Closed Ideal with Respect a Binary Operation * On BCK-Algebra

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Abstract
In this paper, we define a new ideal of BCK-algebra, we call it a closed ideal with respect a binary operation *,
and denoted by (* -closed ideal). We stated and proved some properties on closed ideal and give some examples on it.

Indexing Terms/Keywords: BCK-algebra, Closed Ideal, A Binary Operation * on BCK-Algebra.

1) Introduction
The notion of BCK- algebras was introduced and formulated first in 1966 by Y.Imai and K.Iseki [Y.Imai and K.Iseki, 1966]. In the same year, K.Iseki [K.Iseki , 1966] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras where the class of BCK-algebras is a proper subclass of the class of BCI-algebras. The notion of a BCI-algebra is a generalization of a BCK-algebra. The general development of BCK/ BCI-algebra the ideal theory plays an important role. We introduce a new ideal of BCK-algebra is called a closed ideal with respect a binary operation *, then we study and prove some properties of them.

2) Preliminary
In this section we review some concepts we needed in this paper

Definition 2.1 [Z.M.Samaei , M.A.Azadani and  L.N. Ranjbar, 2011]
Let X be a non-empty set with binary operation “*” and 0 is a constant an algebraic system (X, *, 0) is called a BCK-algebra if it satisfies the following conditions:
1) ((x * y) * (x * z)) * (z * y) = 0,
2) (x * (x * y)) * y = 0,
3) x * x = 0,
4) If x * y = 0 and y * x = 0 then x = y, \forall x, y, z \in X
5) 0 * x = 0.

Remarks 2.2 [A.A.A. Agboola1 and B. Davvaz2, 2015]
Let X be a BCK-algebra then:

a) A partial ordering” ≤” on X can be defined by x ≤ y if and only if
x * y = 0.
b) A BCK-algebra X has the following properties:
1) x * 0 = x.
2) If x * y = 0 implies (x * z) * (y * z) = 0 and (z * y) * (z * x) = 0.
3) (x * y) * z = (x * z) * y.
4) (x * y) * (x * z) ≤ (x * z).

Example 2.3
The set X = {0, 1, 2} with binary operation ”*” defined by the following table is a BCK-algebra.
Table 1. BCK-algebra

<table>
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<tr>
<th>*</th>
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Definition 2.4 [Sun Shin Ahn and Keumseong Bang, 2003]
Let $(X, *, 0)$ and $(X', *', 0')$ be two BCK-algebras. A mapping $f: X \rightarrow Y$ is called a homomorphism from $X$ to $X'$ if for any $x, y \in X$, $f(x * y) = f(x) *' f(y)$.

Note that if $f: X \rightarrow Y$ is a homomorphism of BCK-algebras, then $f(0) = 0$.

Definition 2.5:
A mapping $f: (X, *, 0) \rightarrow (Y, *', 0)$ of BCK-algebras is called an epimorphism if $f$ is a homomorphism and surjective.

Definition 2.6 [Young Bae Jun, and Kyoung Ja Lee, 2012]
A BCK-algebra is said to be commutative if $x * (x * y) = y * (y * x)$ for any $x, y \in X$.

Example 2.7
The set $X = \{0, 1, 2\}$ with binary operation " * " defined by the following table is commutative BCK-algebra.

Table 2. commutative BCK-algebra

<table>
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Definition 2.8 [Young Bae Jun, and Kyoung Ja Lee, 2012]
A nonempty subset $S$ of a BCK-algebra $X$ is called a BCK sub algebra of $X$ if $x * y \in S$ for all $x, y \in S$.

Definition 2.9 [Young Bae Jun, and Kyoung Ja Lee, 2012]
A nonempty subset $A$ of a BCK-algebra $X$ is called a BCK ideal of $X$ if it satisfies:
1) $0 \in A$
2) $x * y \in A$, $y \in A$ then $x \in A$ and $x, y \in X$

Proposition 2.10 [Sajda Kadhum Mohammed & Azal Taha Abdul Wahab, 2015]
Let $I$ and $J$ are BCK-algebra of $X$, then $I \times J$ is BCK-algebra of $X \times X$.

