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The Effect of Inverse Transformation on the Unit Mean and Constant Variance Assumptions of a Multiplicative Error Model Whose Error Component has a Gamma Distribution.

By

Ohakwe J.
Department of Mathematicsa and Statistics/ Computer Science/Physics, Faculty of Science, Federal University Otuoke , P.M.B. 126, Yenagoa, Bayelsa State, Nigeria.
Email: ohakwe.johnson@yahoo.com ; Phone No.: +2348050560750

Abstract

In this paper, the effect of inverse transformation on the unit mean and constant variance assumptions of a multiplicative error model whose error component is Gamma distributed was studied. From the results of the study, it was discovered that the unit mean assumption is violated after inverse transformation. The mean and variance of the inverse-transformed gamma error component were found to be smaller than those of the untransformed error. Furthermore this change in mean, ∇µ was modeled and was found to increase per unit increase in Ǔ the shape parameter while that of the variance ∇ů was found to decrease per unit increase in the shape parameter and their relationships (predictive equations) were determined. Finally, it was discovered that in order to achieve the unit mean condition after inverse transformation, the condition $\beta = \alpha - 1$ is unavoidable where Ǔ and б are respectively the shape and location parameters of the Gamma density function.

Key words: Multiplicative Error Model; Gamma distribution; Inverse Transformation; Mean; Variance

1. Introduction

Suppose the model of interest is a multiplicative error model given as

$$X_{t,t\in N} = \Psi\left(X_{t-1}\right)\xi_t \tag{1}$$

where $X_{t,t\in N}$ is a discrete time series process defined on [0, Ð), $\Psi\left(X_{t-1}\right)$, the information available for forecasting $X_{t,t\in N}$ and ξ_t, a random variable defined over a [0, +Ð) support with unit mean and unknown constant variance, σ_1^2. That is

$$\xi_t \sim V^+\left(1,\sigma_1^2\right) \tag{2}$$

By the definition given in (2) it is very clear that ξ_t in (1) can be specified by means of any probability density function (pdf) having the characteristics in (2). Examples are Gamma, Log-Normal, Weibull, and mixtures of them (Brownlees et al., (2011)). Engle and Gallo (2006) favor a Gamma (ϕ,ϕ) (which implies $\sigma^2 = 1/\phi$); Bauwens and Giot (2000), in Autoregressive Conditional Duration (ACD) model framework considered a Weilbull $\Gamma\left((1+\phi)^{-1},\phi\right)$ (in this case, $\sigma^2=\Gamma(1+2\phi)/\Gamma\left((1+\phi)^2-1\right)$).

In statistical modeling, the familiar application of the normal linear model involves a response variable that is assumed normally distributed with constant variance. In other applications a response variable may occur in a form that suppresses an underlying normal linear structure (Fraser (1967)). Sometimes in these applications the context may suggest a logarithm or inverse or square root transformation and so on, which reveals the normal linear form. Transformation may also be necessary to either stabilize the variance component of a model or to normalize it. Details on the reasons for transformations are found in; Box and Jenkings (1964); Iwueze et al., (2011).

Recently there are various studies on the effects of transformation on the error component of the multiplicative error model whose error component is classified under the characteristics given in (2) of which the multiplicative time series model is a subclass. The aim of such studies is to establish the conditions for successful transformation. A successful transformation is achieved when the desirable properties of a data set remains unchanged after transformation . These basic properties or assumptions of interest for this study are; (i) Unit mean and (ii) constant variance. In this area of research, Iwueze (2007) investigated the effect of logarithmic transformation on the error component (e_t) of a multiplicative time series model where ($e_t \sim N\left(1,\sigma_1^2\right)$) and discovered that the logarithm transform; Y = Log e_t can be assumed to be normally distributed with mean, zero and the same variance, σ^2 for $\sigma < 0.1$. Similarly Nwosu et al., (2010) and Otuonye et al., (2011) had studied the effects of inverse and square root transformation on the error component of the same model . Nwosu et al., (2010) discovered that the inverse transform $Y=\dfrac{1}{e_t}$ can be assumed to be normally distributed with mean, one and the same variance provided $\sigma_1 \le 0.07$. Similarly Otuonye et al., (2011) discovered that the square root transform; $Y=\sqrt{e_t}$ can be assumed to be normally distributed with unit mean and variance, 4 σ_1^2 for $\sigma_1 \le 0.59$, where σ_1^2 is the variance of the original error component before transformation.

The application of inverse transformation to model (1) gives

$$X^*_{t,t\in N} = \Psi^*\left(X_{t-1}\right)\xi^*_t \tag{3}$$

where $X^*_{t,t\in N}=\dfrac{1}{X^*_{t,t\in N}}$, $\Psi^*\left(X_{t-1}\right)=\dfrac{1}{\Psi\left(X_{t-1}\right)}$ and $\xi^*_t=\dfrac{1}{\xi_t}$. Model (3) is still a multiplicative error model and therefore ξ^*_t must also be characterized with unit mean and some constant variance, σ_2^2 which may or may not be equal to σ_1^2. In this paper we want to study the effect of inverse transformation on a non-normal distributed error component of a multiplicative error model whose distributional characteristics belong to the Gamma distribution. The purpose is to determine if the assumed fundamental structure (unit mean and constant variance) is maintained after inverse transformation and also to investigate what happens to σ_1^2 and σ_2^2 in terms of equality or non-equality. The overall reason for concentrating on the error component of model (3) is as plane as the nose on the face: the reason is that the assumptions for model analysis are always placed on the error component, ξ_t , thus

$$\xi_t^* \sim V^+\left(1, \sigma_2^2\right) \tag{4}$$

The paper is organized into six Sections. The introduction is contained in section one while some of the basic distributional characteristics of this study are given in Section two. The relationship between the means and variances of the transformed and untransformed Gamma distributed error component would be determined in Section 3 while the Summary of the results and conclusion are contained in section four. Finally the references and Appendix are contained in Sections five and six respectively.

2.0 Some Basic Distributional Characteristics of the Study

Given that ξ_t in (1) has a gamma distribution, its probability density function (Freund (2000)) is given by

$$f\left(\xi_t\right) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \xi_t^{\alpha-1} e^{-\frac{\xi_t}{\beta}}, \xi_t > 0, \Gamma(\alpha) > 0 \tag{5}$$

with

$$E\left(\xi_t^k\right) = \mu_1^k = \frac{\beta^k \Gamma(\alpha + k)}{\Gamma(\alpha)} \tag{6}$$

where $E\left(\xi_t^k\right)$ is the kth moment of the distribution (k = 1, 2, 3, . . .). From (6), the following results are true

$$E\left(\xi_t\right) = \mu_1 = \alpha\beta \tag{7}$$

$$E\left(\xi_t^2\right) = \alpha(\alpha+1)\beta^2 \tag{8}$$

$$\sigma_1^2 = \alpha\beta^2 \tag{9}$$

$$E\left(\xi_t - \mu_1\right)^3 = 2\alpha\beta^3 \tag{10}$$

$$E\left(\xi_t - \mu_1\right)^4 = 3\alpha(\alpha+2)\beta^4 \tag{11}$$

where $E\left(\xi_t - \mu_1\right)^3$ and $E\left(\xi_t - \mu_1\right)^4$ are measures of skewness and kurtosis.

Suppose inverse transformation is deemed appropriate in a data set whose model can be suitably represented by model (1) to either stabilize the variance or remedy the presence of outlier(s) in a data set, we therefore obtain model (3) whose error component, ξ_t^* has the probability density function given by

$$f^*\left(\xi_t\right) = \frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)}\xi_t^{-\alpha-1}e^{-\frac{\beta}{\xi_t}}, \xi_t > 0, \Gamma\left(\alpha\right) > 0 \tag{12}$$

as obtained by Cook (2008). The distributional characteristics of (12) is given in (4).

Given (12), the following results are true

$$E^*\left(\xi_t^k\right) = \mu_2^k = \frac{\beta^k}{\left(\alpha-1\right)\left(\alpha-2\right)\ldots\left(\alpha-k\right)}, k=1,2,3,\ldots \tag{13}$$

$$E^*\left(\xi_t\right) = \mu_2 = \frac{\beta}{\left(\alpha-1\right)} \tag{14}$$

$$\sigma_2^2 = Var\left(\xi_t^*\right) = \frac{\beta^2}{\left(\alpha-1\right)^2\left(\alpha-2\right)} \tag{15}$$

$$E^*\left(\xi_t - \mu_2\right)^3 = \frac{4\beta^4}{\left(\alpha-1\right)^3\left(\alpha-2\right)\left(\alpha-3\right)} \tag{16}$$

$$E^*\left(\xi_t - \mu_2\right)^4 = \frac{3\beta^4\left(3\alpha+5\right)}{\left(\alpha-1\right)^4\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)} \tag{17}$$

In practice, the required general assumptions for modeling (1) are unit mean and constant variance and these would be the major focus of this study. Considering that (3) (Model (1) after inverse transformation) is also a multiplicative error model, the unit mean and constant variance assumptions still remain valid even though the variances of the two models may or may not be equal. Given (1), the condition for unit mean from (7) is either

$$\beta = \frac{1}{\alpha} \tag{18}$$

or

$$\alpha = \frac{1}{\beta} \tag{19}$$

However considering that Ŭ is the shape parameter, we shall be interested in (18). On applying the unit mean condition of (18) in the results of (7) through (11) we obtain the results given in Table 1. Also included in Table 1 is the ratio of the moments of the untransformed Gamma to those of the inverse-transformed distribution subject to the application of the unit mean condition.

Having obtained the moments of the inverse-transformed Gamma distributed error component subject to the unit-mean condition, the next task would be to model the relationship between the untransformed and transformed Gamma distributed error component in terms of the mean and variance and these would be investigated in Section 3.

3.0 Relationship between the Mean and Variance of the Gamma Error Component before and after Inverse Transformation

In this Section, the relationship between the means and variances of the transformed and untransformed Gamma distributed error component would be determined with a view to ascertain the unit increase or decrease in the mean and variance of the inverse transformed error component per unit increase in the value of Ǔ, the shape parameter. For this purpose, the differences $E\left(\xi_t\right)-E\left(\xi_t^*\right)=\nabla\mu$ and $Var\left(\xi_t\right)-Var\left(\xi_t^*\right)=\nabla\sigma$ are computed and the relationships between the computed differences and the shape parameter, Ǔ are obtained. The results of the computations of $E\left(\xi_t^*\right), Var\left(\xi_t\right)\ and\ Var\left(\xi_t^*\right)$ using the expressions in Table 1 are given in Table 2.

Furthermore $\nabla\mu$ and $\nabla\sigma$ would be regressed on Ǔto obtain a predictive function given by

$$\nabla\mu=f\left(\alpha\right) \tag{20}$$

and

$$\nabla\sigma=g\left(\alpha\right) \tag{21}$$

where $f\left(\alpha\right)$ and $g\left(\alpha\right)$ are finite functions of Ǔ. The reasons for determining the predictive functions are to enable an analyst determine the increase/decrease in the mean/variance of a gamma distributed error component after an inverse transformation.

For the purpose of the regression analysis, considering that in Table 2, $Var\left(\xi_t^*\right)=\sigma_2^2=0$ for all values of Ǔ > 12, the regression analysis would be constrained to the values of Ǔ= 2, 3, 4, . . ., 12. Goodness of the regression fit would be assessed using the coefficient of determination, R^2 (Draper and Smith (1981)) . The results of the regression analysis are given in Figures 1a, 1b and 2. From Figures 1a and 1b, the cubic predictive equation given by

$$\nabla\mu=-0.1148+0.4340*\alpha-0.0544*\alpha^2+0.0022*\alpha^3 \tag{22}$$

whose R^2 = 93.5% is a better predictive equation of the increase in mean of a Gamma distribution after inverse transformation than that of the quadratic, whose R^2 = 79.8%. However from Figure 2, the decrease in variance is given by

$$\nabla\sigma=0.4256-0.10570*\alpha-0.0024*\alpha^2 \tag{23}$$

whose R^2 = 99.7% .

4.0: Summary and Conclusion

In this paper, the effect of inverse transformation on the unit mean and constant variance assumptions of a multiplicative error model whose error component is Gamma distributed was studied. From the results of the study, it was discovered that the unit mean assumption is violated after inverse transformation. The mean and variance of the inverse-transformed gamma error component were found to be smaller than those of the untransformed error. Furthermore the decrease in mean, $\nabla\mu$ was found to increase per unit increase in Ų the shape parameter while that of the variance $\nabla\sigma$ was found to decrease per unit increase in the shape parameter and the relationships (predictive equations) are respectively given by

(i) $\nabla\mu = -0.1148 + 0.4340*\alpha - 0.0544*\alpha^2 + 0.0022*\alpha^3,\ \alpha = 2,3,\ldots,12$ and

(ii) $\nabla\sigma = 0.4256 - 0.10570*\alpha - 0.0024*\alpha^2, \alpha = 3,4,\ldots,12$

Furthermore, the unit mean condition of the untransformed gamma error component is achieved when $\beta = \frac{1}{\alpha}$, however, for the inverse transformed gamma error term, $\beta = \alpha - 1$.

In conclusion, inasmuch as there is a decreased error variance after inverse transformation, the unit mean violation where the mean of the transformed error component is $\mu_2 \leq 0.5$ (μ_2 is the mean after the inverse transformation) is of a major concern. In order to achieve the unit mean condition after inverse transformation, the condition $\beta = \alpha - 1$ is unavoidable. Finally based of the results of this study I recommend that inverse transformation is not appropriate for a multiplicative error model with a Gamma distributed error component.

5.0 References

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6.0 Appendix

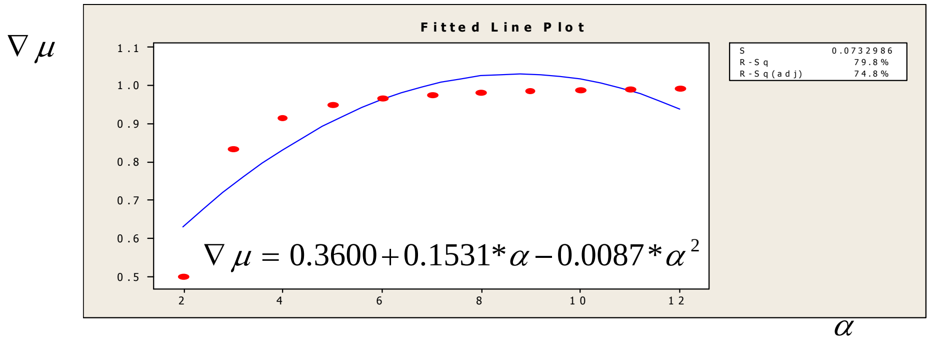

Figure 1a: Quadratic Predictive Equation of the Change in Mean of the Gamma Distributed Error Component after Inverse transformation

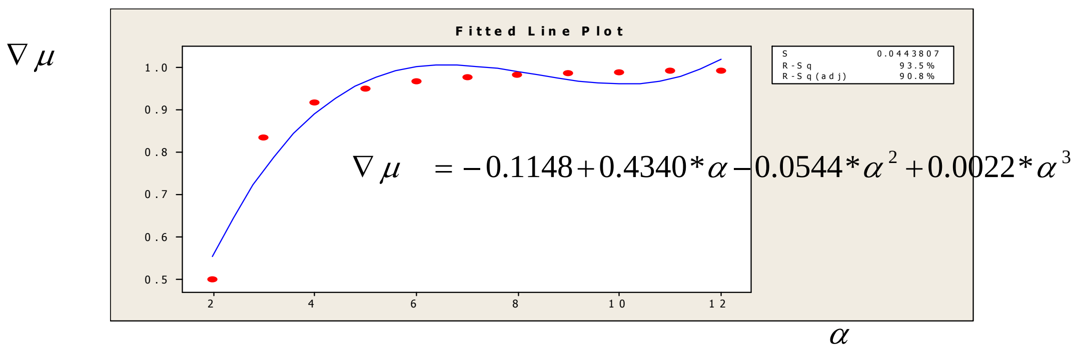

Figure 1b: Cubic Predictive Equation of the Change in Mean of the Gamma Distributed Error Component after Inverse transformation

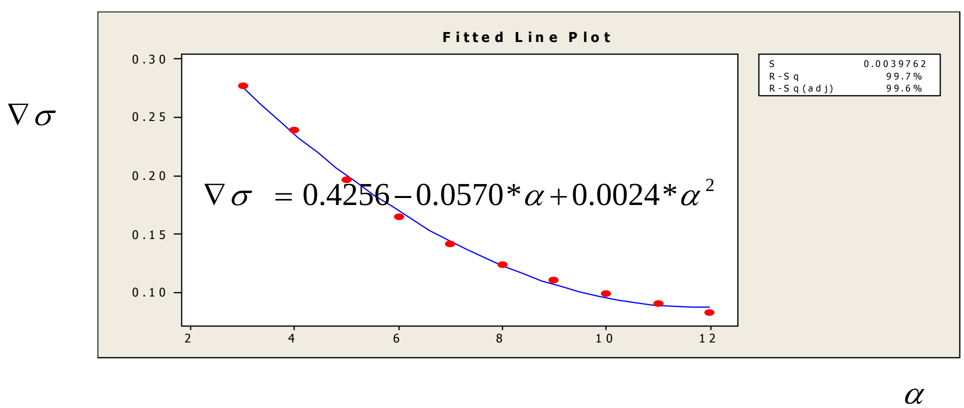

Figure 2: Predictive Equation of the Change in Variance of the Gamma Distributed Error Component after Inverse transformation

