Construction of new Non-standard Finite Difference Schemes for the Solution of a free un-damped Harmonic Oscillator Equation

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Abstract

In this paper we discussed the numerical solution to the model of a freely suspended spring with a mass attached. Suitable nonstandard finite difference schemes were developed for the solution of the resulting dynamical system which follows a free un-damped harmonic oscillator. The result of the numerical experiment on the schemes are illustrated with 3D graphs .

Keywords: Numerical methods , Non-Local Approximations, Non-standard finite difference schemes, Qualitative properties, Free un-damped Harmonic Oscillator, Monotonicity of solution.

1. Introduction

The solutions of continuous dynamical systems given by systems of ordinary differential equations are typically computed by various numerical procedures defined on discretized time meshes. It is a well known fact that finite difference scheme is one of the oldest and popular techniques for numerical solution of ordinary differential equations. In most of the equations in mathematical physics, engineering and in some physical sciences, finite difference schemes have been designed and investigated both from the theoretical point view, which is the convergence aspect, and the practical point of view which is the consistency and stability point of view (Anguelov and Lubuma, 2000).

Non-standard finite difference schemes (NSFD) have emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential and biological models as well as chaotic systems (Mickens 2005). These techniques have many advantages over classical techniques and provide an efficient numerical solution. In fact, the non-standard finite difference method is an extension of the standard finite difference method. Non-standard schemes as introduced by Mickens (1989,1990,1994) are used to help resolve some of the issues related to numerical instabilities. Furthermore, Mickens (1999,2000,2005) introduced certain rules for obtaining the best difference equations, one of the most important is the selection of a suitable denominator function $\phi(h)$ for the dicrete derivatives

1.1 Equation of motion of a spring suspended freely with a mass attached

The major reference for this model is Zill (2005). When a spring is suspended vertically from a rigid support, then we can assume that the spring do not exert any force. We can then consider a body of mass M attached to the free end so that the amount of stretch depend on the value of M. If we displace the body by a distance S then the spring exerts a restoring force F= -kS , where k>0 is the spring constant following Hooke’s Law. Let us neglect t the air resistance and there are no retardation force acting against the direction of motion(i.e free and un-damped motion). At the equilibrium point the weight M x g and the initial force exerted by the stiffness of the spring are equal in magnitude i.e Mg = kS. If the body is displaced by a distance x from the equilibrium position then the restoring force is k(X+S). Since there are no external forces acting on the body against the direction of motion , then we can use the Newton's second Law to represent the free motion of the displacement from the equilibrium point by the expression

$$M\frac{d^2x}{dt^2} = -k(X+S) + Mg \qquad (1)$$

If we substitute the fact that Mg = kS then the equation may be written as

$$\frac{d^2x}{dt^2} + k\,x = 0 \qquad (2)$$

Which can also be written as

$$\frac{d^2x}{dt^2} + w^2x = 0 \qquad (3)$$

Where $w^2 = k/M$

Any dynamic system whose function obey the equation above is said to perform a simple free undamped harmonic motion

2. Derivation of the Nonstandard Finite Difference Schemes

There are quite a large number of numerical methods that may be used for obtaining approximate solution of a second order ordinary differential equation among which Finite difference schemes have been in the fore front.

While the aforementioned techniques are rather successful at dealing with generic differential equations, they can often come unstuck for some particular problems of interest that arise in applications, particularly when such problems exhibit special properties such as symmetries or conservation laws, or when there are solutions with some special structure.

Non-standard schemes as introduced by Mickens (1989,1990,1994) are used to help resolve some of the issues related to numerical instabilities. Furthermore, Mickens (1999,2000,2005) introduced certain rules for obtaining the best difference equations. Many authors including [1], [2], [5], have been contributing to this new technique. Such techniques produce numerically stable schemes which provide reliable solution to the differential equation and carried along the dissipative & qualitative properties of the original equation. In this paper we will employ Mickens five point rule to obtain a new numerical scheme for the solution of the model of a suspended spring with mass attached on a free motion.

2.1 Construction

Consider the equation in (3)

The analytic solution is of the form c1coswt + c2sinwt

The component of equation (3) may be approximated in the following manner:

$$\frac{d^2x}{dt^2} \equiv \frac{X_{k+1} - 2X_k + X_{k-1}}{h} \qquad (4)$$

$$w^2x \equiv w^2(X_k) \qquad (5)$$

Using Micken's rules 2 and 3,and the subsequent non-local approximation technique introduced by Angueluv and Lubume (2003), the denominator function h must be replaced by a much more complex function of the step size h. Experience has shown that such functions perform better when it is related to a particular solution of the ordinary differential equation. This is referred to as the renormalization of the denominator function. Mickens (1994) in his paper has shown that such function must obey the following conditions to avoid numerical instability:

$\phi(h) = h + O(h^2)$ for first derivative

$\phi(h) = h^2 + O(h^4)$ for second derivative

For the first derivative the following are suitable $\phi(h)$ functions:

Sin(h) , 1-e^{-h} etc.

