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On The Relationship Between Aboodh Transform and New Integral Transform " ZZ Transform"

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Abstract:
In this paper we discusses some relationship between Aboodh transform and the new integral transform called ZZ transform, we solve first and second order ordinary differential equations with constant and non-constant coefficients , using both transforms ,and showing ZZ transform is closely connected with Aboodh transform.

Keywords: Aboodh Transform, ZZ Transform, Differential equations

1.Introduction

In the literature there are numerous integral transforms [10] and widely used in physics, astronomy as well as in engineering. In order to solve the differential equations, the integral
transform were extensively used and thus there are several works on the theory and application of integral transform such as the Laplace, Fourier, Mellin, and Hankel, Fourier Transform, Sumudu Transform, Elzaki Transform and Aboodh Transform .
Aboodh Transform [1,2] was introduced by Khalid Aboodh in 2013, to facilitate the process of solving ordinary and partial differential equations in the time domain. This transformation has deeper connection with the Laplace and Elzaki Transform. [3,4, 5]. New integral transform, named as ZZ Transformation [6-9] intrpduce by Zain Ul Abadin Zafar [2016] , ZZ transform was successfully applied to integral equations, ordinary differential equations . the main objective is to introduce a comparative study to solve differential equations by using Aboodh transform and ZZ transform. The plane of the paper is as follows: In section 2, we introduce the basic idea of Aboodh transform, and ZZ Transform , Application in 3 and conclusion in 4, respectively.

2.Definition and standard Results :

2.1. Aboodh transform :

Definition :

A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set **A,** defined by:

$$A = \{f(t): \exists\ M , k_1, k_2 > 0 , |f(t)| < Me^{-vt}$$

For a given function in the set M must be finite number, k_1, k_2 may be finite or infinite. Aboodh transform which is defined by the integral equation

$$A[f(t)] = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}\,dt \quad t \geq 0 , k_1 \leq v \leq k_2 \qquad (1)$$

Aboodh transform of some functions :

$$A(1)=\frac{1}{v^2} \quad , \quad A(t^n)=\frac{n!}{v^{n+2}} \quad , A(e^{at})=\frac{1}{v^2-av}$$

$$A(\sin(at))=\frac{a}{v(v^2+a^2)} \qquad , \quad A(\cos(at))=\frac{1}{(v^2+a^2)}$$

.Aboodh transform of derivatives :

$$A[f\,'(t)]=vK(v)-\frac{f(0)}{v} \quad , \quad A[f\,''(t)]=v^2K(v)-\frac{f'(0)}{v}-f(0)$$

$$A\left[f^{(n)}(t)\right]=v^nK(v)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{2-n+k}} \ . \ ` A\{tf(t)\}=-\frac{d}{dv}k(v)-\frac{1}{v}k(v)$$

$$A\{tf'(t)\}=-\frac{d}{dv}\left[vk(v)-\frac{f(0)}{v}\right]-\frac{1}{v}\left[vk(v)-\frac{f(0)}{v}\right],$$

$$A\{tf''(t)\}=-\frac{d}{dv}\left[v^2k(v)-\frac{f'(0)}{v}-f(0)\right]-\frac{1}{v}\left[v^2k(v)-\frac{f'(0)}{v}-f(0)\right]$$

2.2 The ZZ Transform:

Definition:

Let f(t) be a function defined for all $t \geq 0$. The ZZ transform of $f(t)$ is the function $Z(u,s)$
defined by

$$Z(u,s)=H\{f(t)\}=s\int_0^{\infty} f(ut)e^{-st}dt \tag{2}$$

provided the integral on the right side exists. The unique function $f(t)$ in (2) is called the inverse
transform of $Z(u,s)$ is indicated by

$$f(t)=H^{-1}\{Z(u,s)\}$$

Equation (2) can be written as

$$H\{f(t)\}=\frac{s}{u}\int_0^{\infty} f(t)e^{-\frac{s}{u}t}dt \tag{3}$$

ZZ transform of some functions :

$$H\{1\}=1 \quad , \quad H\{t^n\}=n!\frac{u^n}{s^n}, \quad H\{e^{at}\}=\frac{s}{s-ua}$$

$$E(\sin(at))=\frac{aus}{s^2+a^2u^2} \qquad , \quad E(\cos(at))=\frac{s^2}{s^2+a^2u^2}.$$

