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# Fixed Point and Common Fixed Point Theorem for Expansive Mapping in Fuzzy Metric Space

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***ABSTRACT***

*In this paper, we have endeavored to establish fixed point and common fixed theorem for expansive mapping in fuzzy metric space.*

***Keywords:*** *Fixed Point, Fixed Point Theorem, Metric Space, Fuzzy Metric Space, Expansive Mappings.*
***AMS Subject Classification [2000]:*** *47H10, 54H25, 46J10, 46J15.*

## *1. INTRODUCTION AND PRELIMINARIES*

*In 1965, Prof. Lofty Zadeh [4] introduced the concept to fuzzy sets as a new way to represent vagueness in our everyday life. However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable, we can divide the into following two groups. The first group involves those results in which a fuzzy metric on a set X is treated as a map where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy.*

*Fuzzy metric spaces have been introduced by Kramosil and Michalek [7], George and Veersamani [1] modified the notion of fuzzy metric with help of continuous t-norms. Recently many have proved fixed point theorems involving fuzzy sets [1, 8, 9, 10-13].*

*The purpose of this paper is to prove fixed point theorem in fuzzy metric spaces for using expansive mapping in fuzzy metric space.*

*To prove of our results we need some definitions which are as follows:*

***Definition 1.1:*** *A fuzzy set A in X is a function with domain X and values in* $[0, 1]$.

***Definition 1.2:*** *A triangular norm* $*$ *(shortly t− norm) is a binary operation on the unit interval* $[0, 1]$*such that for all* $a, b, c, d \in [0, 1]$*the following conditions are satisfied:*

**1.2 (a)** $*$ *is commutative and associative;*

**1.2 (b)** $*$ *is continuous;*

**1.2 (c)** $a * 1 = a, \quad \forall\, a \in [0,1]$;

**1.2 (d)** $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$.

*Two typical examples of continuous t-norm are* $a * b = ab$ *and* $a * b = min\{a, b\}$.

***Definition 1.3:*** *A 3-tuple* $(X, M,*)$*is said to be a fuzzy metric space, if X is an arbitrary set,* $*$ *is a continuous t-norm and M is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions, for all* $x, y, z \in X$*and* $s, t > 0$:

**1.3 (a)** $M(x, y, 0) > 0$;

**1.3 (b)** $M(x, y, t) = 1, \forall\, t > 0$, *if and only if* $x = y$;

**1.3 (c)** $M(x, y, t) = M(y, x, t)$;

**1.3 (d)** $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

**1.3 (e)** $M(x, y,*): (0, \infty) \to (0,1]$ *is continuous.*

*Where* $M(x, y, t)$ *denote the degree of nearness between* $x$ *and* $y$ *with respect to* $t$. *Then* $M$ *is called a fuzzy metric on* $X$.

***Definition 1.4:*** *Let* $(X, M,*)$*be a fuzzy metric space. Then*

**1.4 (a)** *A sequence* $\{x_n\}$*in X is said to converges to x in X if for each* $\varepsilon > 0$ *and each* $t > 0$, *there exists* $n_0 \in N$ *such that*

$$M(x_n, x, t) > 1 - \varepsilon\ \forall\, n \geq n_0.$$

**1.4 (b)** *A sequence* $\{x_n\}$*in X is said to Cauchy if for each* $\varepsilon > 0$*and each* $t > 0$,*there exists* $n_0 \in N$*such that*

$$M(x_n, x_m, t) > 1 - \varepsilon\ \forall\, m > n \text{ and } m, n \geq n_0.$$

**1.4 (c)** *A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

***Definition 1.5:*** *Two mappings* $f$ *and* $g$ *of a fuzzy metric space* $(X, M,*)$ *into itself are said to be weakly commuting if*

$$M(fgx, gfx, t) \geq M(fx, gx, t),\ \forall\, x \in X, \text{and } t > 0.$$

***Definition 1.6:*** *Two mappings $f$ and $g$ of a fuzzy metric space $(X, M, *)$ into itself are $R$-weakly commuting provided there exists some positive real number $R$ such that*

$$M(fgx, gfx, t) \geq M\left(fx, gx, \frac{t}{R}\right), \forall\, x \in X, R > 0, \text{ and } t > 0.$$

***Definition 1.7:*** *Two self maps $f$ and $g$ of a fuzzy metric space $(X, M, *)$ are called reciprocally continuous on $X$,*

$$\text{if } lim_{n\to\infty} fgx_n = fx \text{ and } lim_{n\to\infty} gfx_n = gx,$$

