ORIENTED MANIFOLDS WITH COMPACT SUPPORT AND COHOMOLOGY ALGEBRA

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Abstract
The cohomology of $M$ with compact supports is the graded algebra $\Omega(M_c, \delta)$ and is given by $\Omega_c(M) = \sum_{k=0}^{n} \Omega^k_c(M)$. The bilinear map $\Omega(M) \times \Omega_c(M) \to \Omega_c(M)$ is induced by $G(M) \times G_c(M) \to \Omega_c(M)$ and makes $\Omega_c(M)$ into a left graded $\Omega(M)$-module. $\Omega(S^n)$, which is the cohomology of $S^n$, is determined by $\Omega^0(S^n) \cong \Omega^n(S^n) \cong \mathbb{R}$ and $\Omega^k(S^n) = 0$ for $n \geq 1$. Also, we determine the cohomology of $\mathbb{R}^n$ with compact supports. Finally, it is shown that the map $i_M: \Omega(M) \to \Omega_c(M)$ is a linear isomorphism.

Keywords: Compact manifold, cohomology, graded algebra, isomorphism, bilinear map.

1. Introduction
Let $M$ be an $n$-manifold, then the graded algebra of differential forms on $M$ is defined as $G(M) = \sum_{k=0}^{n} G^k(M)$ and $G(M)$ is converted into a graded differential algebra by the exterior derivative (Greub et al., 1972). The differential forms $\Phi$ satisfying the condition $\delta \Phi = 0$ construct cocycles in this differential algebra and this differential form is closed. The closed forms are graded subalgebra $Z(M)$ of $G(M)$ as $\delta$ is an antiderivation (Bott and Tu, 1982). The subset $H(M) = \delta G(M)$ is a graded ideal in $Z(M)$. The differential forms in $G(M)$ are called coboundaries and the corresponding cohomology algebra is defined by $\Omega(M) = Z(M)/H(M)$ and this cohomology algebra is called the de Rham cohomology algebra of $M$ (Iversen, 1986).

The cohomology of $M$ with compact supports is the graded algebra $\Omega(M_c, \delta)$ (Grivaux, 2010). It is denoted by $\Omega_c(M)$ and is defined by

$$\Omega_c(M) = \sum_{k=0}^{n} \Omega^k_c(M), \; n = \dim M.$$ 

Multiplication in $G(M)$ is restricted to a real bilinear map as $G_c(M)$ is an ideal (Kobayashi and Nomizu, 1963). $G_c(M)$ is confined into a left graded $G(M)$-module by this multiplication which is given by

$$G(M) \times G_c(M) \to \Omega_c(M).$$

The bilinear map $\Omega(M) \times \Omega_c(M) \to \Omega_c(M)$ is induced by the above map and makes $\Omega_c(M)$ into a left graded $\Omega(M)$-module (Sternberg, 1964). This map can be written as

$$(\lambda, \mu) \mapsto \alpha \ast \beta, \lambda \in \Omega(M), \mu \in \Omega_c(M).$$

In the same way, $\Omega_c(M)$ can be converted into a right graded $\Omega(M)$-module and we can write $\mu \ast \lambda, \mu \in$
\[ \Omega_c(M), \lambda \in \Omega(M). \] Also, the algebra homomorphism
\[ (\tau_M)_\#: \Omega_c(M) \to \Omega(M) \]
is induced by the inclusion map \( \tau_M: G_c(M) \to G(M). \) The above module structures can be converted to ordinary multiplication by this homomorphism (Haller and Rybicki, 1999).

2. Preliminaries and Auxiliary Results

Let \( \Omega: \mathbb{R} \times M \to N \) be a smooth map. Two smooth maps \( f, g: M \to N \) are said to be homotopic (Eilenberg and Maclane, 1950) if \( \Omega(0, x) = f(x) \) and \( \Omega(1, x) = g(x) \). We can define a linear map \( h: \Gamma(N) \to \Gamma(M) \) homogeneous of degree \(-1\) for such a homotopy \( \Omega \) by

\[ h = I^0 \circ i(T) \circ \Omega^*. \]

Consider the spaces \( \Omega^k(M) \) having finite dimension, then the \( k \)th Betti number of \( M \) is defined by \( b_k = \dim \Omega^k(M) \) and the Poincaré polynomial of \( M \) is defined by

\[ p_M(t) = \sum b_k t^k. \]

If \( M \) consists of a single point, then \( \Omega^k(M) = 0 \) \((k \geq 1)\) and \( \Omega^0(M) = \mathbb{R} \).

