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An Efficient Sumudu Decomposition Method to Solve System of Pantograph Equations

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Abstract: This paper is the witness of the coupling of decomposition method with the efficient Sumudu transform known as Sumudu decomposition method to build up the exact solutions of the linear and nonlinear system of Pantograph model equations. Three mathematical models are tested to elucidate effectiveness of the method. The obtained numerical results re-confirm the potential of the proposed method. In nonlinear cases this method uses He's Polynomials for solving the non-linear terms. It is observed that suggested scheme is highly reliable and may be extended to other highly nonlinear delay differential models.

Keywords: Decomposition method, Sumudu transform, System of multi-Pantograph delay differential equations, He's polynomials

1. INTRODUCTION

Many physical phenomenons are mathematically modeled by differential equations which are ordinary and Delay Differential equations. Delay differential equations differ from ordinary differential equation in two ways i.e. solution and initial data. Both depend in delay differential equations at previous state of time. Pantograph equation is a kind of Delay Differential equations. In 1851, Taylor was the first who gave name to these equations. This type of equations have been studied due to various applications which arises in many fields of sciences like electric systems, population dynamics, environmental science, natural science and life science, electro-dynamics, number-theory, engineering and mathematics. Liu and Li solved multi-pantograph delay equation by Runge-Kutta methods [2]. Evans and Raslan solved the delay differential equation by ADM [3]. Keskin et al. got approximate solution by via method of differential-transform [4]. Sezer & Dascioglu established Taylor method and advanced case or retarded case of pantograph equations generalized type solved through this technique. Yu solved Multi-pantograph equation by VIM [6]. Sezer et al. get solution of multi-pantograph equation of approximate type by using coefficients variable [7]. Singular Perturbed Multi-Pantograph Equations were solved by S. P. Qian and F. Z. Geng, with

Method of Reprociding Kernel [8]. Cherruault, Adomian, Abbaoui, and Rach, controlled Convergence of Decomposition Method [9]. Delay dynamic system was studied by El-Safty et al. with the spline function of 3-h step for getting approximate solution [11].

Numerous schemes have been established for explaining advanced as well as retarded pantograph equations. First time Watugala introduced Sumudu transform in his effort of work (Watugala, 1993). Many people then further developed it and used it to get solution of many problems. Belgacem et al recognized its fundamentals properties in (2003, 2006). Its Properties are very different and valuable that can help in science and engineering for solving many complicated applications.

On view of the above approaches we are intent to solve system of Pantograph equations by Sumudu Decomposition method, because this method yields an approximate solution in a small number of terms and is easy to compute.

2. ANALYSIS OF THE METHOD

Let A be a space of functions as follows

$$A=\{f(t):\tau_1,\tau_2>0,\left|f(t)\right|<Me^{t/\tau_j},if\ t\in(-1)^j\times[0,\infty)\}, \tag{1}$$

Now the well-defined Sumudu-transform of function is

$$S[f(t)]=\int_0^\infty f(ut)e^{-t}dt. \tag{2}$$

To demonstrate basic idea of SDM for system of Multi-Pantograph equations

$$\begin{aligned}
&y_1{}'(t)=\alpha_1 y_1(t)+h_1\big(t,y_i(t),y_i(p_j t)\big),\\
&y_2{}'(t)=\alpha_2 y_2(t)+h_2\big(t,y_i(t),y_i(p_j t)\big),\\
&\qquad\vdots\\
&y_m{}'(t)=\alpha_m y_m(t)+h_m\big(t,y_i(t),y_i(p_j t)\big),\\
&y_i(0)=y_{i0},\qquad i=1,\dots,n,\qquad j=1,2,\dots,
\end{aligned} \tag{3}$$

Where $\alpha_i, y_{i0}\in C$ and h_i are functions of analytical type, $0<q_j<1$.