*whenever $\{x_n\}$ is a sequence in $X$ such that*

$$lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = x,$$

*for some $x$ in $X$.*

***Definition 1.8:*** *Two self mappings $f$ and $g$ of a fuzzy metric space $(X, M, *)$ are called compatible,*

$$\text{if } lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1.$$

*whenever $\{x_n\}$ is a sequence in $X$ such that*

$$lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = x,$$

*for some $x$ in $X$.*

***Definition 1.9:*** *Two self maps $f$ and $g$ of a fuzzy metric space $(X, M, *)$ are called reciprocally continuous on $X$, if*

$$lim_{n\to\infty} fgx_n = fx \text{ and } lim_{n\to\infty} gfx_n = gx,$$

*whenever $\{x_n\}$ is a sequence in $X$ such that,*

$$lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = x,$$

*for some $x$ in $X$.*

***Definition 1.10:*** *Let $(X, M, *)$ be a fuzzy metric space. If there exists $q \in (0,1)$, such that*

$$M(x, y, qt) \geq M(x, y, t)$$

*for all $x, y \in X$ and $t > 0$.*

***Definition 1.11:*** *Let $X$ be a set, $f$, $g$ self maps of $X$. A point $x$ in $X$ is called coincidence point of $f$ and $g$, if $fx = gx$.*

*We say $w = fx = gx$, a point of coincidence of $f$ and $g$.*

***Definition 1.12:*** *Two self maps $A$ and $B$ of a fuzzy metric space $(X, M, *)$ are called weak-compatible (or coincidentally commuting) if they commute at their coincidence point, i.e., if $Ax = Bx$ then $ABx = BAx$ for some $x \in X$.*

***Lemma 1.13:*** *Let $X$ be a set, $f$, $g$ owc self maps of $X$. If $f$ and $g$ have a unique point of coincidence, $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.*

***Lemma 1.14:*** *Let* $(X, M,*)$ *be a fuzzy metric space, then for all* $x, y$ *in* $X, M(x, y, .)$ *is non-decreasing.*

***Lemma 1.15:*** *Let*$(X, M,*)$ *be a fuzzy metric space. If there exists* $q \in (0,1)$ *such that* $M(x, y, qt) \geq M(x, y, \frac{t}{q^n})$ *for positive integer* $n$*. Taking limit as* $n \to \infty, M(x, y, t) \geq 1$ *and hence* $x = y$.

***Lemma 1.16:*** *Let*$(X, M,*)$ *be a fuzzy metric space and let* $A$ *and* $S$ *be continuous mappings of* $X$*, then* $A$ *and* $S$ *are compatible if and only if they are compatible of type* $(P)$.

***Lemma 1.17:*** *Let*$(X, M,*)$ *be a fuzzy metric space and let* $A$ *and* $S$ *be compatible mappings of type* $(P)$ *and* $Az = Sz$ *for some* $z \in X$*, Then*

$$AAz = ASz = SAz = SSz.$$

## *2. MAIN RESULT*

*In this section we prove some common fixed point theorem for expansive mapping in fuzzy metric spaces.*

***Theorem 2.1:*** *Let* $(X, M,*)$ *be a complete fuzzy metric space and let* $A, B, S$ and $T$ *be self-mappings of* $X$ *satisfying the following conditions:*

***2.1(a)*** $A(X) \subset T(X), B(X) \subset S(X)$;

***2.1(b)*** $S$ *and* $T$ *are continuous,*

***2.1(c)*** *The pair* $\{A, S\}$ *and* $\{B, T\}$ *are expansive mappings of type* $(P)$*on* $X$.

***2.1(d)*** *There exists* $q > 1$ *such that for every* $x, y \in X$ *and* $t > 0$,

$$M(Ax, By, qt) \leq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t).$$

*Then* $A, B, S$ *and* $T$ *have a unique common fixed point in* $X$.

***Proof****: Since* $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$*, for any* $x_0 \in X$*, there exists* $x_1 \in X$ *such that* $Ax_0 = Tx_1$ *and for this* $x_1 \in X$*, there exists* $x_2 \in X$ *such that* $Bx_1 = Sx_2$*. Inductively, we define a sequence* $\{y_n\}$ *in X, such that*

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-1} and y_{2n} = Sx_{2n} = Bx_{2n-1},$$

*for all* $n = 0,1,2, \dots \dots$

*from 2.1(d),*

$$M(y_{2n+1}, y_{2n+2}, qt) = M(Ax_{2n}, Bx_{2n+1}, qt).$$

$$\leq M(Sx_{2n}, Tx_{2n+1}, t) * M(Ax_{2n}, Sx_{2n}, t) * M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Ax_{2n}, Tx_{2n}, t)$$

$$= M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+1}, t)$$

$$\leq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t).$$

 