Proposition 2.11 [Sajda Kadhum Mohammed & Azal Taha Abdul Wahab, 2015]
Let $A$ and $B$ are BCK-algebra of $X$, then $A \cap B$ is BCK-algebra of $X$. 
Proposition 2.12 [Sajda Kadhum Mohammed & Azal Taha Abdul Wahab, 2015] Let A and B are BCK-algebra of X, then A∪B is BCK-algebra of X if A ⊆ B or B ⊆ A.

3) Main Results:
In this section, we define a closed ideal with respect a binary operation * of BCK-algebra. We stated and proved some properties on closed ideal and give some examples on it.

Definition 3.1
Let X is a BCK-algebra. A non empty subset I of X is said closed ideal with respect a binary operation * and denoted by (* -closed ideal) on X if satisfies the following conditions :
1) a * b ∈ I ∀ a, b ∈ I
2) I * X ⊆ I

Example 3.2:
Let X = \{0, 1, 2\} with binary operations '∗' defined by the following tables is BCK-algebra:

\[
\begin{array}{ccc}
\ast & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
\end{array}
\]

Then by usual calculation we can prove that I = \{0, 1\} ⊆ X is (* -closed ideal)

Example 3.3:
Let X = \{0, 1, 2, 3\} with binary operations '∗' defined by the following tables is BCK-algebra:

\[
\begin{array}{cccc}
\ast & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
1 & 0 & 3 & 2 \\
2 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 \\
\end{array}
\]

Then I = \{0, 1, 2\} ⊆ X is not (∗ -closed ideal) since 1 ∈ I and 2 ∈ I but 1 ∗ 2 = 3 \notin I

Remark 3.4
If I is (∗ -closed ideal) of BCK-algebra, then, 0 ∈ I

Proof
Let I be \((\ast\)-closed ideal) so \(I \neq \emptyset\). Then \(\exists a \in I\),
then \(a \ast x \in I \ \forall \ x \in X\) [by 2 of definition 3.1]
So, \(0 = a \ast a \in I\), and therefore \(0 \in I\).

**Remark 3.5**
If I is \((\ast\)-closed ideal) of BCK-algebra, then I is sub algebra.

**Proof**
Let I is \((\ast\)-closed ideal) of BCK-algebra and let \(a, b \in I\)
\(\Rightarrow a \ast b \in I \Rightarrow I\) is sub algebra.

**Remark 3.6**
The converse of above remark in general is not true.

**Proof**
We will prove it by using the example (3.3):
Take \(I = \{0, 1\} \subseteq X\) it is clear that is a sub algebra but I is not \((\ast\)-closed ideal)
since \(I \ast x \not\subset I\) where \(1 \in I\) and \(3 \in X\) but \(1 \ast 3 = 2 \notin I\).

**Proposition 3.7**
Let \(X\) is BCK-algebra and let \(A, B\) \((\ast\)-closed ideal) of \(X\) Then \(A \cap B\) is \((\ast\)-closed ideal) of \(X\)

**Proof**
Let \(X\) is BCK-algebra and since \(A \cap B \neq \emptyset\) by (3.4)
1) Let \(a, b \in A \cap B \Rightarrow a, b \in A\) and \(a, b \in B\)
Since \(A, B\) are \((\ast\)-closed ideal) then \(a \ast b \in A\) and \(a \ast b \in B\)
\(\Rightarrow a \ast b \in A \cap B\)
2) Let \(a \in A \cap B\) and \(x \in X \Rightarrow a \in A\) and \(a \in B\) and \(x \in X\)
\(\Rightarrow a \ast x \in A\) and \(a \ast x \in B\); [since \(A\) and \(B\) \((\ast\)-closed ideal)]
\(\Rightarrow a \ast x \in A \cap B \Rightarrow (A \cap B) \ast x \subseteq (A \cap B)\),
then \(A \cap B\) is \((\ast\)-closed ideal).

