On M-Compact Ring

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Abstract: In the present paper, we have introduced some new definitions On M-cover ring, M-compact ring, weakly M-compact ring, weakly M-compact c. ring, M-compact locally ring and M-compact strong locally ring, we obtain some examples and results related to M-cover ring, M-compact ring, weakly M-compact ring, weakly M-compact c. ring, M-compact c. ring, M-compact locally ring and M-compact strong locally ring.

Keywords: rings, M-cover ring, M-compact ring, ring homomorphism, ring isomorphism.

AMS 2017: 54A25, 45B05.

1- Introduction:

The notion of groupoid was introduced by H. Brandt [Math. Ann., 96(1926), 360-366; MR 1512323]. A groupoid \((G,\ast)\) is a set on which is defined a non associative binary operation which is closed on \(G\), the groupoid \((G,\ast)\) is a semigroup if the binary operation \(\ast\) is associative [4].

Milne [2], introduced details on a ring. We call \((R,\ast,\circ)\) to be a ring if the following conditions are satisfied.

1) \((R, \ast)\) is a group.
2) \((R, \circ)\) is a semigroup.
3) (i) \(a \times (b + c) = a \times b + a \times c\)
   (ii) \((a + b) \times c = a \times c + b \times c\)

for all \(a, b, c \in R\), a nonempty subset \(P\) of \(R\) is said to be a subring of \(R\) if \(P\) is a ring under the operations of \(R\).

We investigated on M-cover ring, M-compact ring, weakly M-compact ring, weakly M-compact c. ring, M-compact c. ring, M-compact locally ring and M-compact strong locally ring, we obtain some good examples and results related to these concepts above.

2- Definitions:

Definition (1): Let \((R,\ast,\circ)\) be a ring , and I be an indexed set.
Let \( M = \{ R_i ; R_i \subset R, (R_i, *, o) \text{ is a proper subring of } (R, *, o), \forall i \in I \} \) be a family of proper subrings of \((R, *, o), (I \text{ is a finite or an infinite set}), \) we say that \( M \) is a \textit{M-cover ring} of \((R, *, o)\) if \( R = \bigcup_{i \in I} R_i \).

**Definition (2):** Let \((R, *, o)\) be a ring, we say that \((R, *, o)\) is a \textit{weakly M-compact ring} if there is a finite \textit{sub-M-cover ring} of \((R, *, o)\).

**Definition (3):** Let \((R, *, o)\) be a ring, we say that \((R, *, o)\) is \textit{M-compact ring} if for every \textit{M-cover ring} of \((R, *, o)\) there exists a finite \textit{sub-M-cover ring} of \((R, *, o)\).

**Definition (4):** Let \((R, *, o)\) be a ring, we say that \((R, *, o)\) is \textit{weakly M-compact} \textit{c. ring} if there is a countable \textit{M-cover ring} of \((R, *, o)\).

**Definition (5):** Let \((R, *, o)\) be a ring, we say that \((R, *, o)\) is \textit{M-compact} \textit{c. ring} if for every \textit{M-cover ring} of \((R, *, o)\) there exists a countable \textit{sub-M-cover ring} of \((R, *, o)\).

**Definition (6):** Let \((R, *, o)\) be a ring, we say that \((R, *, o)\) is \textit{M-compact} \textit{locally ring} if for every element \(x\) of \(R\) there is a subring (proper) of \(R\) include \(x\).

**Definition (7):** Let \((R, *, o)\) be a ring, we say that \((R, *, o)\) is a \textit{M-compact strong} \textit{locally ring} if for every element \(x\) of \(R\) (except the unite element) there is a unique subring (proper) of \(R\) include \(x\).

**Definition (8):** Let \((R, *, o)\) be a ring, the subring \((H, *, o)\) of the ring \((R, *, o)\) is called a \textit{M-compact subring} (\textit{weakly M-compact} \textit{subring, M-compact c. subring, M-compact strongly} \textit{locally subring}), if \((H, *, o)\) is a \textit{M-compact} \textit{ring} (\textit{weakly M-compact} \textit{c. ring, M-compact c. ring, M-compact} \textit{locally ring, M-compact strongly} \textit{locally ring}), respectively.

