linear derivative operator with differential subordination of meromorphic ε -valent functions

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Abstract: The present paper is to investigate some inclusion relations between the linear derivative operator and differential subordination with other interesting properties for meromorphic ε -valent Functions in the puncture unit disk $© ^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

Keywords: Meromorphic functions , differential subordination, the linear derivative operator.

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Introduction.

Let μ_ε be the class of analytic and ε -valent meromorphic functions defined on

$© ^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

$$f(z) = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} a_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon} , (\varepsilon \in \mathbb{N} = \{1,2, \dots\}) \tag{1}$$

For function $f \in \mu_\varepsilon$ given by (1) and $q \in \mu_\varepsilon$ defined by

$$q(z) = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} b_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon} , (\varepsilon \in \mathbb{N} = \{1,2, \dots\}) \tag{2}$$

the hadamard product of f and q defined by

$$(f * q)(z) = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} a_{\mathcal{S}-\varepsilon} b_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon} \tag{3}$$

Let $\mathfrak{D}_*^{t,\varepsilon} f$ denote the linear derivative operator of Ruschwey typ [9][6],

$f \in \mu_\varepsilon$ defined by:

$$\mathfrak{D}_*^{t,\varepsilon} f(z) = \frac{z^{-\varepsilon}}{(1-z)^{t+\varepsilon}} * f(z), \qquad t > -\varepsilon, (z \in © ^*) \tag{4}$$

The (4) can be written by binomial coefficients

$$\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z) = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} \binom{t+\mathcal{S}}{\mathcal{S}} a_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon}, \quad t > -\varepsilon. \tag{5}$$

The class of functions $\mathbb{h}$ with $\mathbb{h}(0) = 1$, is, which are convex univalent and analytic in © ={ z ∈ $\mathbb{C}$: $|z| < 1$}.

Recently some authors studied differential subordination of meromorphic functions of different subclasses [1],[2],[3],[4] and [5]

Definition (1): If satisfies the subordination condition the function $\mathfrak{f} \in \mu_\varepsilon$ is said to be in the

class $\mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$:

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{p+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)'''' \prec \mathbb{h}(z). \tag{6}$$

Where $t \in \mathbb{C}, \mathbb{h} \in \Psi$.

It is necessary to put the restrictions on the operator $\mathfrak{D}_*^{t,\varepsilon}$ such that

$$\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1 * \mathfrak{f}_2) = (\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_1) * \mathfrak{f}_2 = \mathfrak{f}_1 * (\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_2), \tag{7}$$

if $\mathfrak{f}_1, \mathfrak{f}_2 \in \mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$, we get the convolution results of the class of multivalent analytic functions $\mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$.

Lemma 1[8]: let q be analytic and convex univalent in © and Let $\mathbb{h}$ be analytic in ©

with $q(0) = \mathbb{h}(0)$. If

$$q(z) + \frac{1}{\mathfrak{M}} z'(z) \prec \mathbb{h}(z), \tag{8}$$

Where $Re\ \mathfrak{M} \geq 0$ and $\mathfrak{M} \neq 0$, then

$$q(z) \prec \mathbb{h}^{\neg}(z) = \mathfrak{M} z^{-\mathfrak{M}} \int_0^z L^{\mathfrak{M}-1}\, \mathbb{h}(L) dL \prec \mathbb{h}(z).$$

And $\mathbb{h}^{\neg}(z)$ is the best dominant of (7).

Lemma (2)[10]: let $\mathfrak{f}(z) \prec \emptyset(z) (z \in ©)$ and $q(z) \prec \mathfrak{I}(z)$ $(z \in ©)$ if the function $\emptyset(z)$

and $\mathfrak{I}(z)$ are convex in ©. Then $(\mathfrak{f} * q)(z) \prec (\emptyset * \mathfrak{I})(z)$ $(z \in ©)$.