ZZ transform of derivatives :

1) let $H\{f(t)\}=Z(u,s)$ then

$$H\left\{f^{(n)}(t)\right\}=\frac{s^n}{u^n}Z(u,s)-\sum_{k=0}^{n-1}\frac{s^{n-k}}{u^{n-k}}f^{(k)}(0)$$

2) (i) $H\{tf(t)\}=\frac{u^2}{s}\frac{d}{du}\big(Z(u,s)\big)+\frac{u}{s}Z(u,s)$

(ii)) $H\{tf'(t)\}=\frac{u^2}{s}\frac{d}{du}\left(\frac{s}{u}Z(u,s)\right)+Z(u,s)$

(iii) $H\{tf''(t)\}=s\frac{d}{du}\big(Z(u,s)\big)-\frac{s}{u}Z(u,s)+\frac{s}{u}f(0)$

3 Application :

Example 3.1 consider the first order differential equation

$$\frac{dy}{dx}+y=0 \tag{4}$$

With the initial condition; $, y(0) = 1$ (5)

Solution:

1: Applying the Aboodh transform of both sides of Eq. (4),

$$A\left\{\frac{dy}{dx}\right\} + A\{y\} = A\{0\} \tag{6}$$

Using the differential property of Aboodh transform Eq.(6) can be written as:

$$vK(v) - \frac{y(0)}{v} + K(v) = 0 \tag{7}$$

Using initial condition (5), Eq. (7) can be written as:

$$k(y) = \frac{1}{v(1+v)} \tag{8}$$

The inverse Aboodh transform of this equation is simply obtained as

$$y(x) = e^{-x} \tag{9}$$

Where $K(v)$Is the Aboodh transform of the function $y(x)$

2: Applying the ZZ transform of both sides of Eq. (4),

$$H\left\{\frac{dy}{dx}\right\} + H\{y\} = H\{0\}, \text{ so} \tag{10}$$

Using the differential property of ZZ transform Eq.(6) can be written as:

$$\frac{s}{u}z(u,s) - \frac{s}{u}y(0) + z(u,s) = 0 \tag{11}$$

Using initial condition (5), Eq. (11) can be written as

$$z(u,s)\left[\frac{s}{u} + 1\right] = \frac{s}{u} \quad , \quad z(u,s) = \frac{s}{s+u} \tag{12}$$

The inverse ZZ transform of this equation is simply obtained as

$$y(x) = e^{-x} \tag{13}$$

Where H is the ZZ transform of the function $y(x)$

Example 3.2 Let us consider the second-order differential equation

$$y'' + y = 0 \quad , \tag{14}$$

With the initial condition; $, y(0) = y'(0) = 1$ (15)

Solution:

1: Applying the Aboodh transform of both sides of Eq. (14),

$$\mathrm{A}\{y''\} + \mathrm{A}\{y\} = \mathrm{A}\{0\} \tag{16}$$

Using the differential property of Aboodh transform Eq.(16) can be written as

$$v^2K(v) - 1 + K(v) - \frac{1}{v} = 0 \tag{17}$$

Using initial condition (15), Eq. (17) can be written as

$$k(v) = \frac{1}{v^2+1} + \frac{1}{v(v^2+1)} \tag{18}$$

The inverse Aboodh transform of this equation is simply obtained as

$$y(x) = \cos x + \sin x \tag{19}$$

2: Applying the ZZ transform of both sides of Eq. (14),

$$\mathrm{H}\{y''\} + \mathrm{H}\{y\} = \mathrm{H}\{0\} \qquad , \text{so} \tag{20}$$

Using the differential property of ZZ transform Eq.(20) can be written as:

$$\frac{s^2}{u^2}Z(u,s) - \frac{s^2}{u^2}y(0) - \frac{s}{u}y`(0) + Z(u,s) = 0 \qquad (21)$$

Using initial condition (15), Eq. (21) can be written as

$\frac{s^2}{u^2}Z(u,s) - \frac{s^2}{u^2} - \frac{s}{u} + Z(u,s) = 0,$ therefore

$$Z(u,s) = \frac{s^2}{s^2+u^2} + \frac{su}{s^2+u^2} \qquad (22)$$

The inverse ZZ transform of this equation is simply obtained as

$$y(x) = \cos x + \sin x \qquad (23)$$

Example 3.3 Consider the second-order differential equation

$$y'' - 3y' + 2y = 0 \qquad (24)$$

With the initial condition $, y(0) = 1\ ,\ y'(0) = 4$ (25)

