Exact Solution of Coupled Nonlinear PDEs Via Sumudu Decomposition Method

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Abstract: In this paper, we apply the Sumudu Decomposition Method on system of coupled nonlinear partial differential equations to calculate the analytical solutions in closed form. The nonlinear term can easily be handled with the help of He's polynomials. The proposed technique is tested on four problems. Calculated results show the potential of the technique.

Keyword: Nonlinear PDEs, He's polynomials, Sumudu transform, Adomian decomposition method

1. Introduction

Most of phenomena in nature are described by nonlinear differential equations. Different analytical methods have been applied to find a solution to them. For example, Adomian decomposition method has been applied for solving algebraic, differential, integro differential, differential-delay and partial differential equations. In the nonlinear case for ordinary differential equations and partial differential equations, the method has the advantage of dealing directly with the problem [1, 2]. These equations are solved without transforming them to more simple ones. The method avoids linearization, perturbation, discretization, or any unrealistic assumptions [3-5]. In the present paper, the intimate connection between the Sumudu transform theory and decomposition method arises in the solution of nonlinear partial differential equations is demonstrated.

Sumudu transform is defined over the set of the following functions:

$$A = \{ f(t) : \tau_1, \tau_2 > 0, |f(t)| < Me^{t/\tau_j}, if\ t \in (-1)^j \times [0, \infty) \} \tag{1}$$

by the following formula:

$$G(u) = S[f(t); u] := \int_0^\infty f(ut)e^{-t}dt, \qquad u \in (-\tau_1, \tau_2). \tag{2}$$

In [6, 7], some fundamental properties of the Sumudu transform were established. In [8] this new transform was applied to the one-dimensional neutron transport equation. Further in [9] the Sumudu transform was extended to the distributions and some of their properties were also studied in [10]. Recently Ahmad et al. applied this transform to solve the fuzzy differential equations [11]. A very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except a factor $n!$. i.e.

$$f(t) = \sum_{n=0}^{\infty} a_n t^n, \tag{3}$$

and

$$F(u) = \sum_{n=0}^{\infty} n! a_n t^n, \tag{4}$$

Similarly, the Sumudu transform sends combinations, $C(m,n)$, into permutations $P(m,n)$, and hence it will be useful in the discrete systems. Further

$$\begin{aligned} S(H(t)) &= L(\delta(t)) = 1, \\ L(H(t)) &= S(\delta(t)) = \frac{1}{u}. \end{aligned} \tag{5}$$

Theorem: [12] let $f(t)$ be in A, and let $G^n(u)$ denote the Sumudu transform of the nth derivative $f^n(t)$ of $f(t)$ then for $n \geq 1$

$$G^n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{n-k}}. \tag{6}$$

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2. Analysis of the Method

We consider the general inhomogeneous nonlinear equation with initial conditions given below:

$$LU + RU + NU = h(x,t), \tag{7}$$

where L is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, NU represents the nonlinear terms and $h(x,t)$ is the source term. First we explain the main idea of SDM, the method consists of applying Sumudu transform

$$S[LU] + S[RU] + S[NU] = S[h(x,t)]. \tag{8}$$

Using the differential property of Sumudu transform and initial conditions we get

$$\frac{1}{u^n}S[u(x,t)] - \frac{1}{u^n}u(x,0) - \frac{1}{u^{n-1}}u'(x,0) - \ldots - \frac{u^{n-1}(x,0)}{u} + S[RU] + S[NU] = S[h(x,t)], \tag{9}$$

$$S[u(x,t)] = u(x,0) + uu'(x,0) + \ldots + u^{n-1}u^{n-1}(x,0) - u^n S[RU] - u^n S[NU] + u^n S[h(x,t)] \tag{10}$$

Assuming the solution as an infinite series

$$u(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t) \ , \tag{11}$$

and the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{i=0}^{\infty} H_i, \tag{12}$$

where H_i are He's Polynomials of U_0, $U_1, \ldots U_n$ and it can be calculated by formula

$$H_i = \frac{1}{i!}\frac{d^i}{dp^i}[N\sum_{i=0}^{\infty} p^i u_i]_{p=0}, \quad i = 0,1,2,\ldots \tag{13}$$

Thus

$$S\left[\sum_{i=0}^{\infty} U_i(x,t)\right] = U(x,0) + uU'(x,0) + \ldots + u^{n-1}U^{n-1}(x,0)$$
$$- u^2 S[RU] - u^n S[RU(x,t)] - u^n S[\sum_{i=0}^{\infty} H_i] + u^n S[h(x,t)]. \tag{14}$$

