A Generalized Higher Reverse Left (respectively Right) Centralizer on Prime \( \Gamma \)-Rings

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Abstract:
This study introduces the concepts of generalized higher reverse left (respectively right) centralizer, Jordan generalized higher reverse left (respectively right) centralizer and Jordan triple generalized higher reverse left (respectively right) centralizer of Gamma-rings. In this paper we prove the following main results. Every Jordan generalized higher reverse left (respectively right) centralizer of a 2-torsion free prime \( \Gamma \)-ring \( M \) into itself is a Jordan triple generalized higher reverse left (respectively right) centralizer of \( M \) and every Jordan generalized higher reverse left (respectively right) centralizer of a 2-torsion free \( \Gamma \)-ring \( M \), into itself, such that \( x \alpha y \beta x = x \beta y \alpha \), \( \alpha \in \Gamma \) is a Jordan triple generalized higher reverse left (respectively right) centralizer of \( M \), for all \( x , y \in M \) and \( \alpha , \beta \in \Gamma \).

Key Words: prime \( \Gamma \)-ring, left centralizer, right centralizer, Jordan centralizer

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1-INTRODUCTION:
Let \( M \) and \( \Gamma \) be two additive abelian groups. Suppose that there is a mapping from \( M \times \Gamma \times M \rightarrow M \) (the image of \( (x , \alpha , y) \) denoted by \( x \alpha y \), where \( x , y \in M \) and \( \alpha \in \Gamma \)) satisfying the following properties for all \( x , y , z \in M \) and \( \alpha , \beta \in \Gamma \):

(i) \( (x + y) \alpha z = x \alpha z + y \alpha z \)
   \( x (\alpha + \beta) z = x \alpha z + x \beta z \)
   \( x \alpha (y + c) = x \alpha y + x \alpha z \)

(ii) \( (x \alpha y) \beta z = x \alpha (y \beta z) \).

Then \( M \) is called a \( \Gamma \)-ring. [1] .

\( M \) is called a prime if \( x \Gamma M \) \( y = (0) \) implies that \( x = 0 \) or \( y = 0 \), where \( x , y \in M \) [5].

\( M \) is called a semiprime if \( x \Gamma M \) \( x = (0) \) implies that \( x = 0 \), where \( x \in M \) [5].

\( M \) is called a 2-torsion free if \( 2x = 0 \) implies that \( x = 0 \), for all \( x \in M \) [5].

If \( M \) is a \( \Gamma \)-ring, then \( [x , y]_\alpha = x \alpha y - y \alpha x \), for all \( x , y \in M \) and \( \alpha \in \Gamma \), is known as a commutator. [5]

An additive mapping \( d : M \rightarrow M \) is called a derivation if the following holds:

\( d(x \alpha y) = d(x) \alpha y + x \alpha d(y) \), for all \( x , y \in M \) and \( \alpha \in \Gamma \) [2].
Additionally, \( d \) is called a Jordan derivation if the following property holds:

\[
d(x \cdot x) = d(x) \cdot x + x \cdot d(x), \text{ for all } x \in M \text{ and } \alpha \in \Gamma. \quad [2]
\]

An additive mapping \( F : M \longrightarrow M \) is called a generalized derivation associated with the derivation \( d : M \longrightarrow M \) if the following equation holds:

\[
F(x \cdot y) = F(x) \cdot y + x \cdot d(y), \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma. \quad [2]
\]

In addition, \( F \) is called a Jordan generalized derivation associated with the Jordan derivation \( d : M \longrightarrow M \) if the following property is satisfied:

\[
F(x \cdot x) = F(x) \cdot x + x \cdot d(x), \text{ for all } x \in M \text{ and } \alpha \in \Gamma. \quad [2]
\]

A left (respectively right) centralizer of a \( \Gamma \)-ring \( M \), is an additive mapping, \( T : M \longrightarrow M \) which satisfies the following equation:

\[
T(x \cdot y) = T(x) \cdot y \text{ (respectively } T(x \cdot y) = x \cdot T(y) \text{) }, \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma
\]

\( T \) is called a centralizer of \( M \) if it is both a left and right centralizer \([4]\). A left (respectively right) Jordan centralizer of a \( \Gamma \)-ring \( M \), is an additive mapping, \( T : M \longrightarrow M \) which satisfies the following equation:

\[
T(x \cdot x) = T(x) \cdot x \text{ (respectively } T(x \cdot x) = x \cdot T(x) \text{) }, \text{ for all } x \in M \text{ and } \alpha \in \Gamma
\]

\( T \) is called a Jordan centralizer of \( M \) if it is both a left and right Jordan centralizer \([4]\).

An additive mapping \( F : M \longrightarrow M \) is called a generalized centralizer of \( M \) associated with the centralizer \( T : M \longrightarrow M \) if the following equation holds:

\[
F(x \cdot y + y \cdot \beta x) = F(x) \cdot y + y \cdot \beta T(x), \text{ for all } x, y \in M \text{ and } \alpha, \beta \in \Gamma. \quad [3]
\]

\( F \) is called a Jordan generalized centralizer of \( M \) associated with the Jordan centralizer \( T : M \longrightarrow M \) if the following equation holds:

\[
F(x \cdot x + x \cdot x) = F(x) \cdot x + x \cdot T(x), \text{ for all } x \in M \text{ and } \alpha, \beta \in \Gamma. \quad [3]
\]