*From (lemma 1.14) and (lemma 1.16), we have*

$M(y_{2n+1}, y_{2n+2}, qt) \leq M(y_{2n}, y_{2n+1}, t).$ *2.1(i)*

*Similarly, we have*

$M(y_{2n+2}, y_{2n+3}, qt) \leq M(y_{2n+1}, y_{2n+2}, t).$ *2.1(ii)*

*From 2.1(i) and 2.1(ii), we have*

$M(y_{n+1}, y_{n+2}, qt) \leq M(y_n, y_{n+1}, t).$ *2.1(iii)*

*From 2.1(iii), we have*

$$M(y_n, y_{n+1}, t) \leq M\left(y_n, y_{n-1}, \frac{t}{q}\right)$$

$$\leq M\left(y_{n-2}, y_{n-1}, \frac{t}{q^2}\right) \leq \cdots \ldots .. \leq M\left(y_{1,} y_2, \frac{t}{q^n}\right) \to 1,$$

*as* $n \to \infty$.

*So,* $M(y_n, y_{n+1}, t) \to 1$ *as* $n \to \infty$ *for any* $t > 0$.

*For each* $\varepsilon > 0$ *and* $t > 0$,

*we can choose* $n_0 \in \mathbb{N}$ *such that*

$M(y_n, y_{n+1}, t) < 1 - \varepsilon$ *for all* $n > n_0$.

*For* $m, n \in \mathbb{N}$ *we suppose* $m \geq n$.

*Then we have that*

$$M(y_n, y_m, t) \leq M\left(y_n, y_{n+1}, \frac{t}{m-n}\right)$$

$$* M\left(y_{n+1}, y_{n+2}, \frac{t}{m-n}\right) * \ldots \ldots * M\left(y_{m-1}, y_m, \frac{t}{m-n}\right)$$

$$\leq (1-\varepsilon) * (1-\varepsilon) * \ldots \ldots (m-n) \text{ times.}$$

$$\leq (1-\varepsilon).$$

*And hence* $\{y_n\}$ *is a Cauchy sequence in X.*

 

*Since* $(X, M, *)$ *is complete,* $\{y_n\}$ *converges to some point* $z \in X$, *and so* $\{Ax_{2n-2}\}, \{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ *and* $\{Tx_{2n-1}\}$ *also converges to z.*

*From (lemma 1.16) and (lemma 1.17), we have*

$AAx_{2n-2} \to Sz$ *and* $SSx_{2n} \to Az$ 2.1 (iv)

$BBx_{2n-1} \to Tz$ *and* $TTx_{2n-1} \to Bz$. 2.1(v)

*From 2.1(d), we get*

$$M(AAx_{2n-2}, BBx_{2n-1}, qt) \leq$$

$$M(SAx_{2n-2,}TBx_{2n-1}, t) * M(AAx_{2n-2,}SAx_{2n-2}, t) * M(BBx_{2n-1,}TBx_{2n-1}, t)$$

$$* M(AAx_{2n-2,}TBx_{2n-1}, t).$$

*Taking limit as* $n \to \infty$ *and using 2.1(iv) and 2.1(v), we have*

$$M(Sz, Tz, qt) \leq M(Sz, Tz, t) * M(Sz, Sz, t) * M(Tz, Tz, t) * M(Sz, Tz, t)$$
$$\leq M(Sz, Tz, t) * 1 * 1 * M(Sz, Tz, t)$$
$$\leq M(Sz, Tz, t).$$

*It follows that* $Sz = Tz$. 2.1(vi)

*Now, from 2.1(d),*

$$M(Az, BTx_{2n-1}, qt) \leq$$
$$M(Sz, TTx_{2n-1}, t) * M(Az, Sz, t) * M(BTx_{2n-1,}TTx_{2n-1}, t)$$
$$* M(Az, TTx_{2n-1}, t)$$

*Again taking limit as* $n \to \infty$ *and using 2.1(v) and 2.1(vi), we have*

$$M(Az, Tz, qt) \leq M(Sz, Sz, t) * M(Az, Tz, t) * M(Tz, Tz, t) * M(Az, Tz, t)$$
$$\leq M(Az, Tz, t).$$

*And hence* $Az = Tz$. 2.1(vii)

*From 2.1(d), 2.1(vi) and 2.1(vii),*

$$M(Az, Bz, qt) \leq M(Sz, Tz, t) * M(Az, Sz, t) * M(Bz, Tz, t) * M(Az, Tz, t)$$
$$= M(Az, Az, t) * M(Az, Az, t) * M(Bz, Az, t) * M(Az, Az, t)$$
$$\leq M(Az, Bz, t).$$

*And hence* $Az = Bz$. 2.1(viii)

*From 2.1(vi), 2.1(vii) and 2.1(viii), we have*

$Az = Bz = Tz = Sz.$ *2.1(ix)*

*Now, we show that* $Bz = z$.