On comparing both sides and by using standard ADM

$$S[U_0(x,t)] = U(x,0) + uU'(x,0) + \ldots + u^{n-1}U^{n-1}(x,0) + u^n S[h(x,t)] = K(x,u), \tag{15}$$

Then it follows that

$$S[U_1(x,t)] = -u^n S[RU_0(x,t)] - u^n S[H_0],$$
$$S[U_2(x,t)] = -u^n S[RU_1(x,t)] - u^n S[H_1]. \tag{16}$$

In more general, we have

$$S[U_{i+1}(x,t)] = -u^n S[RU_i(x,t)] - u^n S[H_i],\ i \geq 0. \tag{17}$$

On applying the inverse Sumudu transform

$$U_0(x,t) = K(x,t),$$
$$U_{i+1}(x,t) = -S^{-1}\left[u^n S[RU_i(x,t)] + u^n S[H_i]\right],\ i \geq 0, \tag{18}$$

where $K(x,t)$ represents the term that is arising from source term and prescribed initial conditions. We now consider the inhomogeneous nonlinear partial differential equations:

$$S[LU] + S[RU] + S[NU] = S[h(x,t)], \tag{19}$$

with the initial conditions

$$u(x,0) = f(x),\ u_t(x,0) = g(x). \tag{20}$$

Where $L = \frac{\partial^2}{\partial t^2}$ is second-order differential operator, NU represents a general non-linear differential operator where as $h(x,t)$ is source term. The methodology consists of applying Sumudu transform first on both sides,

$$S[u(x,t)] = f(x) + ug(x) - u^2 S[RU] - u^2 S[NU] + u^2 S[h(x,t)], \tag{21}$$

Then by taking inverse Sumudu transform

$$u(x,t) = f(x) + tg(x) - S^{-1}\left[u^2 S[RU] - u^2 S[NU] + u^2 S[h(x,t)]\right] \tag{22}$$

3. Numerical Applications

Example3.1 Consider a nonlinear partial differential equation

$$u_{tt} + u^2 - u_x^2 = 0,\ t > 0, \tag{23}$$

with initial conditions

$$u(x,0) = 0,\ u_t(x,0) = e^x. \tag{24}$$

By taking Sumudu transform for (23) and (24), we obtain

$$\frac{1}{u^2} S[u(x,t)] - \frac{u(x,0)}{u^2} - \frac{u_t(x,0)}{u} = S[u_x^2 - u^2], \tag{25}$$

$$S[u(x,t)] = ue^x + u^2 S[u_x^2 - u^2], \tag{26}$$

applying the inverse Sumudu transform

$$u(x,t) = te^x + S^{-1}\left[u^2 S[u_x^2 - u^2]\right], \tag{27}$$

which assumes a series solution of the function $u(x,t)$ and is given by

$$u(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t). \tag{28}$$

Thus

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = te^x + S^{-1}\left[u^2 S[\sum_{i=0}^{\infty} H_i(u) - \sum_{i=0}^{\infty} B_i(u)]\right], \tag{29}$$

H_i And B_i are He's polynomials that represent nonlinear terms.

Thus by comparing the coefficients of p , we get

$$p^0; u_0(x,t) = te^x, \tag{30}$$

$$p^1; u_1(x,t) = S^{-1}\left[u^2 S[\sum_{i=0}^{\infty} H_0(u) - \sum_{i=0}^{\infty} B_0(u)]\right] = 0, \tag{31}$$

$$\vdots$$

$$p^{i+1}; u_{i+1}(x,t) = S^{-1}\left[u^2 S[\sum_{i=0}^{\infty} H_i(u) - \sum_{i=0}^{\infty} B_i(u)] \right] = 0 \quad i \geq 1. \tag{32}$$

Therefore the exact solution when $p \to 1$ is

$$u(x,t) = te^x. \tag{33}$$

Example 3.2 Consider the system of nonlinear coupled partial differential equation

$$\begin{aligned} u_t(x,y,t) - v_x w_y &= 1, \\ v_t(x,y,t) - w_x u_y &= 5, \\ w_t(x,y,t) - u_x v_y &= 5. \end{aligned} \tag{34}$$

with initial conditions

$$\begin{aligned} u(x,y,0) &= x + 2y, \\ v(x,y,0) &= x - 2y, \\ w(x,y,0) &= -x + 2y. \end{aligned} \tag{35}$$

Applying the Sumudu transform we obtain

$$\begin{aligned} \frac{1}{u} S[u(x,y,t)] - \frac{u(x,y,0)}{u} &= 1 + S[v_x w_y], \\ \frac{1}{u} S[v(x,y,t)] - \frac{v(x,y,0)}{u} &= 5 + S[w_x u_y], \\ \frac{1}{u} S[w(x,y,t)] - \frac{w(x,y,0)}{u} &= 5 + S[u_x v_y]. \end{aligned} \tag{36}$$