Solution:

1: Applying the Aboodh transform of both sides of Eq. (24),

$$\mathrm{A}\{\,y''\} - \mathrm{A}\{3y'\} + \mathrm{A}\{2y\} = \mathrm{A}\{0\} \qquad (26)$$

Using the differential property of Aboodh transform Eq.(26) can be written as

$$v^2K(v) - \frac{4}{v} - 1 - 3vk(v) + \frac{3}{v} + 2k(v) = 0$$

$$k(v) = \frac{-2}{v(v-1)} + \frac{3}{v(v-2)} \qquad (27)$$

Then take the inverse of Aboodh transform we get

$$y(t) = -2e^{t} + 3e^{2t}\ . \qquad (28)$$

3: Applying the ZZ transform of both sides of Eq. (24),:

$$\mathrm{H}\{\,y''\} - \mathrm{H}\{3y'\} + \mathrm{H}\{2y\} = \mathrm{H}\{0\} \quad \text{, so} \qquad (29)$$

Using the differential property of ZZ transform Eq.(29) can be written as:

$$\frac{s^2}{u^2}Z(u,s) - \frac{s^2}{u^2}f(0) - \frac{s}{u}f'(0) - 3\left(\frac{s}{u}z(u,s) - \frac{s}{u}f(0)\right) + 2Z(u,s) = 0 \quad (30)$$

Using initial condition (25), Eq. (30) can be written as

$$\frac{s^2}{u^2}Z(u,s) - \frac{s^2}{u^2} - 4\frac{s}{u} - 3\frac{s}{u}Z(u,s) - 3\frac{s}{u} + 2Z(u,s) = 0$$

$$Z(u,s) = \frac{su+s^2}{(s-u)(s-2u)} \qquad (31)$$

Solve equation (31) Then take the inverse of ZZ transform we get

$$y(t) = -2e^{t} + 3e^{2t} \qquad (32)$$

Example 3.4

Consider the initial value problem

$$y''(t) + 2y'(t) + 5y(t) = e^{-t}sin\ t \qquad (33)$$

With the initial conditions

$$y(0) = 0\ ,\ y'(0) = 0 \qquad (34)$$

Solution:

1: Applying the Aboodh transform of both sides of Eq. (33),

$$\mathrm{A}\{y''(t)\} + \mathrm{A}\{2y'(t)\} + \mathrm{A}\{5y(t)\} = \mathrm{A}\{e^{-t}sin\ t\} \qquad (35)$$

Using the differential property of Aboodh transform Eq.(35) can be written as

$$v^2k(v)-\frac{f'(0)}{v}-f\ (0)+2vk(v)-2\frac{f(0)}{v}+5k(v)=\frac{1}{v[(1+v)^2+1]} \quad (36)$$

Now applying the initial condition to obtain

$$v^2k(v)+2vk(v)+5k(v)=\frac{1}{v[(1+v)^2+1]}$$

$$k(v)=\frac{1}{3}\left(\frac{1}{v((1+v)^2+1)}-\frac{1}{v((1+v)^2+4)}\right) \quad (37)$$

Now applying the inverse Aboodh transform, we get

$$y(t)=\frac{1}{3}e^{-t}\sin t+\frac{1}{3}e^{-t}\sin 2t$$

$$y(t)=\frac{1}{3}e^{-t}(\sin t+\sin 2t). \quad (38)$$

2: Applying the ZZ transform to both sides of (33) we have

$$H\{\,y''(t)\}+2H\{y'(t)\}+5H\{y(t)\}=H\{e^{-t}sin\ t\} \quad (39)$$

Using the differential property of ZZ transform Eq.(29) can be written as:

$$\frac{s^2}{u^2}Z(u,s)-\frac{s^2}{u^2}y(0)-\frac{s}{u}y'(0)+2\left(Z(u,s)-\frac{s}{u}y(0)\right)+5Z(u,s)=\frac{\frac{s}{u}}{\left(\frac{s}{u}+1\right)^2+1} \quad (40)$$