Let \( T = (t_i)_{i \in N} \) be a family of additive mappings of a ring \( R \), into itself. Then \( T \) is called a higher left centralizer, we have that

\[
t_n(xy) = \sum_{i=1}^{n} t_i(x)t_{i-1}(y), \text{ for all } x, y \in R \text{ and } n \in N. \quad [6]
\]

In addition, \( T \) is called a Jordan higher left centralizer if the following equation holds:

\[
t_n(x^2) = \sum_{i=1}^{n} t_i(x)t_{i-1}(x), \text{ for all } x \in R \text{ and } n \in N. \quad [6]
\]

Let \( F = (f_i)_{i \in N} \) be a family of additive mappings of a ring \( R \), into itself. Then \( F \) is called a generalized higher left centralizer associated with the higher left centralizer \( T = (t_i)_{i \in N} \) of \( R \), if the following equation is satisfied:

\[
f_n(xy) = \sum_{i=1}^{n} f_i(x)t_{i-1}(y), \text{ for all } x, y \in R \text{ and } n \in N. \quad [6]
\]

Moreover, \( F \) is called a Jordan generalized higher left centralizer associated with the Jordan higher left centralizer \( T = (t_i)_{i \in N} \) of \( R \), we have that

\[
f_n(x^2) = \sum_{i=1}^{n} f_i(x)t_{i-1}(x), \text{ for all } x \in R \text{ and } n \in N. \quad [6]
\]

Jarullah and Salih introduced the concepts of higher reverse left (resp. right) centralizer and Jordan higher reverse left (respectively right) centralizer a \( \Gamma \)-ring as follows:
Let \( t = (t_i)_{i\in \mathbb{N}} \) be a family of additive mappings of a \( \Gamma \)-ring \( M \) into itself. Then \( t \) is called a higher reverse left (respectively right) centralizer of \( M \), we have that

\[
 t_n(x \alpha y) = \sum_{i=1}^{n} t_i(y) \alpha t_{i-1}(x)
\]

(respectively \( t_n(x \alpha y) = \sum_{i=1}^{n} t_{i-1}(y) \alpha t_i(x) \)), for all \( x, y \in M, \alpha \in \Gamma \) and \( n \in \mathbb{N} \) [7].

Let \( t = (t_i)_{i\in \mathbb{N}} \) be a family of additive mappings of a \( \Gamma \)-ring \( M \) into itself. Then \( t \) is called a Jordan higher reverse left (respectively right) centralizer of \( M \) if the following equation holds:

\[
 t_n(x \alpha x) = \sum_{i=1}^{n} t_i(x) \alpha t_{i-1}(x)
\]

(respectively \( t_n(x \alpha x) = \sum_{i=1}^{n} t_{i-1}(x) \alpha t_i(x) \)), for all \( x \in M \), \( \alpha \in \Gamma \) and \( n \in \mathbb{N} \) [7].

Jarullah and Salih introduced the concepts of higher reverse left (respectively right) centralizer, Jordan higher reverse left (respectively right) centralizer and generalization on Rings as follows:

Let \( t = (t_i)_{i\in \mathbb{N}} \) be a family of additive mappings of a ring \( R \) into itself. Then \( t \) is called a higher reverse left (respectively right) centralizer of \( R \), we have that

\[
 t_n(xy) = \sum_{i=1}^{n} t_i(y) t_{i-1}(x)
\]

(respectively \( t_n(xy) = \sum_{i=1}^{n} t_{i-1}(y) t_i(x) \)), for all \( x, y \in R \) and \( n \in \mathbb{N} \) [8].

Let \( t = (t_i)_{i\in \mathbb{N}} \) be a family of additive mappings of a ring \( R \) into itself. Then \( t \) is called a Jordan higher reverse left (respectively right) centralizer of \( R \), if the following equation holds:

\[
 t_n(x^2) = \sum_{i=1}^{n} t_i(x) t_{i-1}(x)
\]

(respectively \( t_n(x^2) = \sum_{i=1}^{n} t_{i-1}(x) t_i(x) \)), for all \( x \in R \) and \( n \in \mathbb{N} \) [8].

Let \( T = (T_i)_{i\in \mathbb{N}} \) be a family of additive mappings of a ring \( R \) into itself. Then \( T \) is called a generalized higher reverse left (respectively right) centralizer of a ring \( R \) into itself associated with the higher reverse left (respectively right) centralizer \( t = (t_i)_{i\in \mathbb{N}} \) of \( R \), such that

\[
 T_n(xy) = \sum_{i=1}^{n} T_i(y) t_{i-1}(x)
\]

(respectively \( T_n(xy) = \sum_{i=1}^{n} t_{i-1}(y) T_i(x) \)), for all \( x, y \in R \) and \( n \in \mathbb{N} \) [9].
Let \( T = (T_i)_{i \in \mathbb{N}} \) be a family of additive mappings of a ring \( R \) into itself. Then \( T \) is called a generalized Jordan higher reverse left (respectively right) centralizer of a ring \( R \) into itself associated with Jordan higher reverse left (respectively right) centralizer \( t = (t_i)_{i \in \mathbb{N}} \) of \( R \), such that

\[
T_n(x^2) = \sum_{i=1}^{n} T_i(x) t_{i-1}(x)
\]

(respectively \( T_n(x^2) = \sum_{i=1}^{n} t_{i-1}(x) T_i(x) \)), for all \( x \in R \) and \( n \in \mathbb{N} \) [9].