**Definition (9) [3]:** Let \((R, *, o)\) and \((\bar{R}, \bar{*, o})\) are two rings, we say that

1- \(f : (R, *, o) \rightarrow (\bar{R}, \bar{*, o})\) is a \textit{homomorphism} if \(f(x * y) = f(x) \bar{*} f(y)\) and \(f(x \circ y) = f(x) \bar{\circ} f(y), \forall x, y \in G.\)

2- \(f : (R, *, o) \rightarrow (\bar{R}, \bar{*, o})\) is an \textit{isomorphism} if \(f\) is a \textit{bijective homomorphism}.

**Definition (10) [3]:** Let \((R, *, o)\) and \((\bar{R}, \bar{*, o})\) are two rings, we say that \((R, *, o)\) is an isomorphic to \((\bar{R}, \bar{*, o})\), denoted that \((R, *, o) \cong (\bar{R}, \bar{*, o}), \) if there is an \textit{isomorphism} \(f : (R, *, o) \rightarrow (\bar{R}, \bar{*, o})\).

### 3- Examples:

**Example (1):** The ring \((\mathbb{Z}_2, +, \cdot)\), has no \textit{M-cover ring}. The ring \((\mathbb{Z}_2, +, \cdot, 2)\) is not \textit{M-cover ring}, while the ring \((\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus, \otimes)\) has \textit{M-cover ring} \{\{(0,0),(1,1)\},\{(0,0),(1,0)\},\{(0,0),(0,1)\}\}. Such that \((a, b)\oplus(c, d) = (a+c, b+d), (a, b)\otimes(c, d) = (a \cdot c, b \cdot d).\)

**Example (2):** Let \(R = \{0, 1, 2, \ldots \}\), defined a binary operator \(\prec\) as follows;

\[
a \prec b = \begin{cases} \max\{a, b\} & a \neq b \\ 0 & a = b, \ \forall a, b \in R.\end{cases}
\]

It is easy to show that \((R, \prec)\) is a group. The ring \((R, <, \ast)\) is a \textit{M-compact C. ring} (* define by \(a \ast b = 0\) for all \(a, b \in R).\) The ring \((R, <, \ast)\) is not \textit{M-cover ring}, since the family of subrings \(\{(0, a, b), <, \ast) ; a, b \in \mathbb{N}\}\) is a \textit{M-cover ring} of \((R, <, \ast)\) has no finite \textit{sub-M-cover ring} of \((R, <, \ast)\).
**Example (3):** Let \((\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus, \otimes)\) be a ring. Then \((\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus, \otimes)\) is a \(M\)-compact strongly locally ring(also \(M\)-compact locally ring), since there are only three subrings of \((\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus, \otimes)\) which are \((M_1, \oplus, \otimes), (M_2, \oplus, \otimes)\) and \((M_3, \oplus, \otimes)\), (except the trivial subrings) where
\[
M_1 = \{(0,0), (0,1)\}, \quad M_2 = \{(0,0), (1,0)\}, \quad M_3 = \{(0,0), (1,1)\}.
\]

**Example (4):** Let \(R = \{-n, \ldots, -2, -1, 0\}\), defined a binary operator \(\succ\) as follows;
\[
a \succ b = \begin{cases} 
\min \{a, b\} & a \neq b, \forall a, b \in R. \text{ It is easy to show that } (R, \succ) \text{ is a group.} \\
0 & a = b
\end{cases}
\]
The ring \((R, \succ, *)\) (* define by \(a * b = 0 \text{ for all } a, b \in R\)) is \(M\)-compact ring.

**Example (5):** Let \(X = \{0\} \cup \mathbb{R}^+\), defined a binary operator \(\prec\) as follows;
\[
a \prec b = \begin{cases} 
\max \{a, b\} & a \neq b, \forall a, b \in X. \text{ It is easy to show that } (X, \prec) \text{ is a group.} \\
0 & a = b
\end{cases}
\]
The ring \((X, \prec, *)\) (* define by \(a * b = 0 \text{ for all } a, b \in X\)) is \(M\)-compact locally ring.
The ring \((X, \prec, *)\) is not \(M\)-compact c. ring, since the family of subrings \(\{(\{0, a, b\}, \prec, *)\}; a, b \in \mathbb{R}^+\) is a \(M\)-cover ring of \((X, \prec, *)\) has no countable \(sub-M\)-cover ring of \((X, \prec, *)\).

4- **Main Results:**

The prove of all the following lemmas are direct from definitions.

**Lemma (1):** If \((R, *, \circ)\) is a \(M\)-compact ring, then \((R, *, \circ)\) is a \(M\)-compact c. ring.

**Lemma (2):** If \((R, *, \circ)\) is a \(M\)-compact ring, then \((R, *, \circ)\) is a weakly \(M\)-compact ring.

**Lemma (3):** If \((R, *, \circ)\) is a \(M\)-compact c. ring, then \((R, *, \circ)\) is a weakly \(M\)-compact c. ring.