This method contains 1stly applying Sumudu transform

$$\begin{aligned} S[y_1(t)] &= S[R_1(y_1(t), y_2(t),...., y_n(t)) + N_1(y_1(t), y_2(t),...., y_n(t)) + h_1], \\ S[y_2(t)] &= S[R_2(y_1(t), y_2(t),...., y_n(t)) + N_2(y_1(t), y_2(t),...., y_n(t)) + h_2], \\ &\vdots \\ S[y_n(t)] &= S[R_n(y_1(t), y_2(t),...., y_n(t)) + N_n(y_1(t), y_2(t),...., y_n(t)) + h_n], \end{aligned} \tag{4}$$

Using initial conditions

$$y_i(0) = y_{i0}, \qquad i = 1,2,..,n, \tag{5}$$

Wherever N_i , R_i and h_i are non-linear, linear operators and analytical functions respectively.

By the Differentiating property & initial conditions, to become

$$\begin{aligned} S[Y_1(t)] &= \frac{1}{u}[Y_1(0)] + uS[R_1(y_1(t), y_2(t),...., y_n(t)) + N_1(y_1(t), y_2(t),...., y_n(t)) + h_1], \\ S[Y_2(t)] &= \frac{1}{u}[Y_2(0)] + uS[R_2(y_1(t), y_2(t),...., y_n(t)) + N_2(y_1(t), y_2(t),...., y_n(t)) + h_2], \\ &\vdots \\ S[Y_n(t)] &= \frac{1}{u}[Y_n(0)] + uS[R_n(y_1(t), y_2(t),...., y_n(t)) + N_n(y_1(t), y_2(t),...., y_n(t)) + h_n], \end{aligned} \tag{6}$$

The solution can be decomposed such as

$$y_i(t) = \sum_{j=0}^{\infty} y_{ij}(t), \qquad i = 1,2,..,n, \tag{7}$$

By applying Sumudu Inverse transform, everywhere the terms $y_{ij}(t)$ calculate recursively. The non-linear term can be decomposed as

$$N_i(y_1, y_2,...., y_n) = \sum_{j=0}^{\infty} H_{ij}, \qquad i = 1,2,..,n, \tag{8}$$

H_{ij} are He's Polynomials. It can be computed by formula

$$H_{ij} = \frac{1}{j!}\frac{d^j}{dp^j}[N\sum_{j=0}^{\infty} p^j y_{ij}]_{p=0}, \qquad i, = 1,2,..., n, \tag{9}$$

3. NUMERICAL DEMONSTRATION

3.1 Consider the system of Pantograph Equation

$$y_1'(t) = e^{-t} - e^{t/2} + y_1(\frac{t}{2}) + y_1(t) - y_2(t)$$
$$y_2'(t) = e^{t} + e^{t/2} - y_2(\frac{t}{2}) - y_1(t) - y_2(t), \tag{10}$$

along with conditions

$$y_1(0) = 1, \quad y_2(0) = 1. \tag{11}$$

By means of 1st step of method,

$$S\left[y_1'(t)\right] = S\left[e^{-t} - e^{t/2} + y_1(\frac{t}{2}) + y_1(t) - y_2(t)\right]$$
$$S\left[y_2'(t)\right] = S\left[e^{t} + e^{t/2} - y_2(\frac{t}{2}) - y_1(t) - y_2(t)\right], \tag{12}$$

Using differentiation property with initial condition, Eq.12 becomes

$$S[Y_1(t)] = 1 + uS\left[e^{-t} - e^{t/2}\right] + uS\left[y_1(\frac{t}{2}) + y_1(t) - y_2(t)\right]$$
$$S[Y_2(t)] = 1 + uS\left[e^{t} + e^{t/2}\right] + uS\left[-y_2(\frac{t}{2}) - y_1(t) - y_2(t)\right], \tag{13}$$

Moreover,

$$S[Y_1(t)] = 4 - \frac{1}{1+u} - \frac{2}{1-\frac{u}{2}} + uS\left[y_1(\frac{t}{2}) + y_1(t) - y_2(t)\right]$$
$$S[Y_2(t)] = -2 + \frac{1}{1-u} + \frac{2}{1-\frac{u}{2}} + uS\left[-y_2(\frac{t}{2}) - y_1(t) - y_2(t)\right], \qquad 14)$$

Applying Inverse transform of the method and solution can be decomposed as

$$y_i(t) = \sum_{j=0}^{\infty} p^j y_{ij}(t), \qquad i = 1,2, \qquad 15)$$

It becomes

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$$\begin{aligned}
\sum_{j=0}^{\infty} p^j y_{1j}(t) &= 4 - e^{-t} - 2e^{t/2} + pS^{-1}\left[uS\left[\sum_{j=0}^{\infty} p^j y_{1j}(\frac{t}{2}) + \sum_{j=0}^{\infty} p^j y_{1j}(t) - \sum_{j=0}^{\infty} p^j y_{2j}(t)\right]\right] \\
\sum_{j=0}^{\infty} p^j y_{2j}(t) &= -2 + e^{t} + 2e^{t/2} + pS^{-1}\left[uS\left[-\sum_{j=0}^{\infty} p^j y_{2j}(\frac{t}{2}) - \sum_{j=0}^{\infty} p^j y_{1j}(t) - \sum_{j=0}^{\infty} p^j y_{2j}(t)\right]\right],
\end{aligned} \tag{16}$$

Consequently,

$$p^0; y_{10}(t) = 4 - e^{-t} - 2e^{t/2}$$
$$p^0; y_{20}(t) = -2 + e^{t} + 2e^{t/2},$$

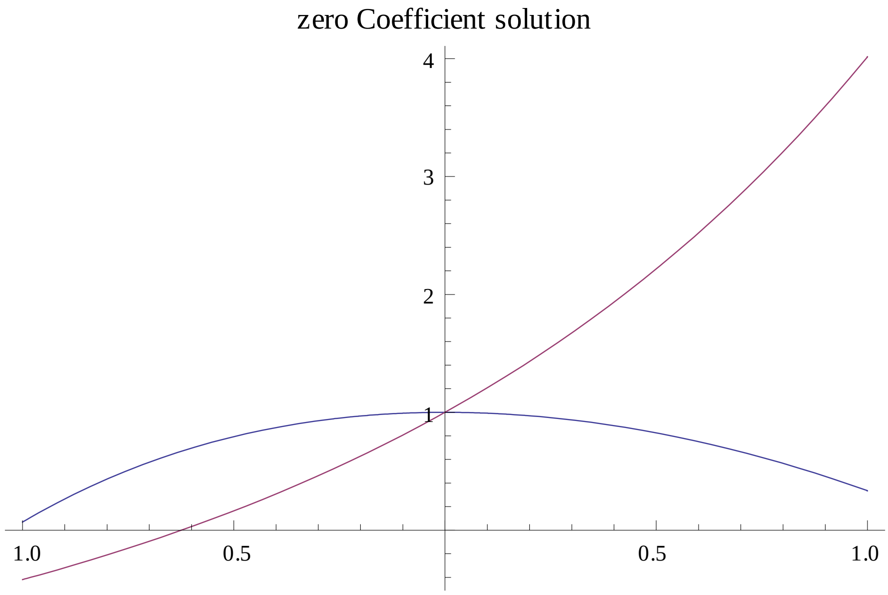

Fig.1: Zeroth coefficient solution

$$p^1; u_{11}(t) = 16 + 10t - e^{-t} - e^{t} + 2e^{-t/2} - 8e^{t/2} - 8e^{t/4}$$
$$p^1; u_{21}(t) = 12 - e^{-t} - e^{t} - 2e^{t/2} - 8e^{t/4},$$

$\vdots,$

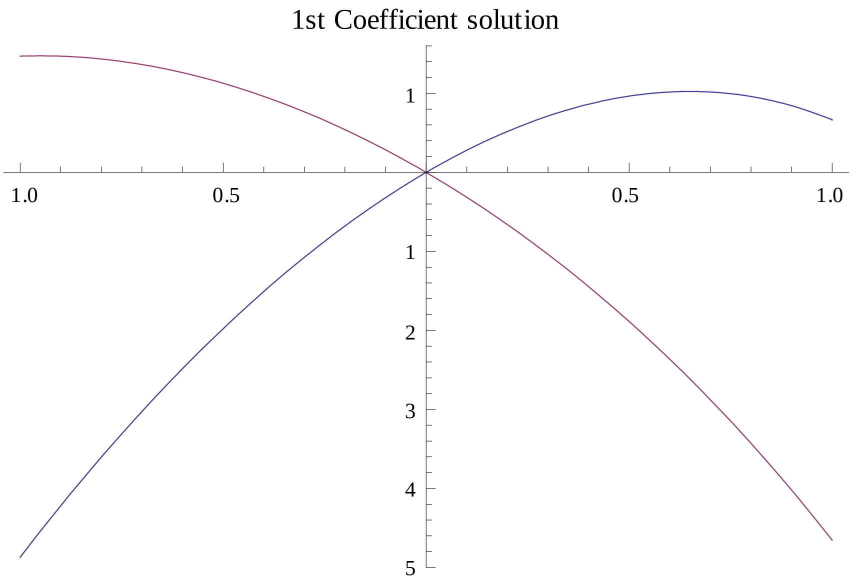

Fig.2: 1st coefficient solution

By calculating other components and $p \to 1$, solution is

$$y_1(t) = e^t,\ \ y_2(t) = e^{-t}. \tag{17}$$

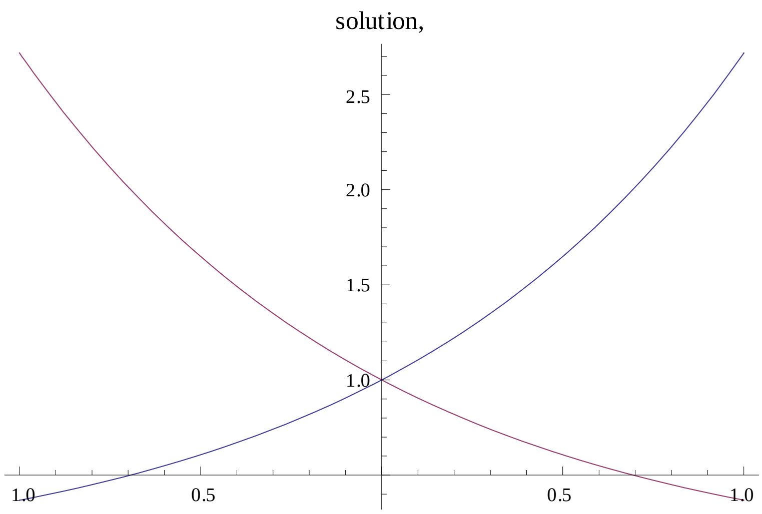

Fig.3: Exact solution

3.2 Consider system of Pantograph equation

$$\begin{aligned} y_1'(t) &= -t\cos\left(\frac{t}{2}\right) + 2y_2\left(\frac{t}{2}\right) + y_3(t) \\ y_2'(t) &= 1 - t\sin(t) - 2y_3^{\,2}\left(\frac{t}{2}\right) \\ y_3'(t) &= -t\cos(t) - y_1(t) + y_2(t), \end{aligned} \tag{18}$$

along with conditions

$$y_1(0) = -1, \quad y_2(0) = 0, \quad y_3(0) = 0, \tag{19}$$

Taking the Sumudu transform of Eq. 18,

$$\begin{aligned}
S\left[y_1'(t)\right] &= S\left[-t\cos\left(\frac{t}{2}\right) + 2y_2(\frac{t}{2}) + y_3(t)\right] \\
S\left[y_2'(t)\right] &= S\left[1 - t\sin(t) - 2y_3^{\,2}(\frac{t}{2})\right] \\
S\left[y_3'(t)\right] &= S\left[-t\cos(t) - y_1(t) + y_2(t)\right],
\end{aligned} \tag{20}$$

Using differentiation property with initial condition, Eq.20 becomes

$$\begin{aligned}
S[Y_1(t)] &= -1 - u\left[\frac{u\left(1-\frac{u^2}{4}\right)}{\left(1+\frac{u^2}{4}\right)^2}\right] + uS\left[2y_2(\frac{t}{2}) + y_3(t)\right] \\
S[Y_2(t)] &= u\left[1 - 2\left(\frac{u^2}{(1+u^2)^2}\right)\right] + uS\left[-2y_3^{\,2}(\frac{t}{2})\right] \\
S[Y_3(t)] &= u\left[-\left(\frac{u(1-u^2)}{(1+u^2)^2}\right)\right] + uS\left[-y_1(t) + y_2(t)\right],
\end{aligned} \tag{21}$$

Applying Inverse transform of the method and solution can be decomposed as

$$y_i(t) = \sum_{j=0}^{\infty} p^j y_{ij}(t), \qquad i = 1,2,3, \tag{22}$$

Eq. 21 gives

$$\begin{aligned}
\sum_{j=0}^{\infty} p^j y_{1j}(t) &= 3 - 4\cos\left(\frac{t}{2}\right) - 2t\sin\left(\frac{t}{2}\right) + S^{-1}\left[uS\left[2\sum_{j=0}^{\infty} p^j y_{2j}(\frac{t}{2}) + \sum_{j=0}^{\infty} p^j y_{3j}(t)\right]\right] \\
\sum_{j=0}^{\infty} p^j y_{2j}(t) &= t - \sin(t) + t\cos(t) + S^{-1}\left[uS\left[-2\sum_{j=0}^{\infty} p^j y_{3j}^{\,2}(\frac{t}{2})\right]\right] \\
\sum_{j=0}^{\infty} p^j y_{3j}(t) &= 1 - t\sin(t) - \cos(t) + S^{-1}\left[uS\left[-\sum_{j=0}^{\infty} p^j y_{1j}(t) + \sum_{j=0}^{\infty} p^j y_{2j}(t)\right]\right],
\end{aligned} \tag{23}$$

Consequently,

$$p^0; y_{10}(t) = 3 - 4\cos\left(\frac{t}{2}\right) - 2t\sin\left(\frac{t}{2}\right)$$
$$p^0; y_{20}(t) = t\cos(t) + t - \sin(t)$$
$$p^0; y_{30}(t) = 1 - t\sin(t) - \cos(t),$$

zeroth order solution,

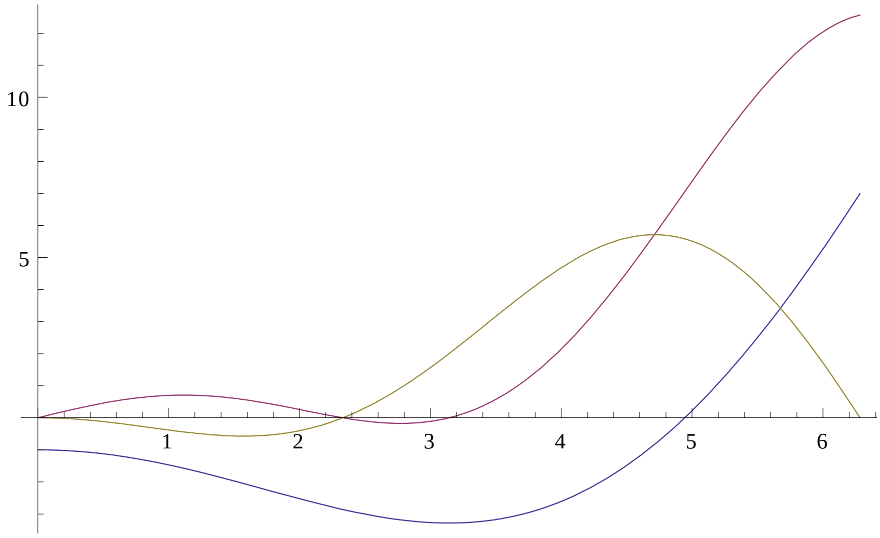

Fig.4: Zeroth coefficient solution

$$p^1; y_{11}(t) = -8 + t + \frac{t^2}{2!} + 8\cos\left(\frac{t}{2}\right) - t\cos(t) + 2t\sin\left(\frac{t}{2}\right)$$
$$p^1; y_{21}(t) = -2t + 8\sin\left(\frac{t}{2}\right) - 2t\cos\left(\frac{t}{2}\right)$$
$$p^1; y_{31}(t) = 30 - 3t + \frac{t^2}{2!} + t\sin(t) + 2\cos(t) + 8\sin\left(\frac{t}{2}\right) - 8t\sin\left(\frac{t}{2}\right) - 32\cos\left(\frac{t}{2}\right),$$

$\vdots,$

By calculating other components and $p \to 1$, it gives exact solution

$$y_1(t) = -\cos(t)$$
$$y_2(t) = t\cos(t) \qquad (24)$$
$$y_3(t) = \sin(t).$$

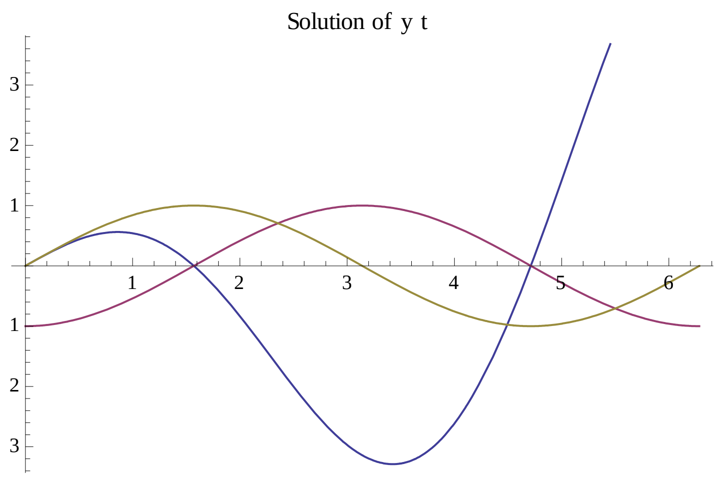

Fig.5: Exact solution of y (t)

3.3 Consider system of Pantograph equations

$$\begin{aligned} y_1'(t) &= y_1(t-1) \\ y_2'(t) &= y_1(t-1) + y_2(t-0.2) \\ y_3'(t) &= y_2(t), \end{aligned} \tag{25}$$

along with conditions

$$y_1(0) = 1, \quad y_2(0) = 1, \quad y_3(0) = 1. \tag{26}$$

By applying the 1st step of Method

$$\begin{aligned} S\left[y_1'(t)\right] &= S[y_1(t-1)] \\ S\left[y_2'(t)\right] &= S[y_1(t-1) + y_2(t-0.2)] \\ S\left[y_3'(t)\right] &= S[y_2(t)], \end{aligned} \tag{27}$$

Using differentiation property with initial condition, Eq. 27 becomes

$$\begin{aligned} S\left[Y_1'(t)\right] &= 1 + uS[y_1(t-1)] \\ S\left[Y_2'(t)\right] &= 1 + uS[y_1(t-1) + y_2(t-0.2)] \\ S\left[Y_3'(t)\right] &= 1 + uS[y_2(t)], \end{aligned} \tag{28}$$

Applying Inverse transform of the method and solution can be decomposed as

$$y_i(t) = \sum_{j=0}^{\infty} p^j y_{ij}(t), \qquad i = 1,2,3, \tag{29}$$

Eq.28 becomes

$$\begin{aligned}
\sum_{j=0}^{\infty} p^j y_{1j}(t) &= 1 + uS\left[\sum_{j=0}^{\infty} p^j y_{1j}(t-1)\right] \\
\sum_{j=0}^{\infty} p^j y_{2j}(t) &= 1 + uS\left[\sum_{j=0}^{\infty} p^j y_{1j}(t-1) + \sum_{j=0}^{\infty} p^j y_{2j}(t-0.2)\right] \\
\sum_{j=0}^{\infty} p^j y_{3j}(t) &= 1 + uS\left[\sum_{j=0}^{\infty} p^j y_{2j}(t)\right],
\end{aligned} \tag{30}$$

By comparing the co-efficient of p , we have

$$\begin{aligned}
p^0 &; y_{10}(t) = 1 \\
p^0 &; y_{20}(t) = 1 \\
p^0 &; y_{30}(t) = 1,
\end{aligned}$$

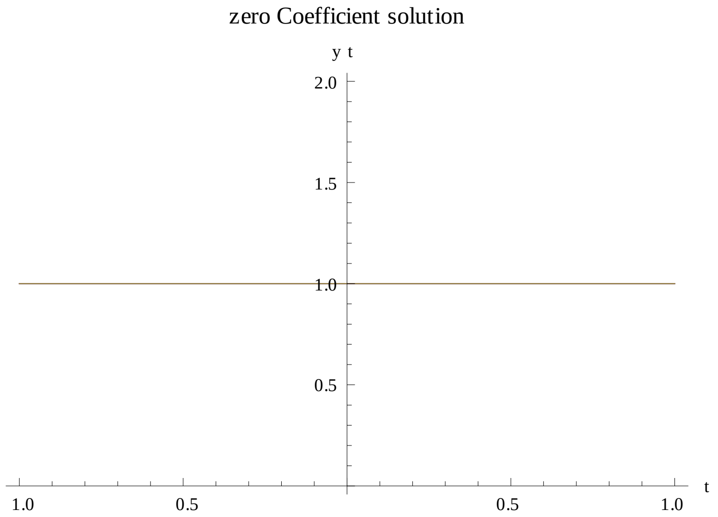

Fig.6: Zeroth solution

$$\begin{aligned}
p^1 &; y_{11}(t) = t \\
p^1 &; y_{21}(t) = 2t \\
p^1 &; y_{31}(t) = t,
\end{aligned}$$

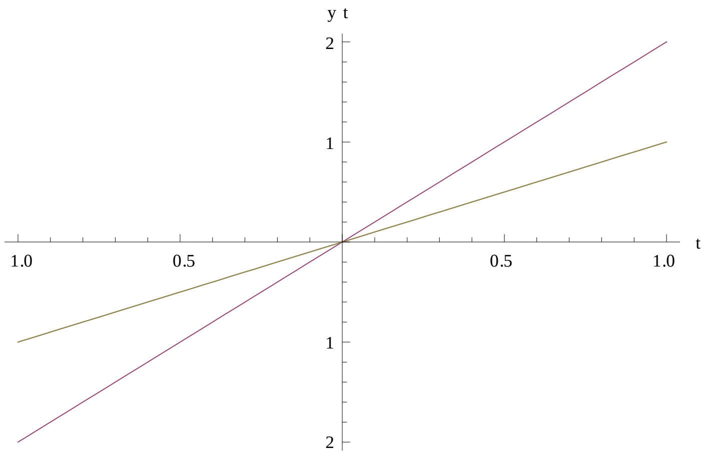

Fig.7:1st coefficient solution

$$p^2; y_{12}(t) = \frac{t^2}{2!} - t,$$
$$p^2; y_{22}(t) = \frac{3t^2}{2!} - 1.4t,$$
$$p^2; y_{32}(t) = t^2,$$

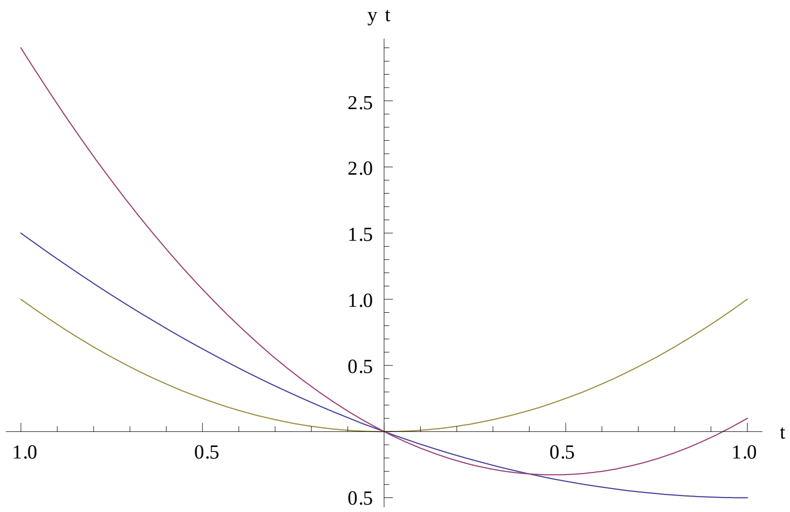

Fig.8: 2nd coefficient solution

$\vdots,$

By calculating other components and $p \to 1$, then we have

$$\begin{aligned} y_1(t) &= e^{t/2} \\ y_2(t) &= e^{2t} \\ y_3(t) &= \frac{1}{2}+\frac{1}{2}e^{2t}, \end{aligned} \tag{31}$$

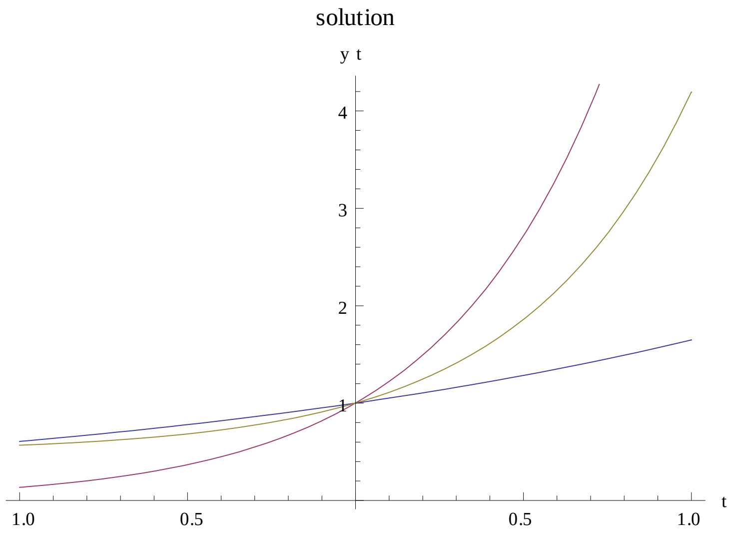

Fig.9: Exact solution

4. CONCLUSION

The applications of the Sumudu decomposition method (SDM) has been extended successfully for solving the linear and nonlinear system of multi-pantograph differential equations. The leading benefit of the SDM is the quick convergence of the solutions. Numerical and graphical representation of the determined results verifies the fully capable to cope with the nonlinearity of the physical problems. It is concluded that the SDM is powerful method to tackle such proposed mathematical models.

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