Table 1: The implication of the Unit Mean Condition on the Gamma Distribution and its counterpart Under Inverse Transformation

Moment	(ξ_t)	(ξ^*_t)	ξ_t / ξ^*_t
Mean	1.0	$\frac{1}{\alpha(\alpha-1)}, \alpha > 1$	$\alpha(\alpha-1)$
Variance	$\frac{1}{\alpha}$	$\frac{1}{\alpha^2(\alpha-1)(\alpha-2)}, \alpha > 2$	$\alpha(\alpha-1)(\alpha-2)$
Skewness	$\frac{2}{\alpha^2}$	$\frac{4}{\alpha^3(\alpha-1)^3(\alpha-2)(\alpha-3)}, \alpha > 3$	$\frac{\alpha(\alpha-1)^3(\alpha-2)(\alpha-3)}{2}$
Kurtosis	$\frac{3(\alpha+2)}{\alpha^3}$	$\frac{4}{\alpha^4(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)}$, $\alpha > 4$	$\frac{3\alpha(\alpha+2)(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)}{4}$, $\alpha > 4$

Table 2: Computations of the Means and Variances of the Transformed and the Untransformed Gamma Distributed Error Component

α	$E(\xi_t)$	$E(\xi^*_t)$	$E(\xi_t) - E(\xi^*_t)$	$Var(\xi_t)$	$Var(\xi^*_t)$	$Var(\xi_t) - Var(\xi^*_t)$
2	1.0000	0.5000	0.5000	0.5000	*	*
3	1.0000	0.1667	0.8333	0.3333	0.0556	0.2778
4	1.0000	0.0833	0.9167	0.2500	0.0104	0.2396
5	1.0000	0.0500	0.9500	0.2000	0.0033	0.1967
6	1.0000	0.0333	0.9667	0.1667	0.0014	0.1653
7	1.0000	0.0238	0.9762	0.1429	0.0007	0.1422
8	1.0000	0.0179	0.9821	0.1250	0.0004	0.1246
9	1.0000	0.0139	0.9861	0.1111	0.0002	0.1109
10	1.0000	0.0111	0.9889	0.1000	0.0001	0.0999
11	1.0000	0.0091	0.9909	0.0909	0.0001	0.0908
12	1.0000	0.0076	0.9924	0.0833	0.0001	0.0832
13	1.0000	0.0064	0.9936	0.0769	0.0000	0.0769
14	1.0000	0.0055	0.9945	0.0714	0.0000	0.0714
15	1.0000	0.0048	0.9952	0.0667	0.0000	0.0666
16	1.0000	0.0042	0.9958	0.0625	0.0000	0.0625
17	1.0000	0.0037	0.9963	0.0588	0.0000	0.0588
18	1.0000	0.0033	0.9967	0.0556	0.0000	0.0555
19	1.0000	0.0029	0.9971	0.0526	0.0000	0.0526
20	1.0000	0.0026	0.9974	0.0500	0.0000	0.0500
21	1.0000	0.0024	0.9976	0.0476	0.0000	0.0476
22	1.0000	0.0022	0.9978	0.0455	0.0000	0.0454
23	1.0000	0.0020	0.9980	0.0435	0.0000	0.0435
24	1.0000	0.0018	0.9982	0.0417	0.0000	0.0417
25	1.0000	0.0017	0.9983	0.0400	0.0000	0.0400
26	1.0000	0.0015	0.9985	0.0385	0.0000	0.0385
27	1.0000	0.0014	0.9986	0.0370	0.0000	0.0370
28	1.0000	0.0013	0.9987	0.0357	0.0000	0.0357
29	1.0000	0.0012	0.9988	0.0345	0.0000	0.0345
30	1.0000	0.0011	0.9989	0.0333	0.0000	0.0333
31	1.0000	0.0011	0.9989	0.0323	0.0000	0.0323
32	1.0000	0.0010	0.9990	0.0313	0.0000	0.0312
33	1.0000	0.0009	0.9991	0.0303	0.0000	0.0303
34	1.0000	0.0009	0.9991	0.0294	0.0000	0.0294
35	1.0000	0.0008	0.9992	0.0286	0.0000	0.0286