2- Generalized Higher Reverse Left (Respectively Right) Centralizer on Prime \( \Gamma \)-Rings

**Definition (2.1)**

Let \( T = (T_i)_{i \in \mathbb{N}} \) be a family of additive mappings of a \( \Gamma \)-ring \( M \) into itself. Then \( T \) is called a generalized higher reverse left (respectively right) centralizer of the \( \Gamma \)-ring \( M \), associated with the higher reverse left (respectively right) centralizer \( t = (t_i)_{i \in \mathbb{N}} \) of the \( \Gamma \)-ring \( M \), into itself if, for all \( x,y \in M \), \( \alpha \in \Gamma \) and \( n \in \mathbb{N} \):

\[
T_n(x \alpha y) = \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(x)
\]

(respectively \( T_n(x \alpha y) = \sum_{i=1}^{n} t_{i-1}(y) \alpha T_i(x) \))

**Example (2.2)**

Let \( T = (T_i)_{i \in \mathbb{N}} \) be a generalized higher reverse left (respectively right) centralizer of a ring \( R \), into itself associated with the higher reverse left (respectively right) centralizer \( t = (t_i)_{i \in \mathbb{N}} \) of \( R \) such that, for all \( x \), \( y \in R \) and \( n \in \mathbb{N} \):

\[
T_n(xy) = \sum_{i=1}^{n} T_i(y) t_{i-1}(x)
\]

(respectively \( T_n(xy) = \sum_{i=1}^{n} t_{i-1}(y) T_i(x) \))

Let \( M = M \times \mathbb{Z}(R) \) and \( \Gamma = \left\{ \binom{n}{0} \mid n \in \mathbb{Z} \right\} \). Then \( M \) is a \( \Gamma \)-ring.

Let \( F = (F_i)_{i \in \mathbb{N}} \) be a family of additive mappings from a \( \Gamma \)-ring \( M \) into itself, such that for all \( (x,y) \in M \):

\[
F(x)(y) = (T_a(x) \quad T_a(y))
\]

Then, there exists a higher reverse left (respectively right) centralizer \( f = (f_i)_{i \in \mathbb{N}} \) of a \( \Gamma \)-ring \( M \) into itself, such that for all \( (x,y) \in M \), the following equation holds:

\[
f_a((x,y)) = (t_a(x) \quad t_a(y))
\]

Therefore, \( F_a \) is a generalized higher reverse left (respectively right) centralizer of \( M \).

**Definition (2.3)**

Let \( T = (T_i)_{i \in \mathbb{N}} \) be a family of additive mappings of a \( \Gamma \)-ring \( M \) into itself. Then \( T \) is called a Jordan generalized higher reverse left (respectively right) centralizer of a \( \Gamma \)-ring \( M \) associated with the Jordan higher
reverse left (respectively right) centralizer \( t = ( t_i )_{i \in N} \) of the \( \Gamma \)-ring \( M \) into itself, if for all \( x \in M , \ \alpha \in \Gamma \) and \( n \in N \), the following equation holds:

\[
T(x \ \alpha \ x) = \sum_{i=1}^{n} T_i(x) \ \alpha \ t_{i-1}(x)
\]

(respectively \( T_n(x \ \alpha \ x) = \sum_{i=1}^{n} t_{i-1}(x) \ \alpha \ T_i(x) \)).

**Definition (2.4)**

Let \( T = ( T_i )_{i \in N} \) be a family of additive mappings of a \( \Gamma \)-ring \( M \) into itself. Then \( T \) is called a Jordan triple generalized higher reverse left (respectively right) centralizer of a \( \Gamma \)-ring \( M \), associated with the Jordan triple higher reverse left (respectively right) centralizer \( t = ( t_i )_{i \in N} \) of the \( \Gamma \)-ring \( M \) into itself if, for all \( x, \ y \in M , \ \alpha , \ \beta \in \Gamma \) and \( n \in N \), the following equation holds:

\[
T_n(x \ \alpha \ y \ \beta \ x) = \sum_{i=1}^{n} T_i(x) \ \beta \ t_{i-1}(y) \alpha t_{i-1}(x)
\]

(respectively resp. \( T_n(x \ \alpha \ y \ \beta \ x) = \sum_{i=1}^{n} t_{i-1}(x) \ \beta \ t_{i-1}(y) \alpha T_i(x) \)).

**Lemma (2.5)**

Let \( T = ( T_i )_{i \in N} \) be a Jordan generalized higher reverse left (respectively right) centralizer of a \( \Gamma \)-ring \( M \) into itself. Then for all \( x, \ y, \ z \in M , \ \alpha , \ \beta \in \Gamma \) and \( n \in N \), the following equation holds:

(i) \( T_n(x \ \alpha \ y + y \ \alpha \ x) = \sum_{i=1}^{n} t_{i-1}(x) \alpha t_{i-1}(y) + \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(y) \)

(respectively \( T_n(x \ \alpha \ y + y \ \alpha \ x) = \sum_{i=1}^{n} t_{i-1}(x) \alpha t_{i-1}(y) + \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(y) \))

(ii) \( T_n(x \ \alpha \ y \ \beta \ x + x \ \beta \ y \ \alpha \ x) = \sum_{i=1}^{n} T_i(x) \ \beta \ t_{i-1}(y) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(x) \ \alpha t_{i-1}(y) \beta t_{i-1}(x) \)

(respectively \( T_n(x \ \alpha \ y \ \beta \ x + x \ \beta \ y \ \alpha \ x) = \sum_{i=1}^{n} T_i(x) \ \beta \ t_{i-1}(y) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(x) \ \alpha t_{i-1}(y) \beta t_{i-1}(x) \))