Then

$$\begin{aligned} S[u(x,y,t)] &= x + 2y + u + uS[v_x w_y], \\ S[v(x,y,t)] &= x - 2y + 5u + uS[w_x u_y], \\ S[w(x,y,t)] &= -x + 2y + 5u + uS[u_x v_y]. \end{aligned} \tag{37}$$

By taking inverse Sumudu transform

$$\begin{aligned} u(x,y,t) &= x + 2y + t + S^{-1}\left[uS[v_x w_y]\right], \\ v(x,y,t) &= x - 2y + 5t + S^{-1}\left[uS[w_x u_y]\right], \\ w(x,y,t) &= -x + 2y + 5t + S^{-1}\left[uS[u_x v_y]\right]. \end{aligned} \tag{38}$$

The recursive relations are

$$
\begin{aligned}
&p^{0}; u_0(x,y,t) = t + x + 2y,\\
&p^{1}; u_1(x,y,t) = 2t,\\
&p^{2}; u_2(x,y,t) = 0,\\
&\vdots\\
&p^{i+1}; u_{i+1}(x,y,t) = S^{-1}\left[uS[\sum_{i=0}^{\infty} H_i(v,w)]\right] = 0, \quad i \geq 2,
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&p^{0}; v_0(x,y,t) = 5t + x - 2y,\\
&p^{1}; v_1(x,y,t) = -2t,\\
&p^{2}; v_2(x,y,t) = 0,\\
&\vdots\\
&p^{i+1}; v_{i+1}(x,y,t) = S^{-1}\left[uS[\sum_{i=0}^{\infty} C_i(u,w)]\right] = 0, \quad i \geq 2,
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&p^{0}; w_0(x,y,t) = 5t - x + 2y,\\
&p^{1}; w_1(x,y,t) = -2t,\\
&p^{2}; w_2(x,y,t) = 0,\\
&\vdots\\
&p^{i+1}; w_{i+1}(x,y,t) = S^{-1}\left[uS[\sum_{i=0}^{\infty} B_i(u,v)]\right] = 0, \quad i \geq 0,
\end{aligned} \tag{41}
$$

where $H_i(v,w)$, $C_i(u,w)$ and $B_i(u,v)$ are Adomian polynomials representing the nonlinear terms in above equations.

The solution of above system $p \to 1$ is given by

$$
\begin{aligned}
u(x,y,t) &= x + 2y + 3t,\\
v(x,y,t) &= x - 2y + 3t,\\
w(x,y,t) &= -x + 2y + 3t.
\end{aligned} \tag{42}
$$

Example 3.3 Consider the following homogeneous linear system of PDEs:

$$
\begin{aligned}
u_t(x,t) - v_x(x,t) - (u - v) = 2,\\
v_t(x,t) + u_x(x,t) - (u - v) = 2,
\end{aligned} \tag{43}
$$

With initial conditions

$$u(x,0)=1+e^{x},\qquad v(x,0)=-1+e^{x}. \tag{44}$$

Taking the Sumudu transform

$$\begin{aligned}\frac{1}{u}S[u(x,t)]-\frac{u(x,0)}{u}&=2+S[v_x+u-v],\\ \frac{1}{u}S[v(x,t)]-\frac{v(x,0)}{u}&=2+S[u-v-u_x].\end{aligned} \tag{45}$$

$$\begin{aligned}S[u(x,t)]&=1+e^{x}+2u+uS[v_x+u-v],\\ S[v(x,t)]&=-1+e^{x}+2u+uS[u-v-u_x].\end{aligned} \tag{46}$$

By taking inverse Sumudu transform

$$\begin{aligned}u(x,t)&=1+e^{x}+2t+S^{-1}\left[uS[v_x+u-v]\right],\\ v(x,t)&=-1+e^{x}+2t+S^{-1}\left[uS[u-v-u_x]\right].\end{aligned} \tag{47}$$

$$u_x(x,t)=\sum_{i=0}^{\infty}p^{i}u_{xi}(x,t),\ \ v_x(x,t)=\sum_{i=0}^{\infty}p^{i}v_{xi}(x,t). \tag{48}$$