**Remark 3.8**
The converse of above remark is not true in general.
Take \(A = \{0, 1\}\) and \(B = \{0, 1, 2\}\) in (example 3.3) then:
\(A \cap B = \{0, 1\}\) is \((\ast\)-closed ideal) but \(B = \{0, 1, 2\}\) is not \((\ast\)-closed ideal); since \(1 \ast 2 = 3 \notin B\)

**Remark 3.9**
Let \(X\) is BCK-algebra and let \(A, B\) \((\ast\)-closed ideal) of \(X\). Then \(A \cup B\) is \((\ast\)-closed ideal) of \(X\) if \(A \subseteq B\) or \(B \subseteq A\), and the converse is not true in general.

**Proof**
Proof is clear now, we show that the converse is not true in general; since if we take \(A, B\) and \(A \cup B\) are \((\ast\)-closed ideal) of \(X\)

**Table 5.** the converse is not true in general.

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</table>
A = {0, 1} is (∗ -closed ideal)
B = {0, 2} is (∗ -closed ideal), A ⊔ B = {0, 1, 2} is (∗ -closed ideal),
but A ∉ B and B ∉ A

**Proposition 3.10**
Let f: X → Y is BCK-algebra homomorphism. Then ker f is (∗ -closed ideal) of X.

**Proof**

Let f: X → Y is BCK-algebra homomorphism. Then

1) a, b ∈ ker f ⇒ f(a) = 0 and f(b) = 0
   ⇒ f(a * b) = f(a) * f(b) = 0 * 0 = 0 ⇒ f(a * b) = 0 ⇒ a * b ∈ ker f
2) Let a ∈ ker f and x ∈ X ⇒ f(a) = 0
   ⇒ f(a * x) = f(a) * f(x); [since f is a homomorphism ]
   = 0 * f(x) = 0; [by 5 of definition 2.1]
   ⇒ f(a * x) = 0 ⇒ a * x ∈ ker f ∀ a ∈ ker f and x ∈ X
   ⇒ ker f * X ⊆ ker f

Then ker f is (∗ -closed ideal)

**Proposition 3.11**
Let f: X → Y is BCK-algebra epimorphism if A is (∗ -closed ideal) of X, then f(A) is (∗ -closed ideal) of Y.

**Proof**

Let f: X → Y is BCK-algebra epimorphism. Let A be (∗ -closed ideal) of X then:

1) Let x', y' ∈ f(A), then ∃ x, y ∈ A such that x'=f(x), y'=f(y),
   since A is (∗ -closed ideal) ⇒ x * y ∈ A ⇒ f(x * y) ∈ f(A)
   but f(x * y) = f(x) * f(y) ⇒ f(x) * f(y) ∈ f(A) so x * y' ∈ f(A)
2) Let a' ∈ f(A) and y ∈ Y since f is an epimorphism
   ⇒ ∃ a ∈ A and x ∈ X such that f(a) = a' and f(x) = y
   ⇒ a * x ∈ A; [since A is (∗ -closed ideal)]
   ⇒ f(a * x) ∈ f(A) ⇒ f(a) * f(x) ∈ f(A); [since f is a homomorphism]
   ⇒ a' * y ∈ f(A) ∀ a' ∈ f(A) and y ∈ Y
   ⇒ f(A) * Y ⊆ f(A)

Then, f(A) is (∗ -closed ideal).

**Proposition 3.12**
Let X is BCK-algebra and let f: X → X' is BCK-algebra homomorphism of X if B is (∗ -closed ideal) of X',
then f -1(B) = {a ∈ X: f(a) ∈ B} is (∗ -closed ideal) of X.