Now applying the initial condition to obtain

$$\left(\frac{s^2}{u^2}+2+5\right)Z(u,s)=\frac{s}{u}+\frac{\frac{s}{u}}{\left(\frac{s}{u}+1\right)^2+1}$$

$$Z(u,s)=\frac{1}{3}\left(\frac{\frac{s}{u}}{\left(\left(\frac{s}{u}+1\right)^2+1\right)}-\frac{\frac{s}{u}}{\left(\left(\frac{s}{u}+1\right)^2+4\right)}\right)+\frac{\frac{s}{u}}{\left(\left(\frac{s}{u}+1\right)^2+4\right)} \quad (41)$$

Now applying the inverse ZZ transform, we get

$$\mathrm{y(t)}=\frac{1}{3}\mathrm{e}^{-\mathrm{t}}\sin\mathrm{t}+\frac{1}{3}\mathrm{e}^{-\mathrm{t}}\sin 2\mathrm{t}$$

$$\mathrm{y(t)}=\frac{1}{3}\mathrm{e}^{-\mathrm{t}}(\sin\mathrm{t}+\sin 2\mathrm{t}) \quad (42)$$

Example 3.5

Consider the initial value problem

$$ty''(t)-ty'(t)+y(t)=2 \quad (43)$$

With the initial conditions

$$y(0)=2\ ,\ y'(0)=-1 \quad (44)$$

Soluton:

1. Applying the Aboodh transform to both sides of (43) we have

$$\mathrm{A}\{ty''(t)\}-\mathrm{A}\{ty'(t)\}+\mathrm{A}\{y(t)\}=\mathrm{A}\{2\} \quad (45)$$

Using the differential property of Aboodh transform Eq.(45) can be written as

$$-\frac{d}{dv}\left[v^2k(v)-\frac{f'(0)}{v}-f(0)\right]-\frac{1}{v}\left[v^2k(v)-\frac{f'(0)}{v}-f(0)\right]$$
$$+\frac{d}{dv}\left[vk(v)-\frac{f\ (0)}{v}\right]+\frac{1}{v}\left[vk(v)-\frac{f\ (0)}{v}\right]+K(v)=\frac{2}{v^2} \quad (46)$$
$$A[y(t)]=K(v)$$

Now applying the initial condition to obtain

$$(v-v^2)k'(v)+3(1-v)k(v)=\frac{2}{v^2}-\frac{2}{v}$$

Or

$$K'(v)+\frac{3}{v}K(v)=-\frac{2}{v^3(1-v)}-\frac{2}{v^2(1-v)}. \quad (47)$$

Equation (47) is a linear differential equation , which has solution in the form

$$k(v) = \frac{c}{v^3} + \frac{2}{v^2} \quad [\text{ c = constant}]$$

Now applying the inverse Aboodh i transform, we get

$$y(t) = ct + 2 \tag{48}$$

2: Applying the ZZ transform to both sides of (43) we have

$$H\{t\,y''(t)\} - H\{ty'(t)\} + H\{y(t)\} = H\{2\} \tag{49}$$

Using the differential property of ZZ transform Eq.(49) can be written as

$$s\frac{d}{du}Z(u,s) - \frac{s}{u}Z(u,s) + \frac{s}{u}f(0) + \frac{u}{s}\frac{d}{du}\frac{s}{u}Z(u,s) + Z(u,s) + Z(u,s) = 2$$

Now applying the initial condition to obtain

$$sZ'(u,s) - \frac{s}{u}Z(u,s) + \frac{2s}{u} - \frac{u^2}{s}\left(\frac{s}{u}Z'(u,s)\right) - \frac{s}{u^2}Z(u,s) = 2$$

$$Z'(u,s) - \frac{1}{u}Z(u,s) = -\frac{2}{u} \tag{50}$$

Equation (50) is a linear differential equation , which has solution in the form

$$Z(u,s) = 2 + cu$$

Now applying the inverse ZZ transform, we get

$$y(t) = 2 + ct \tag{51}$$

4 Conclusions

The main goal of this paper is to conduct a comparative study between Aboodh Transform and new integrals “ZZ Transform” . The Tow methods are powerful and efficient . Aboodh transform and ZZ transform is a convenient tool for solving differential equations in the time domain .

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