Using the decomposition series $u(x,t)$, $v(x,t)$ and u_x, v_x for the linear terms, we obtain

$$\begin{aligned}\sum_{i=0}^{\infty}p^{i}u_i(x,t)&=1+e^{x}+2t+S^{-1}\left[uS[\sum_{i=0}^{\infty}p^{i}v_{ix}+\sum_{i=0}^{\infty}p^{i}u_i-\sum_{i=0}^{\infty}p^{i}v_i]\right],\\ \sum_{i=0}^{\infty}p^{i}v_i(x,t)&=-1+e^{x}+2t+S^{-1}\left[uS[\sum_{i=0}^{\infty}p^{i}u_i-\sum_{i=0}^{\infty}p^{i}u_{ix}-\sum_{i=0}^{\infty}p^{i}v_i]\right].\end{aligned} \tag{49}$$

Thus recursive relation is

$$\begin{aligned}&p^{0};u_0(x,t)=1+e^{x}+2t,\\ &p^{1};u_1(x,t)=te^{x}+2t,\\ &p^{2};u_2(x,t)=\frac{t^{2}}{2!}e^{x},\\ &\vdots\end{aligned} \tag{50}$$

$$\begin{aligned}&p^{0};v_0(x,t)=-1+e^{x}+2t,\\ &p^{1};v_1(x,t)=-te^{x}+2t,\\ &p^{2};v_2(x,t)=\frac{t^{2}}{2!}e^{x},\\ &\vdots\end{aligned} \tag{51}$$

the series solutions are given by

$$u(x,t)=1+e^{x}(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...),$$
$$v(x,t)=-1+e^{x}(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+...). \tag{52}$$

Thus the solutions

$$u(x,t)=1+e^{x+t},$$
$$v(x,t)=-1+e^{x-t}. \tag{53}$$

Example 3.*4* Consider the system of nonlinear partial differential equations

$$u_t+vu_x+u=1,$$
$$v_t-uv_x-v=1, \tag{54}$$

with initial conditions

$$u(x,0)=e^{x}, v(x,0)=e^{-x}. \tag{55}$$

By taking Sumudu transform on the Eq. (54)

$$\frac{1}{u}S[u(x,u)]-\frac{u(x,0)}{u}=1-S[vu_x+u],$$
$$\frac{1}{u}S[v(x,u)]-\frac{u(x,0)}{u}=1+S[uv_x+v], \tag{56}$$

Using initial conditions:

$$S[u(x,u)]=e^{x}+u-uS[vu_x]-uS[u],$$
$$S[v(x,u)]=e^{-x}+u+uS[uv_x]+uS[v], \tag{57}$$

By taking inverse Sumudu transform

$$u(x,t)=e^{x}+t-S^{-1}[uS[vu_x+u]],$$
$$v(x,t)=e^{-x}+t+S^{-1}[uS[uv_x+v]], \tag{58}$$

Then we can write $u(x,t)$ and $v(x,t)$ as an infinite series:

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = e^x + t - S^{-1}\left[uS\left(\sum_{i=0}^{\infty} H_i + \sum_{i=0}^{\infty} u_i \right)\right],$$
$$\sum_{i=0}^{\infty} p^i v_i(x,t) = e^{-x} + t + S^{-1}\left[uS\left(\sum_{i=0}^{\infty} B_i + \sum_{i=0}^{\infty} v_i \right)\right], \tag{59}$$

That represent the nonlinear terms vu_x and uv_x respectively.

By comparing the coefficients of p, we get

$$p^0; u_0 = e^x + t,$$
$$p^0; v_0 = e^{-x} + t, \tag{60}$$

$$p^1; u_1 = -S^{-1}[uS(H_0 + u_0)] = -t - \frac{t^2}{2!} - te^x - \frac{t^2}{2!}e^x,$$
$$p^1; v_1 = -S^{-1}[uS(B_0 + v_0)] = -t + \frac{t^2}{2!} + te^{-x} - \frac{t^2}{2!}e^{-x}, \tag{61}$$

$$p^2; u_2 = -S^{-1}[uS(H_2 + u_2)] = \frac{t^2}{2!} + \frac{t^2}{2!}e^x \ldots,$$
$$p^2; v_2 = -S^{-1}[uS(B_2 + v_2)] = -\frac{t^2}{2!} + \frac{t^2}{2!}e^{-x} \ldots, \tag{62}$$

$$\vdots$$

Thus

$$u(x,t) = e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \ldots\right),$$
$$v(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \ldots\right). \tag{63}$$

Then the solution for the above system is as follows:

$$u(x,t) = e^{x-t}\ , \quad v(x,t) = e^{t-x}. \tag{64}$$

4. Conclusion

The Sumudu Decomposition Method has been applied successfully to linear and nonlinear systems of partial differential equations. Four examples have been presented. The obtained results show that the proposed method is very useful and reliable for nonlinear partial differential equation and systems. Therefore, this method can be applied to many complicated nonlinear problems arising in mathematical physics and engineering.

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