**Lemma (4):** If \((R, *, \circ)\) is a Weakly \(M\)-compact ring, then \((R, *, \circ)\) is a weakly \(M\)-compact c. ring.

**Lemma (5):** If \((R, *, \circ)\) is a \(M\)-compact strong locally ring, then \((R, *, \circ)\) is a \(M\)-compact locally ring.

The following theorems are direct from definitions,

**Theorem (1):** Let \((R, *, \circ)\) be ring, if \(R\) weakly \(M\)-compact ring, then it is \(M\)-compact locally ring.

**Proof:** Let \(x \in R\) and \(R\) weakly \(M\)-compact ring, then there is cover ring of \(R \Rightarrow R = \bigcup_{i \in I} R_i\) \((R_i \text{ is proper sub ring of } R \forall i \in I)\)
\[
x \in \bigcup_{i \in I} R_i \Rightarrow x \in R_i \text{ for some } i. \text{ R is } M\text{-compact locally ring.}
\]

**Corollary (1):** Let \((R, *, \circ)\) be a ring. Then
1) \(M\)-compact ring \(\Rightarrow\) \(M\)-compact locally ring.
2) \(M\)-compact c. ring \(\Rightarrow\) \(M\)-compact locally ring.
3) weakly \(M\)-compact c. ring \(\Rightarrow\) \(M\)-compact locally ring.

**Theorem (2):** Any finite non cyclic ring of order a nonprime number, is a \(M\)-compact locally ring.
Proof: Let \((R, *, \circ)\) is a finite ring, for every element \(x\) of \(G\) the subring \((\langle x\rangle, *, \circ)\) of \((R, *, \circ)\) include \(x\), it is clear that \(R \neq \langle x\rangle\), since \((R, *, \circ)\) is not cyclic ring, and therefore \((R, *, \circ)\) is a \(M\)-compact locally ring.

Theorem (3): If \((R, *, \circ)\) is a finite ring, then the following are equivalent:
1) \((R, *, \circ)\) is a \(M\)-compact ring.
2) \((R, *, \circ)\) is a \(M\)-compact c. ring.
3) \((R, *, \circ)\) is a weakly \(M\)-compact ring.
4) \((R, *, \circ)\) is a weakly \(M\)-compact c. ring.

Theorem (4): Any finite ring has a prime order is not \(M\)-compact locally ring.

Proof: Let \((R, *, \circ)\) is a ring with \(|R| = p\), \(p\) prime number, by "Lagrange theorem" the order of every subgroup of \(R\) divides \(p\), but \(p\) is prime, so there is no proper subgroup of \(R\) except the unit element and hence is no proper subring of \(R\). Then \((R, *, \circ)\) is not \(M\)-compact locally ring.

Corollary (2):
1) Any finite ring has prime order is not \(M\)-compact ring.
2) Any finite ring has prime order is not \(M\)-compact c. ring.
3) Any finite ring has prime order is not weakly \(M\)-compact ring.
4) Any finite ring has prime order is not weakly \(M\)-compact c. ring.

Corollary (3): Any finite ring has prime order \((R, *, \circ)\) is not \(M\)-compact strong locally ring.

Theorem (5): If \((R, *, \circ)\) ring, then \((R, *, \circ)\) is a \(M\)-compact ring is not simple ring.

Proof: If \((R, *, \circ)\) is a \(M\)-compact ring, then there is \(M\)-cover ring and hence there is proper subring of \((R, *, \circ)\) \(\Rightarrow (R, *, \circ)\) is not simple ring.

Theorem (6): Any simple ring \((R, *, \circ)\) is not weakly \(M\)-compact ring.
Proof: Clear, any simple ring has no proper sub ring and has no \(M\)-cover. i.e \((R, *, \circ)\) is not weakly \(M\)-compact ring.

Corollary (4): Any \((R, *, \circ)\) simple ring is not \(M\)-compact ring.

Theorem (7): Any \((R, *, \circ)\) simple ring is not \(M\)-compact c. ring.
Proof: Clear, any simple ring has no proper sub ring and has no \(M\)-cover. i.e \((R, *, \circ)\) is not \(M\)-compact ring.

Theorem (8): Any cyclic ring is not \(M\)-compact locally ring.

