(u)du]h(r(s),s)\right] \end{aligned}$$

Make a transformation $r(t) = \upsilon + X(t)$ and also $r(s) - \upsilon = X(s)$. Which implies $r(u) = \upsilon + X(u)$. This, again gives the transformed model given in equation 3.0.7 where the asset payoff is $h(X(s),s)$ at maturity s.

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Here,

$$
\begin{aligned}
f(r(t),s) &= \mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_t^s \upsilon + X(u)du]h(X(s),s)\right] \\
&= \exp[-\int_t^s \upsilon du]\mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_t^s X(u)du]h(X(s),s)\right] \\
&= \exp[-\upsilon(s-t)]\mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_t^s X(u)du]h(X(s),s)\right] \\
&= \exp[-\upsilon(s-t)]g(X(t),t)
\end{aligned}
$$

where

$$g(X(t),t) = \mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_t^s X(u)du]h(X(s),s)\right] \tag{3.0.13}$$

Equation 3.0.13 is the price of an asset paying $h(X(s),s)$ at maturity s, with an interest rate which follows the Pearson-Sun model given in equation 3.0.7 above. Now solve equation 3.0.13 .

The discounted price of an asset is a martingale, but $g(X(t),t)$ is not a martingale because of the differing discount rate. Then generate a martingale from 3.0.13 by multiplying both sides of the equation by $\exp\left[-\int_0^t X(u)du\right]$

Here,

$$\exp\left[-\int_0^t X(u)du\right]g(X(t),t) = \exp\left[-\int_0^t X(u)du\right]\mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_t^s X(u)du]h(X(s),s)\right]$$

$$\exp\left[-\int_0^t X(u)du\right]g(X(t),t) = \mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_0^s X(u)du]h(X(s),s)\right] \tag{3.0.14}$$

Equation 3.0.14 is now a martingale.

Now, solving Equation 3.0.14, we have,

$$\exp[-X(t)]g(X(t),t) = \mathbb{E}_{\mathbb{Q}}\left[\exp[-X(s)]h(X(s),s)\right]$$

For simplicity, let $g(X(t),t) = g$ and $X(t) = X$. We have

$$\exp(-X)g = \mathbb{E}_{\mathbb{Q}}\left[\exp[-X(s)h(X(s),s)]\right]. \tag{3.0.15}$$

Differentiate the left-hand side of equation 3.0.15. Gives,

$$
\begin{aligned}
d[\exp(-X)g] &= \exp(-X)dg + gd[\exp(-X)] \\
&= \exp(-X)[g_t dt + g_X dX + \frac{1}{2}g_{XX}(dX)^2] + g[-X\exp(-X)dt] \\
&= \exp(-X)[g_t dt + g_X dX + \frac{1}{2}g_{XX}(dX)^2 - Xgdt]
\end{aligned}
$$

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That is

$$d[\exp(-X)g] = \exp(-X)[g_t dt + g_X dX + \frac{1}{2}g_{XX}(dX)^2 - Xg dt] \tag{3.0.16}$$

Let find $(dX)^2$, From 3.0.7, gives,

$$\begin{aligned}(dX)^2 &= (\alpha(\mu - \upsilon - X)dt + \sigma\sqrt{X}d\tilde{\mathscr{B}}(t))^2 \\ &= [\alpha(\mu - \upsilon - X)dt]^2 + 2[\alpha(\mu - X - \upsilon)dt][\sigma\sqrt{X}d\tilde{\mathscr{B}}(t)] + (\sigma\sqrt{X}d\tilde{\mathscr{B}}(t))^2 \\ &= [\alpha(\mu - \upsilon - X)]^2(dt)^2 + 2[\alpha(\mu - \upsilon - X)][\sigma\sqrt{X}]dtd\tilde{\mathscr{B}}(t) + [(\sigma\sqrt{X}]^2(d\tilde{\mathscr{B}}(t))^2\end{aligned}$$

But know that with the fundamental relations (with the help of Itô lemma, $(dt)^2 = 0$, $dtd\tilde{\mathscr{B}}(t) = 0$ and $(d\tilde{\mathscr{B}}(t))^2 = dt$.
Substituting,

$$(dX)^2 = [\sigma\sqrt{X}]^2 dt = \sigma^2 X dt. \tag{3.0.17}$$

Substituting equations 3.0.7 and 3.0.17 into 3.0.16, gives

$$d[\exp(-X)g] = \exp(-X)\left[g_t dt + g_X[\alpha(\mu - \upsilon - X)dt + \sigma\sqrt{X}d\tilde{\mathscr{B}}(t)] + \frac{1}{2}g_{XX}\sigma^2 X dt - Xg dt\right]$$

Expanding and rearranging, gives

$$d[\exp(-X)g(X,t)] = \exp(-X)\left[g_t + \alpha(\mu - \upsilon - X)g_X + \frac{1}{2}g_{XX}\sigma^2 X - Xg\right]dt + \exp(-X)\left[\sigma\sqrt{X}d\tilde{\mathscr{B}}(t)\right]$$

Set the net dt terms to zero, since we differentiated a martingale.

$$g_t(X(t),t) + \alpha(\mu - \upsilon - X(t))g_X(X(t),t) + \frac{1}{2}g_{XX}(X(t),t)\sigma^2 X(t) - X(t)g(X(t),t) = 0$$

The terminal condition is obtained when $t = s$ in equation 3.0.13. That is

$$\begin{aligned}g(X(s),s) &= \mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_s^s X(u)du]h(X(s),s)\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\exp[-\int_s^s X(u)du]h(X(s),s)\right] \\ &= \mathbb{E}_{\mathbb{Q}}[\exp[-X(s) + X(s)]h(X(s),s)] \\ &= \mathbb{E}_{\mathbb{Q}}[h(X(s),s)]\end{aligned}$$

This is an expectation and hence, deterministic. with this, the need to solve the following partial differential equation.

$$g_t(X(t),t) + \alpha(\mu - \upsilon - X(t))g_X(X(t),t) + \frac{1}{2}g_{XX}(X(t),t)\sigma^2 X(t) - X(t)g(X(t),t) = 0 \tag{3.0.18}$$

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subject to the terminal condition

$$g(X(s),s) = \mathbb{E}_{\mathbb{Q}}[h(X(s),s)] \tag{3.0.19}$$

Since the interest rate $X(t)$ is affine, assume that the solution is of the form

$$g(X(t),t) = g(X(s),s)\exp[A(t,s) - B(t,s)X(t)] \tag{3.0.20}$$

For the terminal condition 3.0.19 to hold, we expect the conditions

$$A(s,s) = 0 \quad \text{and} \quad B(s,s) = 0$$

to hold. But 3.0.20 is a solution to 3.0.18, and hence should satisfy it.

For simplicity, let $g(r(t),t) = g$, $A(t,s) = A$, $B(t,s) = B$ and $X(t) = X$.

Here are the following, $g_t = [A_t - B_t X]g$, $g_X = -Bg$ and $g_{XX} = B^2 g$.

Substitution the following into 3.0.18 gives

$$[A_t - B_t X]g - \alpha(\mu - \upsilon - X)Bg + \frac{1}{2}B^2 g\sigma^2 X - Xg = 0$$
$$[A_t - B_t X]g - \alpha(\mu - \upsilon)Bg + \alpha XBg + \frac{1}{2}B^2 g\sigma^2 X - Xg = 0$$
$$[A_t - B_t X]g + (\alpha XBg + \frac{1}{2}B^2 g\sigma^2 X - Xg) - \alpha(\mu - \upsilon)Bg = 0$$

Therefore

$$[A_t - B_t X] = \left[\alpha(\mu - \upsilon)B - (\alpha B + \frac{1}{2}B^2\sigma^2 - 1)X\right]$$

Dividing through by g,

$$[A_t - B_t X]g = \left[\alpha(\mu - \upsilon)B - (\alpha B + \frac{1}{2}B^2\sigma^2 - 1)X\right]g$$

This gives the equations;

$$A_t = \alpha(\mu - \upsilon)B, \qquad A(s,s) = 0 \tag{3.0.21}$$

and

$$B_t = \alpha B + \frac{1}{2}B^2\sigma^2 - 1, \qquad B(s,s) = 0 \tag{3.0.22}$$

The solution to equations 3.0.21 and 3.0.22 can be found in Cairns (2004), Shreve (2004)[9] and other resources and are the same as equations 3.0.9 and 3.0.10, but with a slight modification.

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$A(t,s)$ and $B(t,s)$ are given by equations 3.0.23 and 3.0.24, respectively.

$$A(t,s)=\frac{2\alpha(\mu-\upsilon)}{\sigma^2}\log\left(\frac{2\rho\exp[\frac{\rho+\alpha}{2}(s-t)]}{(\rho+\alpha)(\exp[\rho(s-t)]-1)+2\rho}\right) \tag{3.0.23}$$

$$B(t,s)=\frac{2(\exp[\rho(s-t)]-1)}{(\rho+\alpha)(\exp[\rho(s-t)]-1)+2\rho} \tag{3.0.24}$$

and

$$\rho=\sqrt{\alpha^2+2\sigma^2}$$

This gives

$$\begin{aligned}
f(r(t),t) &= \exp[-\upsilon(s-t)]g(X(t),t)\\
&= \exp[-\upsilon(s-t)]g(X(s),s)\exp[A(t,s)-B(t,s)X(t)]\\
&= g(X(s),s)\exp[A(t,s)-\upsilon(s-t)-B(t,s)X(t)]\\
&= g(X(s),s)\exp[A(t,s)-\upsilon(s-t)-B(t,s)X(t)+\upsilon B(t,s)-\upsilon B(t,s)]\\
&= g(X(s),s)\exp[A(t,s)-\upsilon(s-t)+\upsilon B(t,s)-B(t,s)(\upsilon+X(t))]
\end{aligned}$$

Substituting $r(s)=X(s)+\upsilon$ to get back the Pearson-Sun model given in equation 3.0.1),

$$f(r(t),t)=f(r(s),s)\exp[A_p(t,s)-B(t,s)r(t)], \tag{3.0.25}$$

where $A_p(t,s)=A(t,s)-\upsilon(s-t)+\upsilon B(t,s)$.

Equation 3.0.25 represent the price of a caplet maturing at time s with terminal payoff given by equation 3.0.12 .

The study now have the price of a cap given by

$$F(r(t),t)=h(r(t),T)+p(r(t),T) \tag{3.0.26}$$

where

$$h(r(t),T)=\int_t^T f(r(s),s)p(r(t),s)ds \tag{3.0.27}$$

and

$$p(r(t),s)=\exp[A_p(t,s)-B(t,s)r(s)] \tag{3.0.28}$$

$$p(r(t),T)=\exp[A_C(t,T)-B(t,T)r(t)]. \tag{3.0.29}$$

therefore

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$$F(r(t),t) = \int_t^T f(r(s),s)p(r(t),s)ds + \exp[A_C(t,T) - B(t,T)r(t)].$$

The study also have,
$f(r(s),s) = \mathbb{E}_{\mathbb{Q}}\left[(r(s)-\hat{r})^+\right]$, $\quad A_p(t,s) = A(t,s) - \upsilon(s-t) + \upsilon B(t,s)$ *and*

$$A_C(t,T) = A(t,T) - \upsilon(T-t) + \upsilon B(t,T).$$

Also, $A(t,T)$, $B(t,T)$, $A(t,s)$ *and* $B(t,s)$ are given by equations 3.0.9, 3.0.10, 3.0.23 and 3.0.24 respectively and $r(t)$ is given by 3.0.2.

Equation 3.0.26 is the closed form solution to equation 3.0.3, the price of a cap. It can not be simplifed further save numerical methods, it is possible to simulate its trajectories through a discretization scheme. These discretization schemes can be useful in numerical methods such as Monte Carlo simulation.
Equation 3.0.27 is the price of caplet, which is the price of a European call option integrated over the time interval (t,T). Since for the case of the Pearson-Sun model, to our knowledge, there is no explicit solution of $r(t)$ to the SDE 3.0.1 and we observe that equation 3.0.26 as a function of $r(t)$ among other expressions, the integral given by equation 3.0.27 has, therefore, become complex and hard to find. This has gone to confirm what Kim (2002) stated, that the closed form expression for the price of an option is hard to find under a non-Gaussian interest rate process. If each caplet ends at or out of the money, that is, at the maturity of each calpet s, we have $r(s) \leq \hat{r}$, then $f(r(s),s) = 0$ and the price of the cap would just be the price of the zero coupon bond.

4 CONCLUSION

This study focused on finding an explicit solution for the price of a cap when the Pearson-Sun model describes the evolution of its term structure of interest rates. It found the price to be the sum of discounted payoffs of continuously traded calpets, where the sum is given by an integral, plus the price of a zero coupon bond. The Pearson-Sun model is a modification of the Cox, Ingersoll, and Ross (CIR) interest rate model, hence, instead of finding the parameters of the zero coupon bond under the Pearson-Sun model, the study used the known parameters under the CIR model to obtain it.

The integral in the price of a cap is the price of a European call option written on the interest rate model. It is known from our result, the closed form solution for the price of a European call option written on the Pearson-Sun model. Therefore, the study leaves it's result in terms of the sum of the price of caplets and the price of the zero coupon bond and propose for further studies by comparing these result to numerical methods used to obtaining it.

If each caplet ends at or out of the money, that is, at the maturity of each calpet s, we have $r(s) \leq \hat{r}$, then $f(r(s),s) = 0$ and the price of the cap would just be the price of the zero coupon bond.

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