The Euler-Poincaré characteristic of \( M \) is defined by the alternating sum \( \zeta_M = \sum_{k=0}^{\infty} (-1)^k b_k = p_M(-1). \)

Now, we discuss the axioms for de Rham cohomology. The axioms for de Rham cohomology are given below:

(a) \( \Omega(\text{point}) = \mathbb{R} \)

(b) If \( M \) is the disjoint union of open submanifolds \( M_\alpha \), then

\[ \Omega(M) \cong \prod_{\alpha} \Omega(M_\alpha) \] (disjoint union)

(c) If \( f \sim g: M \to N \), then \( f^\# = g^\# \) (homotopy axiom)

(d) If \( M = U \cup V \) (\( U, V \) are open), there is an exact triangle (Mayer-Vietoris)

\[ \Omega(M) \to \Omega(U) \oplus \Omega(V) \to \Omega(U \cap V) \]

Consider a manifold \( M \) which is the disjoint union \( M = \bigcup_{\nu} M_\nu \) of open submanifolds \( M_\nu \). A
homomorphism \( h^\ast_v : G(M) \rightarrow G(M_v) \) is induced by the inclusion map \( h^\ast_v : M_v \rightarrow M \). We obtain a homomorphism \( h^\ast : G(M) \rightarrow \prod_v G(M_v) \) given by \((h^\ast \Phi)_v = h^\ast_v \), where \( \Phi \in G(M) \) and \( \prod_v G(M_v) \) is the direct product of the algebras \( G(M_v) \).

If \( \delta_v \) denotes the exterior derivative in \( \Omega(M_v) \), then \( \prod_v \Omega(M_v) \) is given by the differential operator \( \prod_v \Omega(M_v) \). As a result, \( h^\ast \) is an isomorphism of graded differential algebras \( \Omega(M_v) \) and \( h^\ast \) induces the following isomorphism

\[
h^\ast : \Omega(M) \xrightarrow{\cong} \prod_v \Omega(M_v)
\]

given by

\[(h^\ast \gamma)_v = h^\ast_v (\gamma), \ \gamma \in \Omega(M).\]

Consider a manifold \( M \) and two open subsets \( X_1, X_2 \) such that \( X_1 \cup X_2 = M \). Let us consider the following inclusion maps

\[
u_1 : X_1 \cap X_2 \rightarrow X_1, \quad u_2 : X_1 \cap X_2 \rightarrow X_2 \quad \nu_1 : X_1 \rightarrow M, \quad u_2 : X_2 \rightarrow M.
\]

which induce a sequence of linear mappings

\[
0 \xrightarrow{} \Omega(M) \xrightarrow{\lambda} \Omega(X_1) \oplus \Omega(X_2) \xrightarrow{\mu} \Omega(X_1 \cap X_2) \xrightarrow{} 0
\]

given by

\[
\lambda \Phi = (v_1^\ast \Phi, v_2^\ast \Phi), \ \Phi \in \Omega(M)
\]

and

\[
\mu(\Phi_1, \Phi_2) = u_1^\ast \Phi_1 - u_2^\ast \Phi_2, \ \Phi_i \in \Omega(U_i), \ i = 1, 2.
\]

Let \( \delta_1, \delta_2, \delta_{12} \) and \( \delta \) be the exterior derivatives in \( \Omega(X_1), \Omega(X_2), \Omega(X_1 \cap X_2) \) and \( \Omega(M) \) respectively, then we have

\[
\lambda \circ \delta = (\delta_1 \oplus \delta_2) \circ \alpha \quad \text{and} \quad \mu \circ (\delta_1 \oplus \delta_2) = \delta_{12} \circ \mu.
\]

Consequently, the following linear maps are induced by \( \lambda \) and \( \mu \):

\[
\lambda_\# : \Omega(M) \rightarrow \Omega(X_1) \oplus \Omega(X_2), \quad \mu_\# : \Omega(X_1) \oplus \Omega(X_2) \rightarrow \Omega(X_1 \cap X_2).
\]

**Lemma 1.** The following sequence of linear mappings is exact

\[
0 \xrightarrow{} \Omega(M) \xrightarrow{\lambda} \Omega(X_1) \oplus \Omega(X_2) \xrightarrow{\mu} \Omega(X_1 \cap X_2) \xrightarrow{} 0.
\]

**Proof.** We have to consider the following three cases:

(a) \( \ker \mu = \text{Im} \lambda \)

(b) \( \lambda \) is injective

(c) \( \mu \) is surjective

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(a) Since it is obvious $\mu \circ \lambda = 0$, so $\text{Im} \lambda \subset \ker \mu$. We need only to show that $\ker \mu \subset \text{Im} \lambda$.