(iii) \( T_n(x \ \alpha \ y \ \beta \ z + z \ \alpha \ y \ \beta \ x) = \sum_{i=1}^{n} T_i(z) \ \beta \ t_{i-1}(y) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(x) \ \beta t_{i-1}(y) \alpha t_{i-1}(z) \)

(respectively \( T_n(x \ \alpha \ y \ \beta \ z + z \ \alpha \ y \ \beta \ x) = \sum_{i=1}^{n} T_i(z) \ \beta \ t_{i-1}(y) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(z) \ \beta t_{i-1}(y) \alpha t_{i-1}(z) \))

(iv) In particular, if \( M \) is a 2-torsion free commutative \( \Gamma \)-ring, then

\[
T_n(x \ \alpha \ y \ \beta \ z) = \sum_{i=1}^{n} T_i(z) \ \beta \ t_{i-1}(y) \alpha t_{i-1}(x)
\]

(respectively \( T_n(x \ \alpha \ y \ \beta \ z) = \sum_{i=1}^{n} t_{i-1}(z) \ \beta \ t_{i-1}(y) \alpha T_i(x) \))

(v) \( T_n(x \ \alpha \ y \ \beta \ z + z \ \alpha \ y \ \alpha \ x) = \sum_{i=1}^{n} T_i(z) \ \alpha t_{i-1}(y) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(x) \ \alpha t_{i-1}(y) \alpha t_{i-1}(z) \)

(respectively \( T_n(x \ \alpha \ y \ \beta \ z + z \ \alpha \ y \ \alpha \ x) = \sum_{i=1}^{n} t_{i-1}(z) \ \alpha t_{i-1}(y) \alpha T_i(x) + \sum_{i=1}^{n} t_{i-1}(z) \ \alpha t_{i-1}(y) \alpha T_i(z) \)).
Proof:

(i) Since \( T \) is Jordan generalized higher reverse left (respectively right) centralizer,

(ii) we have that

\[
T_n((x + y) \alpha (x + y)) = \sum_{i=1}^{n} T_i(x + y) \alpha t_{i-1}(x + y)
\]

\[
= \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(y) + \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(y)
\]

\[\text{... (1)}\]

Meanwhile, we have that

\[
T_n((x + y) \alpha (x + y)) = T_n(x \alpha x + x \alpha y + y \alpha x + y \alpha y)
\]

\[
= T_n(x \alpha x + T_n(y \alpha y) + T_n(x \alpha y + y \alpha x)
\]

\[
= \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(y) + T_n(x \alpha y + y \alpha x) \quad \ldots \ldots (2)
\]

We obtain the following equation by comparing equations \((1)\) and \((2)\)

\[
T_n(x \alpha y + y \alpha x) = \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(y)
\]

(ii) By substituting \((i)\) Lemma \((2.5)\) in \(b\) for \(a\), we have

\[
= \sum_{i=1}^{n} T_i(y) \beta t_{i-1}(x) \alpha t_{i-1}(x) + T_i(x) \beta t_{i-1}(y) \alpha t_{i-1}(x) +
\]

\[
T_i(x) \alpha t_{i-1}(y) \beta t_{i-1}(y) + T_i(x) \alpha t_{i-1}(x) \beta t_{i-1}(y) \ldots (1)
\]

In addition, we obtain that

\[
T_n(x \alpha (x \beta y + y \beta x) + (x \beta y + y \beta x) \alpha x)
\]

\[
= T_n(x \alpha \beta y + x \alpha y \beta x + x \beta y \alpha x + y \beta x \alpha x)
\]

\[
= T_n(y \beta x \alpha x) + T_n(x \alpha x \beta y) + T_n(x \alpha y \beta x + x \beta y \alpha x)
\]

\[
= \sum_{i=1}^{n} T_i(y) \beta t_{i-1}(x) \alpha t_{i-1}(x) + T_i(x) \alpha t_{i-1}(y) \beta t_{i-1}(y) + T_n(x \alpha y \beta x + x \beta y \alpha x) \quad \ldots (2)
\]

We get the following equation by comparing equations \((1)\) and \((2)\)

\[
T_n(x \alpha y \beta x + x \beta y x) = \sum_{i=1}^{n} T_i(x) \beta t_{i-1}(y) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(y) \beta t_{i-1}(x)
\]

(iii) By substituting \((a + c)\) for \(a\) in Definition \((2.4)\), we have

\[
= \sum_{i=1}^{n} T_i(x) \beta t_{i-1}(y) \alpha t_{i-1}(x) + T_i(x) \beta t_{i-1}(y) \alpha t_{i-1}(z) +
\]

\[
T_i(z) \beta t_{i-1}(y) \alpha t_{i-1}(x) + T_i(z) \beta t_{i-1}(y) \alpha t_{i-1}(z) \ldots (1)
\]

Moreover,

\[
T_n((x + z) \alpha y \beta(x + z)) = T_n(x \alpha y \beta x + z \alpha y \beta z + z \alpha y \beta z + z \alpha y \beta x)
\]

\[
= T_n(x \alpha y \beta x) + T_n(z \alpha y \beta z) + T_n(x \alpha y \beta z + z \alpha y \beta x)
\]

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\[\sum_{i=1}^{n} T_i(x) \beta_1 t_{i-1}(y) \alpha_1 t_{i-1}(x) + T_i(z) \beta_1 t_{i-1}(y) \alpha_1 t_{i-1}(z) T_n(x \alpha_1 y \beta z + z \alpha_1 y \beta x)\]

... (2)

The following equation is obtained by comparing equations (1) and (2):

\[T_n(x \alpha_1 y \beta z + z \alpha_1 y \beta x) = \sum_{i=1}^{n} T_i(z) \beta_1 t_{i-1}(y) \alpha_1 t_{i-1}(x) + \sum_{i=1}^{n} T_i(x) \beta_1 t_{i-1}(y) \alpha_1 t_{i-1}(z)\]

(iv) Using Lemma (2.5) (iii) and the fact that \( M \) is a commutative \( \Gamma \)-ring, we have that

\[T_n(x \alpha_1 y \beta z + x \alpha_1 y \beta x) = 2 T_n(x \alpha_1 y \beta z)\]

We obtain the required result, by utilizing the fact that \( M \) is a 2-torsion free.