*From 2.1(d),*

$$M(Ax_{2n}, Bz, qt) \leq M(Sx_{2n}, Tz, t)$$
$$* M(Ax_{2n}, Sx_{2n}, t) * M(Bz, Tz, t) * M(Ax_{2n}, Tz, t)$$

*And taking limit as* $n \to \infty$ *and using 2.1(vi) and 2.1(vii), we have*

$$M(z, Bz, qt) \leq M(z, Tz, t) * M(z, z, t) * M(Bz, Tz, t) * M(z, Tz, t)$$
$$= M(z, Bz, qt) * 1 * M(Az, Az, t) * M(z, Bz, t)$$
$$\leq M(z, Bz, t).$$

*And hence* $Bz = z$. *Thus from 2.1(ix),* $z = Az = Bz = Tz = Sz$ *and* $z$ *is a common fixed point of* $A, B, S$ *and* $T$.

*In order to prove the uniqueness of fixed point, let* $w$ *be another common fixed point of* $A, B, S$ *and* $T$. *Then*

$$M(z, w, qt) = M(Az, Bw, qt)$$
$$\leq M(Sz, Tw, t) * M(Az, Sz, t) * M(Bw, Tw, t) * M(Az, Tw, t)$$
$$\leq M(z, w, t).$$

*From Lemma 1.15,* $z = w$.

*This completes the proof of theorem.*

## *Conclusion*

*In this present article, we prove fixed point and common fixed point theorem satisfying expansive mapping in fuzzy metric space. In fact our main result is more general then other previous known results.*

## *References*

1. *A. George, P. Veeramani, "On some results in fuzzy metric spaces" Fuzzy Sets and Systems, 64 (1994), 395-399.*
2. *B.E. Rhoades, "Contractive definitions and continuity," Contemporary Math. 72 (1988), 233-245.*

**3.** *G. Jungck, "Compatible mappings and common fixed points (2)", Internat. J. Math. Math.Sci. (1988), 285-288.*

**4.** *L.A. Zadeh, "Fuzzy sets" Inform and Control, 8 (1965), 338-353.*

**5.** *M. Grabiec, "Fixed points in fuzzy metric specs, Fuzzy Sets and Systems" 27 (1988), 385-389.*

**6.** *O. Hadzic, "Common fixed point theorems for families of mapping in complete metric space," Math. Japon.29 (1984), 127-134.*

**7.** *O. Kramosil and J. Michalek, "Fuzzy metric and statistical metric spaces," Kybernetika 11 (1975), 326-334.*

**8.** *P. Balasubramaniam, S. Muralisankar, R.P. Pant, "Common fixed' points of four mappings in a fuzzy metric space". J. Fuzzy Math. 10(2) (2002), 379-384.*

**9.** *P. Tiwari, R. Shrivastava, R. K. Dubey "On fixed point theorem in fuzzy metric spaces," Mathematical Theory and Modeling, 3(6), (2013), 148-153.*

**10.** *R. Vasuki, "Common fixed points for R-weakly commuting maps in fuzzy metric spaces," Indian J. Pure Appl. Math. 30 (1999), 419-423.*

**11.** *R.P. Pant, K. Jha, "A remark on common fixed points of four mappings in a, fuzzy metric space," J. Fuzzy Math. 12(2) (2004), 433-437.*

**12.** *S. Kutukcu, S. Sharma, S. H Tokgoz "A fixed point theorem in fuzzy metric spaces" Int .Journal of Math. Analysis vol 1, (2007), 861-872*

**13.** *S. Sharma, "Common fixed point theorems in fuzzy metric spaces", Fuzzy Sets and Systems 127 (2002). 345-352.*

**14.** *S. Sharma, B. Deshpande, "Compatible multivalued mappings satisfying an implicit relation," Southeast Asian Bull.Math.30 (2006), 585-540.*

**15.** *S. Sharma, B. Deshpande, "Discontinuity and weak compatibility in fixed point consideration on non- complete fuzzy metric spaces," J. Fuzzy Math. 11(2) (2003), 671-686.*

**16.** *Y.J. Cho, H.K. Pathak, S.M. Kang. J.S. Jung, "Common fixed points of compatible maps of type* (A) *on fuzzy metric spates," Fuzzy Sets and Systems 93 (1998), 99-111.*