Let $(\Phi_1, \Phi_2) \in \ker \mu$. If $x \in X_1 \cap X_2$, then $\Phi_1(x) = \Phi_2(x)$. Consequently, we can find a differential form $\Phi \in \Omega(M)$ which is given by

$$
\Phi(x) = \begin{cases} 
\Phi_1(x), & x \in X_1 \\
\Phi_2(x), & x \in X_2 
\end{cases}.
$$

Since $\lambda \Phi = (\Phi_1, \Phi_2)$, so $\ker \mu \subset \text{Im} \lambda$. Therefore, $\ker \mu = \text{Im} \lambda$.

(b) Let $x \in X_1 \cup X_2 = M$. If $\lambda \Phi = 0$, then $\Phi(x) = 0$ for $x \in X_1 \cup X_2 = M$.

(c) Consider the covering $X_1, X_2$ of $M$. Let $x_1, x_2$ be subordinate to the covering $X_1, X_2$. Thus, $\{x_1, x_2\}$ is a partition of unity for $M$. Then, $\text{carr } v_1^*x_2, \text{ carr } v_2^*x_1 \subset X_1 \cup X_2$.

For $\Phi \in \Omega(X_1 \cap X_2)$, we define

$$
\Phi_1 = v_1^*x_2 \cdot \Phi \in \Omega(X_1), \quad \Phi_2 = v_2^*x_1 \cdot \Phi \in \Omega(X_2).
$$

Consequently, we have $\Phi = \mu(\Phi_1, -\Phi_2)$.

Consider a compact oriented $n$-manifold $M$. Then, we have

$$
\Omega_*(M) = \Omega(M) \text{ and } i_M : \Omega(M) \xrightarrow{\cong} \Omega(M)^*.
$$

Therefore, the bilinear map $\mathcal{P}_M^k : \Omega^k(M) \times \Omega^{n-k}(M) \to \mathbb{R}$ represents the Poincaré scalar product.

**Theorem 1.** If $M$ is any compact manifold, then the dimension of $\Omega(M)$ is finite.

**Proof.** First we assume that the compact manifold $M$ is orientable. Then the Poincaré scalar product is given by the bilinear map $\mathcal{P}_M^k : \Omega^k(M) \times \Omega^{n-k}(M) \to \mathbb{R}$ and $\mathcal{P}_M^k$ induces the following two linear isomorphisms

$$
\Omega^k(M) \xrightarrow{\cong} \Omega^{n-k}(M)^*,
$$

and

$$
\Omega^{n-k}(M) \xrightarrow{\cong} \Omega^k(M)^*.
$$

Now, from the related results of elementary linear algebra we can observe that each $\Omega^k(M)$ has finite dimension; hence the theorem is proved in this case.

Again, we assume that the compact manifold $M$ is nonorientable. In this case, the double cover $\tilde{M}$ is orientable and compact. Consequently, we have

$$
\dim \Omega(M) = \dim \Omega_*(\tilde{M}) \leq \dim \Omega(\tilde{M}) < \infty.
$$
Thus the dimension of $\Omega(M)$ is finite.

**Lemma 2.** $\int^\#_M : \Omega^n_c(M) \to \mathbb{R}$ is a linear isomorphism if $M$ is a connected oriented $n$-manifold.

**Proof.** Let $\Omega(M)$ be the cohomology of an oriented manifold $M$ and $\Omega_c(M)$ be the cohomology of $M$ with compact support. Then the map

$$i_M : \Omega(M) \to \Omega_c(M)^*$$

is a linear isomorphism. Also, we have

$$\dim \Omega^n_c(M) = \dim \Omega^0(M) = 1.$$  

Moreover, $\int^\#_M$ is surjective. Therefore, $\int^\#_M : \Omega^n_c(M) \to \mathbb{R}$ is a linear isomorphism if $M$ is a connected oriented $n$-manifold.