Proof: Assume \((R, *, \circ)\) is a cyclic ring which is a \(M\)-compact locally ring so for every element \(x\) of \(R\) there is a subring of \(R\) include \(x\), but \(R\) is a cyclic so there is an element say \(g\) such that \(\langle g\rangle = R\) \((R\) generated by \(g\)) and hence any subring contains \(g\) must be equal to \(R\), that is there is no proper subring contains \(g\).

Corollary (5):
1) Any cyclic ring is not \(M\)-compact ring.
2) Any cyclic ring is not \(M\)-compact c. ring.
3) Any cyclic ring is not weakly M-compact ring.
4) Any cyclic ring is not weakly M-compact c. ring.
5) Any cyclic ring is not M-compact strong locally ring.

**Theorem (9):** If \((G,*,^\circ) \cong (\bar{G},\bar{^*},\bar{^\circ})\). Then

\[(G,*,^\circ)\] is a M-compact ring \(\iff (\bar{G},\bar{^*},\bar{^\circ})\) is a M-compact ring.

**Proof:** \((\Rightarrow)\) Let \((\bar{G},\bar{^*},\bar{^\circ})\) be any M-cover ring of \((\bar{G},\bar{^*},\bar{^\circ})\) \(\Rightarrow \bar{G} = \bigcup_{i \in I} \bar{G}_i\), but \(f\) is an isomorphism \(\Rightarrow f(\bar{G}) = \bar{G} = \bigcup_{i \in I} \bar{G}_i\), and \(f^{-1}(\bar{G}_i)\) is a ring \(\forall i \in I\), but \((G,*,^\circ)\) is a M-compact ring so there is a finite set \(J\) such that

\[G = \bigcup_{j \in J} f^{-1}(\bar{G}_j) = f^{-1}(\bigcup_{j \in J} \bar{G}_j) \Rightarrow \bar{G} = f(G) = f\left(f^{-1}(\bigcup_{j \in J} \bar{G}_j)\right) = \bigcup_{j \in J} \bar{G}_j\] and \((\bar{G}_j,\bar{^*}_j,\bar{^\circ}_j)\) is a ring \(\forall j \in J\)

\(\Rightarrow (\bar{G},\bar{^*},\bar{^\circ})\) is a M-compact ring.

\((\Leftarrow)\) Let \((G_i,*,^\circ)\) be any M-cover ring of \((G,*,^\circ)\) \(\Rightarrow G = \bigcup_{i \in I} G_i\), but \(f\) is an isomorphism \(\Rightarrow \bar{G} = f(G) = f(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} f(G_i)\) and is a ring \(\forall i \in I\), but \((\bar{G},\bar{^*},\bar{^\circ})\) is a M-compact ring so there is a finite set \(J\) such that \(\bar{G} = \bigcup_{j \in J} f(G_j) = f\left(\bigcup_{j \in J} G_j\right)\)

\(\Rightarrow G = \bigcup_{j \in J} f^{-1}(\bar{G}_j) = f^{-1}\left(f\left(\bigcup_{j \in J} G_j\right)\right) = \bigcup_{j \in J} G_j\)

and \((G_j,*,^\circ_j)\) is a ring \(\forall j \in J\) \(\Rightarrow (G,*,^\circ)\) is a M-compact ring.

**Theorem (10):** If \(f : (G,*,^\circ) \rightarrow (\bar{G},\bar{^*},\bar{^\circ})\) is an isomorphism and \((H,*,^\circ)\) is a M-compact subring of \((G,*,^\circ)\), then \(f(H)\) is a M-compact subring of \((\bar{G},\bar{^*},\bar{^\circ})\).

**Proof:** Let \(\{W_i : i \in I\}\) is M-cover of \(f(H)\) \(\Rightarrow \bigcup_{i \in I} W_i = f(H)\) Since \(f\) is isomorphism, \(f^{-1}(f(H)) = f^{-1}(\bigcup_{i \in I} W_i) = f(H)\)

\(H = \bigcup_{i \in I} f^{-1}(W_i)\) is sub rings of \((G,*,^\circ)\) for all \(i \in I\)

\(\Rightarrow \exists finite set J \subseteq I \exists H = \bigcup_{i \in J} f^{-1}(W_i)\) [ since \((H,*,^\circ)\) is a M-compact subring of \((G,*,^\circ)\)]

\(\Rightarrow f(H) = f(\bigcup_{i \in J} f^{-1}(W_i)) = \bigcup_{i \in J} f\left(f^{-1}(W_i)\right) = \bigcup_{i \in J} W_i\)

\(\Rightarrow f(H)\) is a M-compact subring of \((\bar{G},\bar{^*},\bar{^\circ})\).