For the second derivative the following are suitable $\phi(h)$ functions:

$\sin^2(h)$, h($\frac{e^{-\lambda h} - 1}{\lambda}$) etc.

Hence we may rewrite equation (4) in the form

$$\frac{d^2x}{dt^2} \equiv \frac{X_{k+1}-2X_k+X_{k-1}}{h} = \frac{X_{k+1}-2X_k+X_{k-1}}{\phi(h)} \quad (6)$$

$$\phi(h) = 4\sin^2(\tfrac{h}{2}) \quad (7)$$

The rule 3 also required that the subsequent terms be approximated as a linear combination of several point on the grid .

Hence we may approximate the other term as follows:

$$w^2x \equiv w^2(aX_k + bX_{k+1}), \quad a+b=1 \quad (8)$$

$$w^2x \equiv w^2(aX_{k+1} + bX_k + cX_{k-1}), \quad a+b+c=1 \quad (9)$$

$$w^2x \equiv w^2\left(\frac{2X_k+X_{k+1}}{3}\right) \quad (10)$$

3. Construction of new schemes

We will now apply the above technique to construction of new scheme s.

Consider a slipping chain under free un-damped motion which obeys a second order ordinary differential equation given by:

$$\frac{d^2x}{dt^2} + 64x = 0 , y(0) = 2/3, y'(0) = -4/3 \quad (11)$$

It can be verified that the analytic solution is

$$X(t) = \tfrac{2}{3}\cos 8t - \tfrac{1}{6}\sin 8t \quad (12)$$

3.1 Construction of Scheme 1

Applying equations (6,7,8 and 11) we can write

$$\frac{X_{k+1}-2X_k+X_{k-1}}{\phi(h)} + 64\,(aX_{k+1} + bX_k) = 0, \quad a+b=1 \quad (13)$$

$$\phi(h) = 4\sin^2(\tfrac{h}{2})$$

$$X_{k+1} - 2X_k + X_{k-1} + 64\,\phi(aX_{k+1} + bX_k) = 0 \quad (14)$$

$$X_{k+1}\,(1+64a\phi) = (2-64b\phi)\,X_k - X_{k-1} \quad (15)$$

(i). $$X_{k+1} = \frac{(2-64b\phi)\,X_k - X_{k-1}}{1+64a\phi} \quad (16)$$

3.2 Construction of Scheme 2

Since $\cos(h^2)$ is very close to unit "1" for small h we may :

(ii). Substitute a=1/4 and b = 3ψ/4, ψ= cos(h) , $\phi = \frac{1}{16}\sin^2(4h)$ in (16) to get a new scheme

$$X_{k+1} = \frac{(2-48\psi)\,X_k - X_{k-1}}{1+64a\phi} \tag{17}$$

3.3 Construction of Scheme 3

Another scheme can be obtained by applying equations 6,7 and 9 in equation (11) thus

$$\frac{X_{k+1}-2X_k+X_{k-1}}{\phi(h)} + 64\,(aX_{k+1} + bX_k + cX_{k-1}) = 0, \quad a=b=c=1/3 \tag{18}$$

So that

(iii). $X_{k+1}\ (1+\frac{64}{3}\phi) = (2-\frac{64}{3}\phi)\,X_k - (1+\frac{64}{3}\phi)X_{k-1}$ (19)

$$X_{k+1} = \frac{(2-\frac{64}{3}\phi)\,X_k - (1+\frac{64}{3}\phi)X_{k-1}}{(1+\frac{64}{3}\phi)} \tag{20}$$

$$\phi = \frac{1}{4}\sin^2(2h) \quad \text{or} \quad h\left(\frac{e^{-\lambda h}-1}{\lambda}\right)$$

3.4 Construction of Scheme 4

We can also use a direct substitution of equation 6 in equation (11) to obtain:

$$\frac{X_{k+1}-2X_k+X_{k-1}}{\phi(h)} + 64\,(X_k) = 0, \tag{21}$$

So that

(iv). $X_{k+1} = (2-64b\phi)\,X_k - X_{k-1}$ (22)

$$\phi = \frac{1}{4}\sin^2(2h) \quad \text{or} \quad h\left(\frac{e^{-\lambda h}-1}{\lambda}\right)$$

The schemes will now be tested for consistency with the analytic solution in a numerical experiment