Consider an oriented $n$-manifold $M$. The linear map $\int^\#_M : G^n(M) \to \mathbb{R}$ satisfies $\int^\#_M \circ \delta = 0$ and it is surjective map. The linear map $\int^\#_M : \Omega^n_c(M) \to \mathbb{R}$ is induced by $\int^\#_M : G^n(M) \to \mathbb{R}$ and this map is also surjective. Let $\lambda \in \Omega^k(M)$ and $\mu \in \Omega^{n-k}_c(M)$. The Poincaré scalar product

$$P^k_M : \Omega^k(M) \times \Omega^{n-k}_c(M) \to \mathbb{R}$$

can be expressed as the following bilinear map $P^k_M(\lambda, \mu) = \int^\#_M \lambda \ast \mu$.

**Lemma 3.** Let $M, N$ be two manifolds, then the following diagram commutes.

![Diagram](attachment:diagram.png)

**Proof.** If $\lambda \in \Omega^k(N)$, $\mu \in \Omega^{n-k}_c$, $\xi \in G^k(N)$, $\xi \in G^{n-k}_c(M)$, then $\lambda$ and $\mu$ are represented by $\xi$ and $\xi$ respectively. Consequently, $(\psi_c)_* \mu \in \Omega^{n-k}_c(N)$ is represented by

$$(\psi_c)_* \xi \text{ and } \psi^* (\xi \land ((\psi_c)_* \xi)) = \psi^* \xi \land \xi.$$  

Since the Poincaré scalar product $P^k_M : \Omega^k(M) \times \Omega^{n-k}_c(M) \to \mathbb{R}$ is the bilinear map given by

$$P^k_M(\lambda, \mu) = \int^\#_M \lambda \ast \mu,$$
thus, $P_M^k(\psi^*\lambda, \mu) = \int_M (\psi^*\lambda) \ast \mu$ and $P_M^k(\lambda, (\psi_\ast)_{\ast}\mu) = \int_M \lambda \ast (\psi_\ast)_{\ast}\mu$. Hence we have

$$P_M^k(\psi^*\lambda, \mu) = \int_M (\psi^*\lambda) \ast \mu = \int_M \psi^*\zeta \wedge \xi = \int_M \xi \wedge (\psi_\ast)_{\ast}\zeta = \int_M \lambda \ast (\psi_\ast)_{\ast}\mu = P_M^k(\lambda, (\psi_\ast)_{\ast}\mu)$$

Since $P_M^k(\psi^*\lambda, \mu) = P_M^k(\lambda, (\psi_\ast)_{\ast}\mu)$, we can conclude that the diagram commutes. Hence the proposition is proved.

3. Main Results

**Theorem 2.** For $n \geq 1$, $\Omega(S^n)$ is determined by $\Omega^0(S^n) \cong \Omega^n(S^n) \cong \mathbb{R}$ and $\Omega^k(S^n) = 0$ ($1 \leq k \leq n - 1$).

**Proof.** First we consider an ($n + 1$)-dimensional Euclidean space $E^{n+1}$. Suppose $S^n$ is embedded in $E^{n+1}$. We know that $S^n$ is connected, thus $\Omega^0(S^n) = \mathbb{R}$. Now let $s \in S^n$ and $\xi \in (0, 1)$ where $\xi$ is fixed. Again, we consider open sets $X_1, X_2 \subset S^n$ defined by

$$X_1 = \{x \in S^n: (x, s) > -\xi\}, \quad X_2 = \{x \in S^n: (x, s) < \xi\}.$$  

As a result, $S^n = X_1 \cup X_2$ and we have the following exact Mayer-Vietoris sequence

$$\cdots \rightarrow \Omega^k(S^n) \rightarrow \Omega^k(X_1) \oplus \Omega^k(X_2) \rightarrow \Omega^k(X_1 \cap X_2) \rightarrow \Omega^{k+1}(S^n) \rightarrow \cdots.$$ 

It is clear that $S^{n-1}$ is contained in $X_1 \cap X_2$. We observe that $X_1$ and $X_2$ are contractible. Consequently, the following exact sequence can be considered as the Mayer-Vietoris sequence

$$\cdots \rightarrow \Omega^k(S^n) \rightarrow \Omega^k(\text{point}) \oplus \Omega^k(\text{point}) \rightarrow \Omega^k(S^{n-1}) \rightarrow \Omega^{k+1}(S^n) \rightarrow \cdots.$$ 