(v) The substitution \( \beta \) for \( \alpha \) in Lemma (2.5) (iii), gives that

\[T_n(x \alpha_1 y \beta z + z \alpha_1 y \beta x) = \sum_{i=1}^{n} T_i(z) \alpha_1 t_{i-1}(y) \alpha_1 t_{i-1}(x) + \sum_{i=1}^{n} T_i(x) \alpha_1 t_{i-1}(y) \alpha_1 t_{i-1}(z)\]

Definition (2.6)

Let \( T = (T_i)_{i \in N} \) be a Jordan generalized higher reverse left (respectively right) centralizer of a \( \Gamma \)-ring \( M \) ring into itself. Then for all \( x, y \in M, \alpha \in \Gamma \) and \( n \in N \), we define

\[\delta_n(x, y) = T_n(x \alpha y) - \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(x)\]

(respectively \( \delta_n(x, y) = T_n(x \alpha y) - \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(x) \))

Lemma (2.7)

Let \( T = (T_i)_{i \in N} \) be a Jordan generalized higher reverse left (respectively right) centralizer of a \( \Gamma \)-ring \( M \) ring into itself. Then for all \( x, y, z \in M, \alpha, \beta \in \Gamma \) and \( n \in N \), we have that the following equations hold:

(i) \( \delta_n(x, y) = - \delta_n(y, x) \)

(ii) \( \delta_n(x + y, z) = \delta_n(x, z) + \delta_n(y, z) \)

(iii) \( \delta_n(x, y + z) = \delta_n(x, y) + \delta_n(x, z) \)

(iv) \( \delta_n(x, y + y) = \delta_n(x, y) + \delta_n(x, y) \)

Proof:

(i) By applying Lemma (2.5) (i), we have that

\[T_n(x \alpha y + y \alpha x) = \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(y)\]

\[T_n(x \alpha y) - \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(x) = -(T_n(y \alpha x) - \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(y))\]

Thus, \( \delta_n(x, y) = - \delta_n(y, x) \).
\( \delta_n (x + y, z) = T_n ((x + y)\alpha z) = \sum_{i=1}^{n} T_i (z)\alpha t_{i-1} (x + y) \)

\[ = T_n (x \alpha z + y \alpha z) - \sum_{i=1}^{n} T_i (z)\alpha t_{i-1} (x) - \sum_{i=1}^{n} T_i (z)\alpha t_{i-1} (y) \]

\( = \delta_n (x, z)_a + \delta_n (y, z)_a \)

\( (i) \delta_n (x + y, z) = T_n ((x + y)\alpha z) = \sum_{i=1}^{n} T_i (z)\alpha t_{i-1} (x + y) \)

\( (ii) \delta_n (x + y, z) = T_n ((x + y)\alpha z) = \sum_{i=1}^{n} T_i (z)\alpha t_{i-1} (x) + \sum_{i=1}^{n} T_i (z)\alpha t_{i-1} (y) \)

\( (iii) \delta_n (x + y, z) = T_n ((x + y)\alpha z) = \sum_{i=1}^{n} T_i (z)\alpha t_{i-1} (x) + \sum_{i=1}^{n} T_i (z)\alpha t_{i-1} (y) \)

\( (iv) \delta_n (x + y, z) = T_n ((x + y)\alpha z) = \sum_{i=1}^{n} T_i (z)\alpha t_{i-1} (x) + \sum_{i=1}^{n} T_i (z)\alpha t_{i-1} (y) \)

\( \delta_n (x + y, z) = \delta_n (x, z) + \delta_n (y, z) \)

Remark (2.8)

It is noteworthy that \( T = (T_i)_{i \in \mathbb{N}} \) is a generalized higher reverse left (respectively right) centralizer of a \( \Gamma \)-ring \( M \), into itself if and only if \( \delta_n (x, y)_a = 0 \), for all \( x, y \in M \), \( \alpha \in \Gamma \) and \( n \in \mathbb{N} \).