4. Performance of the schemes

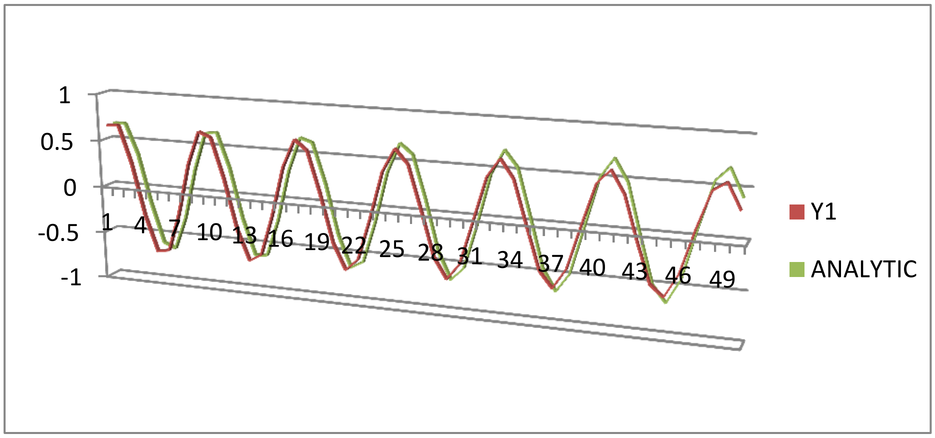

Figure1: Graph showing scheme 1 and the analytic solution

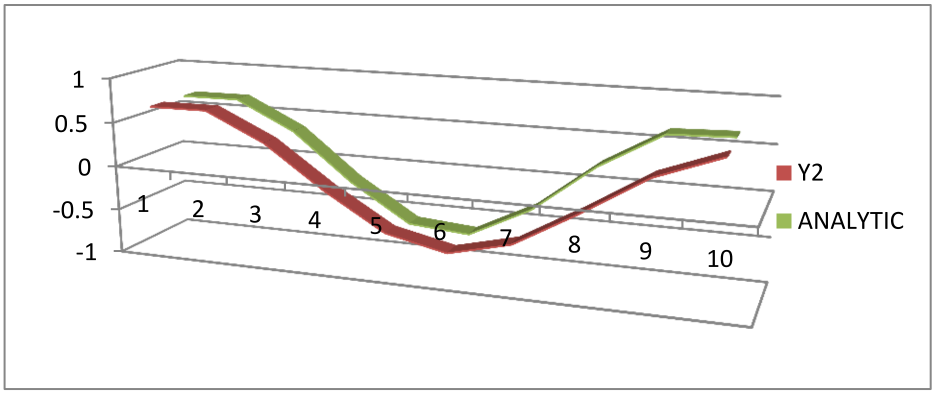

Figure 2 Graph showing scheme 2 and the analytic solution

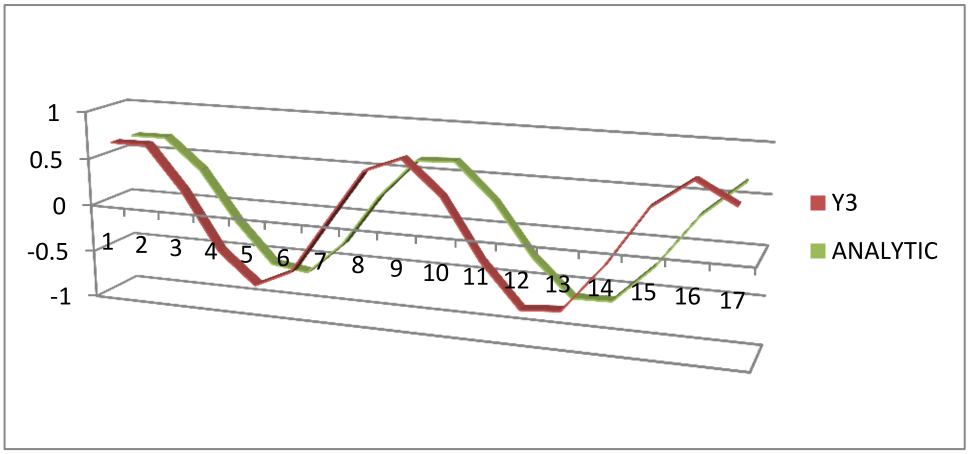

Figure 3: Graph showing scheme 3 and the analytic solution

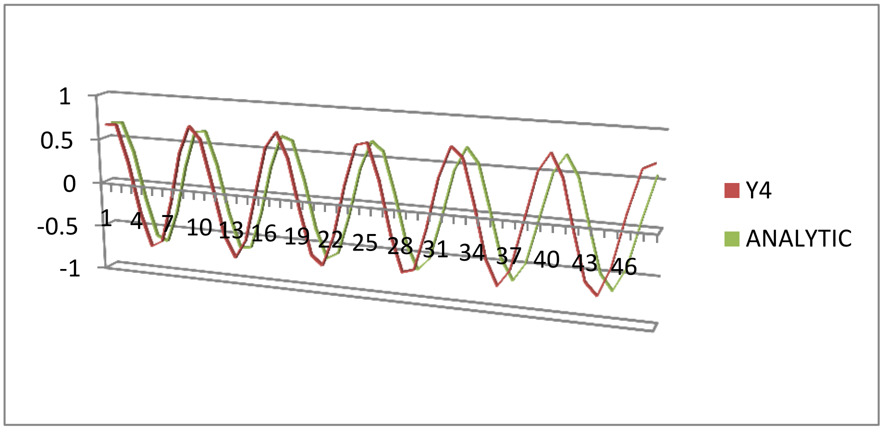

Figure 4: Graph showing scheme 4 and the analytic solution

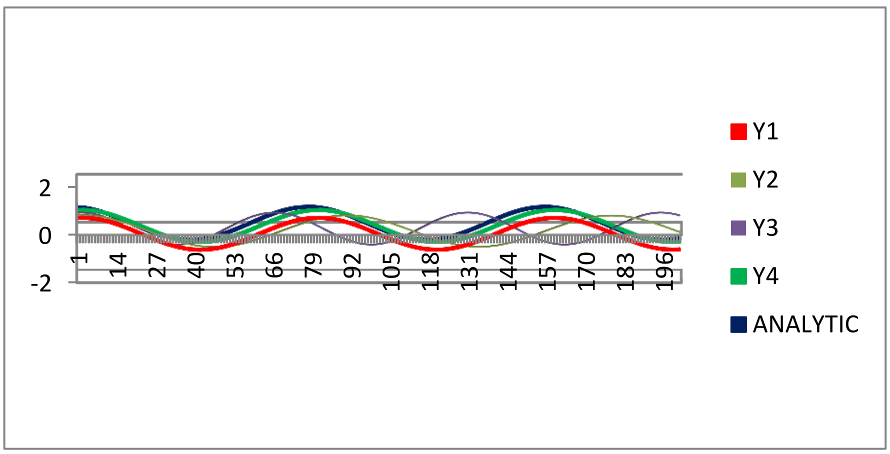

Figure 5: Graph showing all the schemes and the analytic solution

5. Summary and Conclusion

It can be observed from that all the schemes produced solution curves whose behavior is consistent with the analytic solution for the first 20 iterations using a step length of h =0.1. The curves of schemes 1 and 4 are more consistent for a larger number of iterations (see figure 1,4,5) . For all oscillations, the amplitude remains constant with respect to time. We can conclude that the schemes carried along the qualitative properties of the free un-damped harmonic oscillator .

With carefully chosen parameters a,b, and step function ϕ we can get as close as possible to the original curve of the particular harmonic oscillator . We can also conclude that, for a linear second order equation like the free un-damped harmonic motion it is enough to select parameters a and b such that a + b =1 for non-local approximation. It can be observed that the choice of h is restricted to a very small h for scheme '2' because of the term cosine(h).