The above sequence can be split into the following two sequences

$$0 \rightarrow \Omega^0(S^n) \rightarrow \Omega^0(\text{point}) \oplus \Omega^0(\text{point}) \rightarrow \Omega^0(S^{n-1}) \rightarrow \Omega^1(S^n) \rightarrow 0$$

and

$$0 \longrightarrow \Omega^k(S^{n-1}) \longrightarrow \Omega^{k+1}(S^n) \longrightarrow 0, \quad k \geq 1.$$ 

These sequences are exact and from the first sequence we have

$$0 = \dim \Omega^1(S^n) - \dim \Omega^0(S^{n-1}) + 2 \dim \Omega^0(\text{point}) - \dim \Omega^0(S^n).$$

For $n \geq 2$, we observe that $S^{n-1}$ is connected and $S^0$ consists of two points. Thus we can conclude from the above equation

$$\Omega^1(S^n) \cong \begin{cases} \mathbb{R}, & n = 1 \\ 0, & n > 1 \end{cases}.$$ 

Since $0 \longrightarrow \Omega^k(S^{n-1}) \longrightarrow \Omega^{k+1}(S^n) \longrightarrow 0$ for $k \geq 1$, we have

$$\Omega^k(S^n) \cong \Omega^1(S^{n-k+1}) (1 \leq k \leq n).$$

Therefore, $\Omega^0(S^n) \cong \Omega^n(S^n) \cong 0$ and $\Omega^k(S^n) = 0$. Hence, the proposition is proved.
Corollary 1. Consider a connected \( n \)-manifold \( M \). Then \( \Omega^n(M) \cong \mathbb{R} \) when \( M \) is compact and orientable. Otherwise, \( \Omega^n(M) = 0 \).

Proof. First we assume that \( M \) is compact. Then, there are two cases:

(i) \( M \) is orientable

(ii) \( M \) is nonorientable.

If we consider \( M \) to be orientable, then from the consequence of Lemma 2 we can deduce that \( \Omega^n(M) \cong \mathbb{R} \). If \( M \) is nonorientable, then \( \Omega^n(M) = 0 \).

Next we assume that \( M \) is not compact. Then, there are again two cases:

(i) \( M \) is orientable

(ii) \( M \) is nonorientable.

If the manifold \( M \) is orientable, then we have \( \Omega^n(M) \cong \Omega^0_c(M) \). If the manifold \( M \) is nonorientable, then the double cover \( \tilde{M} \) must be orientable, connected and noncompact. Consequently, we have

\[
\Omega^n(M) \cong \Omega^+_n(\tilde{M}) \subset \Omega^n(\tilde{M}) = 0.
\]

Thus, \( \Omega^n(M) \cong \mathbb{R} \) when \( M \) is compact and orientable, otherwise, \( \Omega^n(M) = 0 \).

Corollary 2. \( \Omega^k_c(\mathbb{R}^n) = \begin{cases} 0 & \text{when } k < n \\ \mathbb{R} & \text{when } k = n \end{cases} \) gives the cohomology of \( \mathbb{R}^n \) with compact supports.

Proof. The case \( n = 0 \) is trivial. Assume that \( S^n \) is the one-point compactification of \( \mathbb{R}^n \) for \( n > 0 \). Let \( s \in S^n \) be the compactifying point, thus we can write \( \mathbb{R}^n = S^n - \{s\} \).

The differential forms on \( S^n \) are zero in a neighbourhood of \( s \) and the ideal of differential forms on \( S^n \) is denoted by \( \tau_s \). It is clear that \( \tau_s = G_c(\mathbb{R}^n) \). Consequently, the following sequence is exact

\[
0 \to \tau_s \to G(S^n) \to G_c(S^n) \to 0.
\]

In cohomology, we can derive a long exact sequence from the above short exact sequence. As \( \Omega(G_b(S^n)) = \Omega(\text{point}) \), we can split this long sequence into the following two exact sequences

\[
0 \to \Omega_c^0(\mathbb{R}^n) \to \Omega^0(S^n) \to \mathbb{R} \to \Omega_c^1(\mathbb{R}^n) \to \Omega^1(S^n) \to 0
\]

and

\[
0 \to \Omega_c^k(\mathbb{R}^n) \xrightarrow{=} \Omega^k(S^n) \to 0, \quad k \geq 2.
\]

As \( \Omega^0(S^n) = \mathbb{R} \) and \( \Omega_c^0(\mathbb{R}^n) = 0 \), thus the following exact sequence can be derived from the first sequence

\[
0 \to \Omega_c^1(\mathbb{R}^n) \xrightarrow{=} \Omega^1(S^n) \to 0.
\]
Hence $\Omega^k_c(\mathbb{R}^n) = \begin{cases} 0 \text{ when } k < n \\ \mathbb{R} \text{ when } k = n \end{cases}$ gives the cohomology of $\mathbb{R}^n$ with compact supports. 

**Theorem 3.** Let $\Omega(M)$ be the cohomology of an oriented manifold $M$ and $\Omega_c(M)$ be the cohomology of $M$ with compact support. Then the map $i_M: \Omega(M) \to \Omega_c(M)^*$ is a linear isomorphism.

**Proof.** To prove the theorem, we have to consider the following three cases:

(i) $M = \mathbb{R}^n$

(ii) $M$ is an open subset of $\mathbb{R}^n$

(iii) $M$ is arbitrary

(i) We have to show that the map $i^* : \Omega^n(\mathbb{R}^n) \to \Omega^k_c(\mathbb{R}^n)^*$ is a linear isomorphism to prove $M = \mathbb{R}^n$ since $\Omega^k(\mathbb{R}^n)$ and $\Omega^k_c(\mathbb{R}^n)$ are given by

$$\Omega^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad \text{and} \quad \Omega^k_c(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n \\ 0, & k \neq n \end{cases}$$

Also, in this case it is sufficient to show that $i \neq 0$ as we have

$$\dim \Omega^n(\mathbb{R}^n) = \dim \Omega^k_c(\mathbb{R}^n)^*. $$

Assume that $\varphi \in S(\mathbb{R}^n)$ is a nonnegative function and $\varphi$ is not identically zero. Consider a positive determinant function $\Delta$ in $\mathbb{R}^n$.

Now, $\int_{\mathbb{R}^n} \varphi \cdot \Delta = \int_{\mathbb{R}^n} \varphi(x) \, dx^1 \cdots dx^n > 0$ for a suitable basis of $\mathbb{R}^n$.

Consequently, if $\mu$ is a non-zero element in $\Omega^k_c(\mathbb{R}^n)$, $\mu$ is represented by $f \cdot \Delta$.

From the definitions we have $\langle i(1) , \mu \rangle = \int_{\mathbb{R}^n} 1 \wedge (\varphi \cdot \Delta) = \int_{\mathbb{R}^n} \varphi \cdot \Delta \neq 0$.

Therefore, $\langle i(1) , \mu \rangle \neq 0$ implies that $i(1) \neq 0$ and so $i \neq 0$.

(ii) Assume that $\{b_1, \cdots, b_n\}$ is a basis of $\mathbb{R}^n$. Then, for $v \in \mathbb{R}^n$, we have $v = \sum_{k=1}^n u_k b_k$.

Then an $i$-basis for the topology of $\mathbb{R}^n$ can be represented by the open subsets of the form

$$B = \{ x \in \mathbb{R}^n; a_k < x^k < b_k, \ k = 1, \cdots, n \}.$$

By the definition of diffeomorphism, $B$ is diffeomorphic to $\mathbb{R}^n$. Therefore, with the help of Case (i) and the result of Lemma 3 we conclude that $i_\mu$ is an isomorphism for each such $B$. As a result, for every open subset $M$ of $\mathbb{R}^n$ we have $i_M : \Omega(M) \to \Omega_c(M)^*$ which is an isomorphism.

(iii) Let us assume that every open subset of $M$ is diffeomorphic to open subset of $\mathbb{R}^n$ and $\mathcal{B}$ is the collection of such open subsets of $M$. Consequently, it is obvious that for the topology of $M$ this collection of open subsets $\mathcal{B}$ is an $i$-basis. With the help of the results derived in Case (ii) and Lemma 3, we can conclude that $i_B$ is an isomorphism for every $B \in \mathcal{B}$. Therefore, for
every open subset \( X \subset M \) we can find an \( i_x \) which is an isomorphism. Thus, the map \( i_M: \Omega(M) \rightarrow \Omega_c(M)^* \) is a linear isomorphism.

References