**Theorem (11):** If \(f : (G,*,^\circ) \rightarrow (\bar{G},\bar{^*},\bar{^\circ})\) is an isomorphism and \((S,*,^\circ)\) is a M-compact subring of \((\bar{G},\bar{^*},\bar{^\circ})\), then \(f^{-1}(S)\) is a M-compact subring of \((G,*,^\circ)\).

**Proof:** Let \(\{W_i : i \in I\}\) is M-cover of \(f^{-1}(S)\) \(\Rightarrow \bigcup_{i \in I} W_i = f^{-1}(S)\).
Since $f$ is isomorphism, $f(f^{-1}(S)) = f(U_{i \in I} W_i) = S \Rightarrow S = U_{i \in I} f(W_i)$ [ $f(W_i)$ is sub rings of $(\tilde{G}, \ast, \tilde{\circ})$ for all $i \in I$]

$\Rightarrow \exists$ finite set $J \subseteq I \exists S = U_{i \in J} f(W_i)$ [ since $(S, \ast, \circ)$ is a $M$-compact subring of $(\tilde{G}, \ast, \tilde{\circ})$]

$\Rightarrow f^{-1}(S) = f^{-1}(U_{i \in J} f(W_i)) = U_{i \in J} f^{-1}(f(W_i)) = U_{i \in J} W_i$

$\Rightarrow S$ is a $M$-compact subring of $(G, \ast, \circ)$.

**Theorem (12):** If $(A, \ast, \circ)$ is a ring and $(G, \ast, \tilde{\circ})$ is a $M$-compact ring, then $(A \times G, \oplus, \otimes)$ is a $M$-compact ring. Where

$$(a_1, g_1) \oplus (a_2, g_2) = (a_1 \ast a_2, g_1 \ast g_2), \ (a_1, g_1) \otimes (a_2, g_2) = (a_1 \circ a_2, g_1 \tilde{\circ} g_2),$$

$\forall (a_1, g_1), (a_2, g_2) \in A \times G$.

**Proof:** Let $(A \times G_i, \oplus, \otimes) ; G_i \subseteq G, (A \times G_i, \oplus, \otimes)$ is a subring of $(A \times G, \oplus, \otimes), \forall i \in I$ be any $M$-cover ring of $(A \times G, \oplus, \otimes)$, it is clear that $(G_i, \ast, \circ)$ is a subrings of $(G, \ast, \circ)$, such that $A \times G = U_{i \in I} A \times G_i = A \times (U_{i \in I} G_i) \Rightarrow G = U_{i \in I} G_i$, but $(G, \ast, \circ)$ is a $M$-compact ring, so there is a finite set $J$ such that $G = U_{j \in J} G_j$, and hence

$$A \times G = A \times (U_{j \in J} G_j) = U_{j \in J} A \times G_j \Rightarrow (A \times G, \oplus, \otimes)$$

is a $M$-compact ring.

**Theorem (13):** If $(G, \ast, \circ)$ and $(\tilde{G}, \tilde{\ast}, \tilde{\circ})$ are $M$-compact strong locally rings, then

$$(G \times \tilde{G}, \oplus, \otimes)$$

is a $M$-compact strong locally ring.

**Proof:** Let $(x, y) \in G \times \tilde{G} \Rightarrow x \in G$ and $y \in \tilde{G}$, but $(G, \ast, \circ)$ and $(\tilde{G}, \tilde{\ast}, \tilde{\circ})$ are $M$-compact strong locally rings $\Rightarrow \exists ! G_x$ and $\exists ! \tilde{G}_y$ subrings of $G$ and $\tilde{G}$, respectively, such that $x \in G_x$ and $y \in \tilde{G}_y$ $\Rightarrow (x, y) \in G_x \times \tilde{G}_y$ and $G_x \times \tilde{G}_y$ is a unique subrings of $G \times \tilde{G} \Rightarrow (G \times \tilde{G}, \oplus, \otimes)$ is a $M$-compact strong locally ring.

**Corollary (6):** If $(G, \ast, \circ)$ and $(\tilde{G}, \tilde{\ast}, \tilde{\circ})$ are $M$-compact locally rings, then $(G \times \tilde{G}, \oplus, \otimes)$ is a $M$-compact locally ring.