Lemma (2.9)

Let \( T = (T_i)_{i \in \mathbb{N}} \) be a Jordan generalized higher reverse left (respectively right) centralizer of a prime \( \Gamma \)-ring \( M \), into itself. Then for all \( x, y, z \in M \), \( \alpha, \beta \in \Gamma \) and \( n \in \mathbb{N} \), the following equations hold:

\( (i) \delta_n (x, y)_a \beta t_{a-1}(z) \alpha [ t_{a-1}(x), t_{a-1}(y) ] = 0 \)

\( (ii) \delta_n (x, y)_a \alpha t_{a-1}(z) \alpha [ t_{a-1}(x), t_{a-1}(y) ] = 0 \)

\( (iii) \delta_n (x, y)_\beta \alpha t_{a-1}(z) \alpha [ t_{a-1}(x), t_{a-1}(y) ] = 0 \)

Proof:

(i) The proof is utilizing induction on \( n \in \mathbb{N} \)

If \( n = 1 \), let \( w = x \alpha y \beta z \gamma x + y \alpha x \beta \beta x \alpha y \)

Then, we obtain that

\[ T(w) = T(x \alpha (y \beta z \gamma y \alpha x + y \alpha x \beta \beta x \alpha y) \]

\[ = T(x \alpha y \beta z \gamma y \alpha x + y \alpha x \beta \beta x \alpha y) \]

Moreover, we have that

\[ T(w) = T(x \alpha y \beta z \gamma y \alpha x + y \alpha x \beta \beta x \alpha y) \]

\[ = T(y \alpha x \beta z \gamma y \alpha x + y \alpha x \beta \beta x \alpha y) \]

\[ \cdots (1) \]
The Comparison of equations (1) and (2) yields that

\[ 0 = (T(y \alpha x) - I(x) \alpha y) \beta z \beta y \alpha x + (T(x \alpha y) - I(y) \alpha x) \beta z \beta x \alpha y \]

\[ 0 = \delta(x, y) \alpha \beta z \beta y \alpha x + \delta(x, y) \alpha \beta z \beta x \alpha y \]

\[ 0 = -\delta(x, y) \alpha \beta z \beta y \alpha x + \delta(x, y) \alpha \beta z \beta x \alpha y \]

\[ 0 = \delta(x, y) \alpha \beta z \beta (x \alpha y - y \alpha x) \]

Thus, \( \delta(x, y) \alpha \beta z \beta [x, y]_\alpha = 0 \), for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \).

Now, we can assume the following:

\[ \delta(x, y) \alpha \beta z \beta [t_{s-1}(x), t_{s-1}(y)]_\alpha = 0 \), for all \( x, y, z \in M \), and \( s, n \in \mathbb{N}, s < n \).

\[ T_n(w) = T_n(x \alpha (y \beta z \beta y) \alpha x + y \alpha (x \beta z \beta x) \alpha y) \]

\[ = \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(y \beta z \beta y) \alpha t_{i-1}(x) + \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(x \beta z \beta x) \alpha t_{i-1}(y) \]

\[ = \sum_{i=1}^{n} T_i(x) \alpha t_{i-1}(y) \beta t_{i-1}(z) \beta t_{i-1}(y) \alpha t_{i-1}(x) \]

\[ + \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(x) \beta t_{i-1}(z) \beta t_{i-1}(y) \alpha t_{i-1}(x) + \]

\[ \cdots \]

Thus,

\[ T_n(w) = T_n((x \alpha y) \beta z \beta (y \alpha x) + (y \alpha x) \beta z \beta (x \alpha y)) \]

\[ = \sum_{i=1}^{n} T_i(y \alpha x) \beta t_{i-1}(z) \beta t_{i-1}(x \alpha y) + \sum_{i=1}^{n} T_i(x \alpha y) \beta t_{i-1}(z) \beta t_{i-1}(y \alpha x) \]

\[ = T_n(y \alpha x) \beta t_{n-1}(z) \beta t_{n-1}(y) \alpha t_{n-1}(x) + \sum_{i=1}^{n-1} T_i(y \alpha x) \beta t_{i-1}(z) \beta t_{i-1}(y) \alpha t_{i-1}(x) + \]

\[ \cdots \]

By comparing equations (3) and (4), we have that

\[ 0 = (Ta \alpha x) \beta t_{n-1}(y \alpha x) \beta t_{n-1}(z) \beta t_{n-1}(y) \alpha t_{n-1}(x) + \]
\[
(T_a(x \, \alpha \, y) = \sum_{i=1}^{n} T_i(y) \alpha t_{i-1}(x) \beta t_{n-i}(z) \beta \, t_{n-1}(x) \alpha \, t_{n-1}(y) + \\
+ (x, t \alpha (y))_{i-1} \, t \beta (z, t \beta) y)_{i-1} \, t \alpha (x) iT \rightarrow x \, \alpha y (, iT) \sum_{i=1}^{n} (T_i(y) \alpha t_{i-1}(x) \beta t_{i-1}(z) \beta t_{i-1}(x) \alpha t_{i-1}(y).
\]

It follows that
\[
0 = \delta_a(y, x) a \beta t_{n-1}(z) \beta t_{n-1}(y) \alpha t_{n-1}(x) + \\
\delta_a(x, y) a \beta t_{n-1}(z) \beta t_{n-1}(x) \alpha t_{n-1}(y) \\
0 = \delta_a(y, x) a \beta t_{n-1}(z) \beta (t_{n-1}(x) \alpha t_{n-1}(y) \rightarrow t_{n-1}(y) \alpha t_{n-1}(x))
\]
Thus, \( \delta_a(x, y) a \beta t_{n-1}(z) \beta [t_{n-1}(x), t_{n-1}(y)]_a = 0 \), for all \( x, y, z \in M \),
\( \alpha, \beta \in \Gamma \) and \( n \in N \)

(ii) By substituting \( \beta \) for \( \alpha \) in Lemma (2.9) (i) and applying similar arguments as in the proof of Lemma (2.9) (i), we obtain Lemma (2.9) (ii).

(iii) We get Lemma (2.9) (iii), by Interchanging \( \alpha \) and \( \beta \) in Lemma (2.9) (i).