**Proof**

Let f: X → X' is BCK-algebra homomorphism of X if B is (∗ -closed ideal) of X', then:

1) Let a, b ∈ f -1(B) ⇒ f(a), f(b) ∈ B
   Since B is (∗ -closed ideal) then:
   f(a) * f(b) = f(a * b) ∈ B; [since B is (∗ -closed ideal)]
   ⇒ a * b ∈ f -1(B)
2) Let a ∈ f -1(B) and x ∈ X so f(x) ∈ X' ⇒ f(a) ∈ B and f(x) ∈ X'
   ⇒ f(a) * f(x) = f(a * x) ∈ B, [since B is (∗ -closed ideal)]
   ⇒ a * x ∈ f -1(B) ∀ a ∈ f -1(B) and x ∈ X
   ⇒ f -1(B) * X ⊆ f -1(B) ⇒ f -1(B) is (∗ -closed ideal).

**Proposition 3.13**
Let X is BCK-algebra and let I, J be (∗ -closed ideal) of X. Then I × J is (∗ -closed ideal) of X × X.

**Proof**

Let X is BCK-algebra, and let I, J be (∗ -closed ideal) of X

1) Let x = (a, a') ∈ I × J and y = (b, b') ∈ I × J
   ⇒ x * y = (a, a') * (b, b') = (a * b, a' * b')
then \( a \ast b \in I \) and \( a' \ast b' \in J; \)  
\[ (a \ast b, a' \ast b') \in I \times J \]  
so \( x \ast y \in I \times J \)

2) Let \((x_1, x_2) \in X \times X\) and \((a_1, a_2) \in I \times J\)
\[ a_1 \ast x_1 \in I, a_2 \ast x_2 \in J \]
because \( I \) and \( J \) are (* -closed ideal)

Then \((a_1, a_2) \ast (x_1, x_2) = (a_1 \ast x_1, a_2 \ast x_2) \in I \times J\)
Then \( I \times J \) is (* -closed ideal)

**Proposition 3.14**

Let \( X \) is BCK-algebra and let \( \Gamma' = \{(a , 0) / a \in X \} \) and \( J' = \{(0 , b) / b \in X \}. \)
Then \( \Gamma' \) and \( J' \) are (* -closed ideal) of \( X \times X. \)

**Proof**

Let \( X \) is BCK-algebra to prove that \( \Gamma' \) is (* -closed ideal).

1) Let \( x, y \in \Gamma' \Rightarrow x = (a, 0), y = (b, 0) \)
\[ x \ast y = (a , 0) \ast (b , 0) = (a \ast b , 0) \in \Gamma; \]  
[since \( a \ast b \in X \)]
\[ x \ast y \in \Gamma' \]

2) Let \( x = (a, 0) \in \Gamma' \) and \( t = (r, s) \in X \times X \)
\[ x \ast t = (a , 0) \ast (r , s) = (a \ast r , 0 \ast s) = (a \ast r , 0); \]  
[by 5 of definition 2.1]
\[ x \ast t = (a \ast r , 0) \in \Gamma; \]
\[ \Gamma' \times X \times X \subseteq \Gamma' \] then \( \Gamma' \) is (* -closed ideal) of \( X \times X. \)

In a similar way, we can prove that \( J' \) is (* -closed ideal) of \( X \times X. \)

**Remark 3.15**

Let \( X \) is BCK-algebra and let \( \Gamma' \) and \( J' \) be defined as in the above proposition.
Then \( \Gamma' \cap J' = (0, 0). \)

**Proof**

Let \( X \) is BCK-algebra and let \( \Gamma' \) and \( J' \) is (* -closed ideal) and

let \( x \in \Gamma' \cap J' \Rightarrow x \in \Gamma' \) and \( x \in J' \) then \( x = (a, 0) \) and
\[ x = (0, b) \] where \( a \in X \) and \( b \in X \Rightarrow (a, 0) = (0, b) \Rightarrow a = 0, b = 0 \)

\[ \Rightarrow x = (0, 0) \Rightarrow \Gamma' \cap J' = (0, 0). \]

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