**Theorem (14):** If $(G, \ast, \circ)$ and $(\tilde{G}, \tilde{\ast}, \tilde{\circ})$ are $M$-compact rings, then $(G \times \tilde{G}, \oplus, \otimes)$ is a $M$-compact ring.

**Proof:** Let $(G, \ast, \circ)$ and $(\tilde{G}, \tilde{\ast}, \tilde{\circ})$ are $M$-compact rings $\Rightarrow$ there exists a $M$-cover ring of $(G, \ast, \circ)$ say $\{G_a\}_{a \in A}$ and a $M$-cover ring of $(\tilde{G}, \tilde{\ast}, \tilde{\circ})$ say $\{\tilde{G}_b\}_{b \in B}$ $\Rightarrow G \times \tilde{G} = (U_{a \in A} G_a) \times (U_{b \in B} \tilde{G}_b) = U_{a \in A, b \in B} (G_a \times \tilde{G}_b) \Rightarrow \{G_a \times \tilde{G}_b\}_{a \in A, b \in B}$ is a $M$-cover ring of $(G \times \tilde{G}, \oplus, \otimes)$.

Now, Let $\{W_i\}_{i \in I}$ be any $M$-cover ring of $(G \times \tilde{G}, \oplus, \otimes)$

$\Rightarrow G \times \tilde{G} = U_{i \in I} W_i$, such that $W_i = U_i \times \mathcal{V}_i$, where $\{U_i\}_{i \in I}$ are subrings of $(G, \ast, \circ)$ and $\{\mathcal{V}_i\}_{i \in I}$ are subrings of $(\tilde{G}, \tilde{\ast}, \tilde{\circ})$. But $(G, \ast, \circ)$ is a $M$-compact ring, so there is a $M$-cover ring of
(G,*) contains \{U_i\}_{i \in I} which have a finite sub-M-cover ring (i.e. there is a finite set J ) such that
\[ G = \bigcup_{j \in J} U_j, \quad \text{let } U_{j_1} \in \{U_j\}_{j \in J} \implies \{U_{j_1} \times \mathcal{V}_i\}_{i \in I} \text{ is a M-cover ring of the M-compact ring} \]
\[ \left( U_{j_1} \times \bar{G}, \oplus, \otimes \right) \] (from Theorem 10 since \( \left( U_{j_1},*,\circ \right) \) is a ring and \( (\bar{G},\bar{\oplus},\bar{\otimes}) \) is a M-cover ring), so there is a finite set S such that
\[ U_{j_1} \times \bar{G} = \bigcup_{s \in S} (U_{j_1} \times \mathcal{V}_s) = U_{j_1} \times (\bigcup_{s \in S} \mathcal{V}_s) \]
\[ \implies U_{j \in J}(U_j \times \bigcup_{s \in S} \mathcal{V}_s) = (\bigcup_{j \in J} U_j) \times (\bigcup_{s \in S} \mathcal{V}_s) = G \times \bar{G} \]
\[ \implies G \times \bar{G} = \left( \bigcup_{j \in J} U_j \right) \times \left( \bigcup_{s \in S} \mathcal{V}_s \right) = \bigcup_{j \in J, s \in S} (U_j \times \mathcal{V}_s), \text{ where } U_j \times \mathcal{V}_s \text{ are subrings of} \]
\[ G \times \bar{G}. \text{ And therefore } G \times \bar{G} \text{ is a M-cover ring.} \]

**Corollary (7):** If \( (G,*,\circ) \) is a M-cover ring (M-cover strongly locally ring, M-cover locally ring, weakly M-cover c. ring), then \( (G^2,\oplus,\otimes) \) is a M-cover ring (M-cover strongly locally ring, M-cover locally ring, weakly M-cover ring, weakly M-cover c. ring), respectively.

**Theorem (15):** If \( (G,*,\circ) \) is a M-cover ring (M-cover strongly locally ring, M-cover locally ring, weakly M-cover c. ring), then \( (G^n,\oplus,\otimes) \) is a M-cover ring (M-cover strongly locally ring, M-cover locally ring, weakly M-cover ring, weakly M-cover c. ring), respectively, for each \( n \in \mathbb{N} \).

**Theorem (16):** The product of any finite collection of M-cover rings (M-cover strongly locally rings, M-cover locally rings, weakly M-cover rings, weakly M-cover c. rings), is a M-cover ring (M-cover strongly locally ring, M-cover locally ring, weakly M-cover ring, weakly M-cover c. rings).

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