**Lemma (2.10)**

Let \( T = (T_i)_{i=1}^{n} \) be a Jordan generalized higher reverse left (respectively right) centralizer of a prime \( \Gamma \)-ring \( M \) into itself. Then for all \( x, y, z, u, v \in M \), \( \alpha, \beta \in \Gamma \) and \( n \in N \).

(i) \( \delta_a(x, y) a \beta t_{n-1}(z) \beta [t_{n-1}(x), t_{n-1}(y)]_a = 0 \)

(ii) \( \delta_a(x, y) a \beta t_{n-1}(z) \beta [t_{n-1}(u), t_{n-1}(v)]_a = 0 \)

(iii) \( \delta_a(x, y) a \beta t_{n-1}(z) \beta [t_{n-1}(u), t_{n-1}(v)]_b = 0 \)

**Proof:**

(i) By substituting \( (a + c) \) for \( a \) in Lemma (2.9) (i), we have that
\[
\delta_a(x + u, y) a \beta t_{n-1}(z) \beta [t_{n-1}(x + u), t_{n-1}(y)]_a = 0.
\]

Thus \( \delta_a(x, y) a \beta t_{n-1}(z) \beta [t_{n-1}(x), t_{n-1}(y)]_a + \)
\[
\delta_a(u, y) a \beta t_{n-1}(z) \beta [t_{n-1}(u), t_{n-1}(y)]_a + \\
\delta_a(u, y) a \beta t_{n-1}(z) \beta [t_{n-1}(u), t_{n-1}(v)]_b = 0.
\]

By applying Lemma (2.9) (i), we obtain that
\[
\delta_a(u, y) a \beta t_{n-1}(z) \beta [t_{n-1}(u), t_{n-1}(y)]_a + \\
\delta_a(u, y) a \beta t_{n-1}(z) \beta [t_{n-1}(x), t_{n-1}(y)]_a = 0.
\]

Therefore, we get that
\[
\delta_a(x, y) a \beta t_{n-1}(z) \beta [t_{n-1}(u), t_{n-1}(y)]_a \beta t_{n-1}(z) \beta \delta_a(x, y) a \\
\beta t_{n-1}(z) \beta [t_{n-1}(u), t_{n-1}(y)]_a = 0.
\]

This implies that
\[
0 = - \delta_a(x, y) a \beta t_{n-1}(z) \beta [t_{n-1}(u), t_{n-1}(y)]_a \beta t_{n-1}(z) \beta \delta_a(x, y) a \beta t_{n-1}(z) \beta.
\]
Since $M$ is a prime $\Gamma$-ring, we have that
$$\delta_0(x, y) \beta t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$$ \hspace{1cm} \ldots(1)$$

The substitution of $(y + v)$ for $y$ in Lemma (2.9) (i), gives that
$$\delta_0(x, y + v) \beta t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$$
$$\delta_0(x, y) \alpha t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a +$$
$$\delta_0(x, y) \beta t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a +$$
$$\delta_0(x, v) \beta t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a +$$
$$\delta_0(x, v) \alpha t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$$

By utilizing Lemma (2.9) (i), we obtain that
$$\delta_0(x, y) \beta t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a +$$
$$\delta_0(x, y) \alpha t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$$

Consequently, we have
$$\delta_0(x, y) \alpha t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a \beta t_{\alpha}(z) \beta \delta_0(x, y)$$
$$\beta t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$$

This implies that
$$0 = \delta - \delta(x, y) \alpha t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a \beta t_{\alpha}(z) \beta \delta_0(x, y)$$
$$\beta t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0 \hspace{1cm} \ldots (2)$$

Now, $\delta_0(x, y) \beta t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$
$$\delta_0(x, y) \alpha t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a +$$
$$\delta_0(x, v) \beta t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a +$$
$$\delta_0(x, v) \beta t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$$

By employing equations (1), (2) and Lemma (2.9) (i), we get
$$\delta_0(x, y) \alpha t_{\alpha}(z) \beta [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$$

(ii) We can obtain Lemma (2.10) (ii) by substituting $\beta$ in Lemma (2.10) (i).

(iii) By substituting $\alpha + \beta$ for $\alpha$ in Lemma (2.10) (ii), we have
$$\delta_0(x, y) \alpha t_{\alpha}(z) \alpha [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$$

This implies that
$$\delta_0(x, y) \alpha t_{\alpha}(z) \alpha [t_{\alpha}, t_{\beta}, t_{\gamma}]_a +$$
$$\delta_0(x, y) \alpha t_{\alpha}(z) \alpha [t_{\alpha}, t_{\beta}, t_{\gamma}]_a +$$
$$\delta_0(x, y) \alpha t_{\alpha}(z) \alpha [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$$