Theorem (1): If the function $\mathfrak{f} \in \mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$, then

$$q(z) = \frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)} \prec \mathbb{h}(z), \tag{9}$$

and if $t > 0$, then $q(z) \prec \mathbb{h}^{\neg}(z)$, where

$$\mathbb{h}^{\neg}(z) = \frac{(\varepsilon+3)}{t} z^{\frac{-(\varepsilon+3)}{t}} \int_0^z L^{\frac{(\varepsilon+3)}{t}-1}\, \mathbb{h}(L) dL \prec \mathbb{h}(z) \qquad (z \in ©),$$

$\mathbb{h}^{\neg}(z)$ is the best dominant of subordination $q(z) \prec \mathbb{h}^{\neg}(z)$ $(z \in ©)$

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and $\mathbb{h}^{\neg}(z)$ is convex univalent in ©

Proof. When t=0, trivial.

If $t > 0$, let $f \in \mu_\varepsilon(\lambda, \mathcal{S}: \mathbb{h})$, then

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)^{'''} + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)^{''''} \prec \mathbb{h}(z).$$

By (6) and (9)

$$q(z) + \frac{t}{(\varepsilon+3)} z q'(z) \prec \mathbb{h}(z) \ (z \in \mathrm{U}). \quad (10)$$

During Lemma (1) in (10) with $m = \frac{(\varepsilon+3)}{t}$ and $t > 0$, we give

$$q(z) \prec \mathbb{h}^{\neg}(z) = \frac{(\varepsilon+3)}{t} z^{\frac{-(\varepsilon+3)}{t}} \int_0^z L^{\frac{(\varepsilon+3)}{t}-1} \mathbb{h}(L) dL \prec \mathbb{h}(z)$$

Where q is given by (9).

Theorem (2): $\mu_\varepsilon(t_1, \mathcal{S}: \mathbb{h}), \subset \mu_\varepsilon(t_2, \mathcal{S}: \mathbb{h})$ if $0 \le t_2 < t_1$.

Proof. Let $f \in \mu_\varepsilon(t_1, \mathcal{S}: \mathbb{h})$.

$$\frac{(1+t_2)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)^{'''} + \frac{t_2}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)^{''''}$$

$$= \left[1 - \frac{t_2}{t_1}\right] \frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)^{'''}}{\varepsilon(\varepsilon+1)(\varepsilon+2)} +$$

$$\frac{t_2}{t_1}\left[\frac{(1+t_2)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)^{'''} + \frac{t_2}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)^{''''}\right] \quad (11)$$

since h is a convex set and $0 \le \frac{t_2}{t_1} < 1$. (11) can write as follows:

$$\left[\frac{(1+t_2)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)^{'''} + \frac{t_2}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)^{''''}\right]$$

$$= \left[1 - \frac{t_2}{t_1}\right] q_1(z) + \frac{t_2}{t_1} q_2(z) = \emptyset(z),$$

Where $q_1(z), q_2(z) \prec \mathbb{h}(z)$, by using definition of convex set and by Theorem (1), since $f \in \mu_\varepsilon(t_1, \mathcal{S}: \mathbb{h})$, we get $\emptyset(z) \prec \mathbb{h}(z)$, then $f \in \mu_\varepsilon(t_2, \mathcal{S}: \mathbb{h})$.

Theorem (3): Let ϕ defined by

$$\phi(z) = \frac{(\sigma-\varepsilon)}{z^\sigma} \int_0^z L^{\sigma-1} f(L) dL \qquad (Re\{\sigma\} > -\varepsilon),$$

and let the function $f \in \mu_\varepsilon$

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If

$$\left[1-\frac{\gamma}{\varepsilon}\right]\frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)}+\gamma\frac{z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}\prec \mathbb{h}(z). \quad (12)$$

Then the function $\phi\in\mu_\varepsilon(0,\mathcal{S}:\mathbb{h}^\neg)$ Where

$$\mathbb{h}^\neg(z)=\frac{(\sigma\varepsilon-\varepsilon)}{\gamma}z^{\frac{(\sigma\varepsilon-\varepsilon)(\varepsilon+3)}{\gamma}}\int_0^z L^{\frac{(\sigma\varepsilon-\varepsilon)(\varepsilon+3)}{\gamma}}\mathbb{h}(L)dL\prec\mathbb{h}(z). \quad (13)$$

Proof. Define

$$\mathfrak{I}(z)=\frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)}, \quad (14)$$

then $\mathfrak{I}$ is analytic in ©, $\mathfrak{I}(0)=1$ and

$$\frac{z\mathfrak{I}'(z)}{(\varepsilon+3)}=\mathfrak{I}(z)+\frac{z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)''''}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}. \quad (15)$$

Making use of (12), (14) and (15) and by

$$(\sigma\varepsilon-\varepsilon)\mathfrak{f}(z)=\sigma(\varepsilon+3)\phi'''(z)+z\phi''''(z),$$

then

$$\left[1-\frac{\gamma}{\varepsilon}\right]\frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)}+\frac{\gamma z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}$$

$$=\left[1-\frac{\gamma}{\varepsilon}\right]\frac{z^{p+3}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)}+\frac{\gamma}{(\sigma\varepsilon-\varepsilon)}\left[\frac{\sigma z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)}+\frac{z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)''''}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}\right]$$

$$=\mathfrak{I}(z)+\frac{\gamma}{(\sigma\varepsilon-\varepsilon)(\varepsilon+3)}z\mathfrak{I}'(z)$$

$$=\mathfrak{I}(z)+\frac{\gamma}{(\sigma\varepsilon-\varepsilon)(\varepsilon+3)}z\mathfrak{I}'(z)\prec\mathbb{h}(z).$$

where $\mathbb{h}^\neg(z)$ is given by (13) then $\mathfrak{I}(z)\prec\mathbb{h}^\neg(z)$, and $\phi\in\mu_\varepsilon\ (0,\mathcal{S}:\mathbb{h}^\neg)$.

Theorem (4): Let $\mathfrak{D}_*^{t,\varepsilon}$ satisfy the condition (7) in definition (1) If

$$\mathfrak{f}_i\in\mu_\varepsilon\left(t,\mathcal{S}:\frac{1+\alpha_i z}{1+\beta_i z}\right)(i=1,2).$$

Then the inclusion relationship are hold:

$$\mathfrak{Y}(z) = \frac{(1+t)}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2} z^3\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1{'''} * \mathfrak{f}_2{'''})(z)\right) + \frac{t}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2(\varepsilon+3)^2} z^4\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1{''''} * \mathfrak{f}_2{''''})(z)\right)$$

$$\in \mu_\varepsilon\left(t, k; \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right), \qquad (16)$$

$$h(z) = \frac{z^3\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1{'''} * \mathfrak{f}_1{'''})(z)\right)}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2} \in M_\varepsilon\left(t, \mathcal{S}; \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right), \qquad (17)$$

and

$$\frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)} \prec \left(\left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right). \qquad (18)$$

Proof. Since

$$\mathfrak{f}_1 \in \mu_\varepsilon\left(t, \mathcal{S}: \frac{1+\alpha_1 z}{1+\beta_1 z}\right) \; and \; \mathfrak{f}_2 \in \mu_\varepsilon\left(t, \mathcal{S}: \frac{1+\alpha_2 z}{1+\beta_2 z}\right).$$

Then

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_1(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_1(z)\right)'''' \prec \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right), \qquad (19)$$

and

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_2(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_2(z)\right)'''' \prec \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right). \qquad (20)$$

By Theorem (1), (19) and (20), we give

$$\frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_1(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)} \prec \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right),$$

and

$$\frac{z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_2(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} \prec \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right).$$

By (7), (19) and (20) and Lemma (2), we have

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{y}(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{y}(z)\right)''''$$

$$= \frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\left[\frac{(1+t)}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2} z^3\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1''' * \mathfrak{f}_2''')(z)\right) + \frac{t}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2(\varepsilon+3)^2} z^4\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1'''' * \mathfrak{f}_2'''')(z)\right)\right]\right)'''$$

$$+ \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\left[\frac{(1+t)}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2} z^3\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1''' * f_2''')(z)\right) + \frac{t}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2(\varepsilon+3)^2} z^4\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1'''' * \mathfrak{f}_2'''')(z)\right)\right]\right)''''$$

$$\prec \left(\left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right).$$

Then

$$\mathfrak{y} \in \mu_\varepsilon\left(t, \mathcal{S}; \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right),$$

and

$$\mathbb{h} \in \mu_\varepsilon\left(t, \mathcal{S}; \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right).$$

The proof of (16) and (17) is complete, by same method we obtain

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathbb{h}(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathbb{h}(z)\right)'''' \prec \left(\left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right). \quad (21)$$

Where $\mathbb{h}$ is given by (17). The proof of (18) we get by (21) and Theorem (1).

Theorem 5: let $\mathcal{A} \in \mu_\varepsilon$ and $\mathfrak{f} \in \mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$ and

$$\text{Re}\,\{z^\varepsilon \mathcal{A}(z)\} \geq \tfrac{1}{2}. \quad (22)$$

Then $(\mathfrak{f} * \mathcal{A}) \in \mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$

Proof. If $\mathfrak{f} \in M_\varepsilon(t, \mathcal{S}: \mathbb{h})$ given by (1) and $\mathcal{A} \in \mu_\varepsilon$ we have

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f} * \mathcal{A})(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f} * \mathcal{A})(z)\right)''''$$

$$= \frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)}\left[z^{\varepsilon}\mathcal{A}(z) * \left(z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}(z)\right)^{'''}\right)\right] + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}\left[z^{\varepsilon}\mathcal{A}(z) * \left(z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}(z)\right)^{''''}\right)\right]$$

$$= \{z^{\varepsilon}\mathcal{A}(z)\} * \Psi(z) \qquad (23)$$

$$\Psi(z) = \frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)^{'''} + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)^{''''}.$$

The function $z^{\varepsilon}\mathcal{A}(z)$ has the Herglotz representation [7], by (22).

$$z^{\varepsilon}\mathcal{A}(z) = \int_{|x|=1} \frac{d\varepsilon(x)}{1-xz} \quad (z \in ©) \qquad (24)$$

The probability measure (ε) defined on the unit circle $|x| = 1$, and

$$\int_{|x|=1} d\varepsilon(x) = 1.$$

Because $\mathbb{h}$ is convex univalent in ©.

By (21) and (24) give now

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f} * \mathcal{A})(z)\right)^{'''} + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f} * \mathcal{A})(z)\right)^{''''}$$

$$= \int_{|x|=1} \Psi(xz) d\varepsilon(z) \prec \mathbb{h}(z).$$

Then $(\mathfrak{f} * \mathcal{A}) \in \mu_{\varepsilon}(t, \mathcal{S}: \mathbb{h})$.

Corollary: Suppose $\mathfrak{f} \in \mu_{\varepsilon}(t, \mathcal{S}: \mathbb{h})$ be given by (1) and let

$$Re\left(1 + \sum_{\mathcal{S}=1}^{\infty} \frac{\delta}{k+\delta} z^{\mathcal{S}}\right) > \frac{1}{2}.$$

Then

$$\Phi_{\varepsilon,\delta}(\mathfrak{f}) = \frac{\delta}{z^{\varepsilon+\delta}} \int_0^z L^{\varepsilon+\delta-1}\mathfrak{f}(L)dL \quad (\delta > -\varepsilon),$$

$$and\ \Phi_{\varepsilon,\delta}(\mathfrak{f}) \in \mu_{\varepsilon}(t, \mathcal{S}: \mathbb{h}).$$

Proof. Let $\mathfrak{f} \in \mu_{\varepsilon}(t, \mathcal{S}: \mathbb{h})$ be defined in (1). Then

$$\Phi_{\varepsilon,\delta} \frac{\delta}{z^{\varepsilon+\delta}} \int_0^z L^{\varepsilon+\delta-1}\mathfrak{f}(L)dL = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} \frac{\delta}{\mathcal{S}+\delta} a_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon},$$

$$= \left(z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} a_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon} \right) * \left(z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} \frac{\delta}{\mathcal{S}+\delta} a_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon} \right) = (f * \phi) * z, \quad (25)$$

where

$$\phi(z) = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} \frac{\delta}{\mathcal{S}+\delta} z^{\mathcal{S}-\varepsilon} (\delta > -\varepsilon),$$

and $\phi \in M_\varepsilon$. We give

$$Re\{z^{\varepsilon}\phi(z)\} = Re\left(1 + \sum_{\mathcal{S}=1}^{\infty} \frac{\delta}{\mathcal{S}+\delta} z^{\mathcal{S}} \right) > \frac{1}{2}, \qquad (26)$$

Thus $\phi_{\varepsilon,\delta}(f) \in \mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$, by (25) (26) and theorem (5).

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