References

[1]. Anguelov R , Lubuma J.M.S. (2000). Nonstandard finite difference method, by Non-local Approximation *Keynote address at the South Africa Mathematical Society, Pretoria, South Africa 16-18 October, Notices of S. A. Math. Soc*. 31: 143-543.

[2]. Anguelov R , Lubuma J.M.S. (2003). Nonstandard finite difference method by nonlocal approximation, *Mathematics and Computers in simulation* 6: 465-475.

[3]. Ibijola E.A and Obayomi A.A. (2012). A New Family Of Numerical Schemes for Solving the Combustion Equation. *Journal of Emerging Trends in Engineering and Applied Sciences (JETEAS)* 3 (3): 387-393

[4]. Ibijola E.A and Obayomi A.A. (2012).A New Family Of Numerical Schemes for Solving Equation of the Type $y' = y^n(1-y)$. *Innova Sciencia* vol 4 (6) pages 34-46

[5]. Mickens R.E. (1994). Nonstandard Finite Difference Models of Differential Equations, World *Scientific, Singapore.* Pp 1-115, 144-162

[6]. Potts R.B. (1982) *American Mathematically monthly* 89, 402-407. Differential and Difference equations.

[7]. Zill D.G and Cullen R.M. (2005) *Differential Equations with boundary value problems* (sixth Edition) Brooks