By applying equations (i) and (ii), we get that
$$\delta_0(x, y) \alpha t_{\alpha}(z) \alpha [t_{\alpha}, t_{\beta}, t_{\gamma}]_a +$$
$$\delta_0(x, y) \alpha t_{\alpha}(z) \alpha [t_{\alpha}, t_{\beta}, t_{\gamma}]_a = 0$$
Therefore, we get that
\[ \delta_{\alpha}(x, y) \alpha = \alpha_{\text{t}(z)\alpha} [ t_{\text{t}(u)} , t_{\text{t}(v)} ] \alpha \alpha_{\text{t}(z)\alpha} \delta_{\alpha}(x, y) \alpha \]
\[ \alpha_{\text{t}(z)\alpha} [ t_{\text{t}(u)} , t_{\text{t}(v)} ] \alpha \alpha_{\text{t}(z)\alpha} = 0 \]
This yields that
\[ 0 = - \delta_{\alpha}(x, y) \alpha \alpha_{\text{t}(z)\alpha} [ t_{\text{t}(u)} , t_{\text{t}(v)} ] \alpha \alpha_{\text{t}(z)\alpha} \delta_{\alpha}(x, y) \alpha \]
\[ \alpha_{\text{t}(z)\alpha} [ t_{\text{t}(u)} , t_{\text{t}(v)} ] \alpha \alpha_{\text{t}(z)\alpha} = 0 \]
The fact that \( M \) is a prime \( \Gamma \)-ring gives that
\[ \delta_{\alpha}(x, y) \alpha \alpha_{\text{t}(z)\alpha} [ t_{\text{t}(u)} , t_{\text{t}(v)} ] \alpha = 0 \]
\[ \alpha_{\text{t}(z)\alpha} [ t_{\text{t}(u)} , t_{\text{t}(v)} ] \alpha \alpha_{\text{t}(z)\alpha} = 0 \]

**Theorem (2.11)**

Every Jordan generalized higher reverse left (respectively right) centralizer of a 2-torsion free prime \( \Gamma \)-ring \( M \) into itself is a generalized higher reverse left (respectively right) centralizer of \( M \).

**Proof:**

Let \( T = ( T_i )_{i \in N} \) be a Jordan generalized higher reverse left (respectively right) centralizer of a prime \( \Gamma \)-ring \( M \) into itself. Since \( M \) is a prime \( \Gamma \)-ring, then by employing Lemma (2.10) (i), we have that either
\[ \delta_{\alpha}(x, y) \alpha = 0 \text{ or } [ t_{\text{t}(u)} , t_{\text{t}(v)} ] \alpha = 0 \] for all \( x, y, u, v \in M \), \( \alpha \in \Gamma \) and \( n \in N \).

If \( [ t_{\text{t}(u)} , t_{\text{t}(v)} ] \alpha = 0 \), for all \( u, v \in M, \alpha \in \Gamma, \) then \( \delta_{\alpha}(x, y) \alpha = 0 \) for all \( x, y \in M \) and \( n \in N \). Hence, using Remark (2.8), we obtain that \( T \) is a generalized higher reverse left (respectively right) centralizer of \( M \).

If \( [ t_{\text{t}(u)} , t_{\text{t}(v)} ] \alpha = 0 \), for all \( u, v \in M \) and \( n \in N \), then \( M \) is commutative.

By utilizing Lemma (2.5) (i), we have that
\[ T_n(x \alpha y + x \alpha y) = T_n(2x \alpha y) = 2T_n(x \alpha y) = 2\sum_{i=1}^{n} T_i(y \alpha t_{i-1}(x) \alpha) \]

Since \( M \) is a 2-torsion free \( \Gamma \)-ring, we get that
\[ T_n(x \alpha y) = \sum_{i=1}^{n} T_i(y \alpha t_{i-1}(x)) \alpha \]

Then \( T \) is a generalized higher reverse left (respectively right) centralizer of \( M \).

**Proposition (2.11)**

Let \( T = ( T_i )_{i \in N} \) be a Jordan generalized higher reverse left (respectively right) centralizer of a 2-torsion free \( \Gamma \)-ring \( M \) into itself, such that \( x \alpha y \beta x = x \beta y \alpha x \), for all \( x, y \in M \) and \( \alpha, \beta \in \Gamma \). Then \( T \) is a Jordan triple generalized higher reverse left (respectively right) centralizer of \( M \).

**Proof:**

The substitution of \( b \) for \( x \alpha y + y \alpha x \) in Lemma (2.5) (i), we have that
\[
\sum_{i=1}^{n} T_i(y) \beta t_{i-1}(x) + T_i(x) \alpha t_{i-1}(y) \alpha t_{i-1}(x) + T_i(x) \alpha t_{i-1}(y) = T_i(y) \beta t_{i-1}(x) \alpha t_{i-1}(y) \alpha t_{i-1}(x) + T_i(x) \alpha t_{i-1}(y) \beta t_{i-1}(x) \alpha t_{i-1}(y) \ldots (1)
\]

Moreover, we get that
\[
T_n(x \alpha (x \beta y + y \beta x) + (x \beta y + y \beta x) \alpha x) \equiv t_n(x \alpha x \beta y + x \alpha y \beta x + y \beta x + y \beta x \alpha x) = T_n(y \beta x \alpha x) + T_n(x \alpha \beta y x + x \beta y \alpha x)
\]
\[
= \sum_{i=1}^{n} T_i(y) \beta t_{i-1}(x) \alpha t_{i-1}(x) + T_i(x) \alpha t_{i-1}(x) \beta t_{i-1}(y) + T_n(x \alpha \beta y x + x \beta y \alpha x) \ldots (2)
\]

By comparing equations (1), (2) and the fact that \(x \alpha y \beta x = x \beta y \alpha x\), for all \(x, y \in M\) and \(\alpha, \beta \in \Gamma\), we have that
\[
T_n(x \alpha \beta y x + x \alpha y \beta x) = 2T_n(x \alpha \beta y x)
\]
\[
= 2\sum_{i=1}^{n} T_i(x) \beta t_{i-1}(y) \alpha t_{i-1}(x)
\]

Since \(M\) is a 2-torsion free \(\Gamma\)-ring, we obtain that \(F\) is a Jordan triple generalized higher reverse left (respectively right) centralizer of \(M\).

References: