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On Energy Estimates for Damped String-Like Equation Considering Dirichlet, Neumann and Robin Boundary Conditions

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Abstract

This article provides detailed construction of energy estimates of the viscous damping aspects for axially moving string, which is modeled by a linear homogeneous sting-like equation, will be studied. The nine different boundary conditions are considered for the axially moving continua. The problem at hand describes the damped vertical vibrations of string-like equations, for example, a conveyor belt system and a band-saw blade. In this work, the velocity and coefficient of damping are kept positive and fixed. The stability of the system substantially depends upon change in boundary and subsequently boundary conditions. Also a decay in oscillatory energy is observed in all the considered cases of boundary conditions due to viscous damping. In some cases, the belt energy may increase or may decrease due to variations in different parameters . This exposes the uncertainty in these cases.

Keywords: Belt Conveyor, String, Axially translating, Viscous damping

1. Introduction

The class of the vibratory systems is of great importance in mechanical, structural and many other fields of engineering and applied sciences. Axially moving systems are one of them. Such kind of systems are used to represent many physical phenomena mathematically. The energy dissipation for such kind of axially translating systems can be found by using Reynold's transport theorem. Such energy loss is termed as damping (Darmawijoyo & van Horssen 2003, 2002). Conveyor belts (Sandilo & van Horssen 2012), pipes conveying fluids, elevator cables (Sandilo & van Horssen 2011), crane and mine hoists, and such type of mechanical systems and machines are subject to vibration due to different cause factors.

For few decades it has been observed that study and analysis of axially translating systems with damping has been very important factor in design and manufacture aspects of these mechanical systems. It is necessary and important to study ways and methods to minimize unnecessary noise and vibrations from these axially translating systems. It is common experience that severe vibrations cause construction failures. The construction failure of Tacoma Narrows bridge in Washington, USA, is one of the classic examples of structural collapses. Damaging structures is one aspect of vibrations. The other aspects are they cause human discomfort and anxiety due to noise and unwanted sound energy. Keeping in view these aspects of vibrations, it becomes necessary to study and devise methods and techniques to mitigate vibrations from all such type of systems.

Many studies are devoted for damping devices introduced at boundaries to control the vibration through boundaries (Gaiko & van Horssen 2015). There are few studies where damping is projected along the given space domain of the system (Maitlo *et al* 2016). The control of the oscillation amplitudes in case of two dimensional plates the authors have provided the detailed study of dampers (Zarubinskaya & van Horssen 2006). (Akkaya & van Horssen 2015) is devoted for study about reflection and damping properties for a wave equation. The authors in (Zhu & Ni 2000) have studied the energetics of an elevator cable for different BCs. (Wickert & Mote 1989) is devoted for detailed study of energetics of axially transporting continua for string and beam equations. The string is analyzed for fixed supports and the beam is analyzed for simple supports. It is observed that the authors in their work did not provide any detailed piece of research work for energetics considering all possible cases of BCs. In some particular cases damping devices are connected at the middle of string or middle of the beam (Main & Jones 2007). Spatial position of damping devices play very important role. Introducing damping device at right position can damp the system in true technical sense and suppress unnecessary noise and the vibration. If a damping device is introduced at wrong place then it is possible the vibratory energy increases. In such cases the system becomes unstable (Hagedorn & Seemann 1998).

This research article studies the aspects of the viscous damping of an axially translating string. The string exhibits a second order linear homogeneous partial differential equation (PDE). Nine different physical and natural boundary conditions (BCs) are studied in detail. Dirichlet, Neumann and Robin type of BCs are analyzed in detail. Stability and instability conditions for an axially translating system are shown in systematic way. Stabilizing and destabilizing of a system is depending on the design of the pulley-supports. This research work is presented as given below. Mathematical equations of the problem considered in this article and the energy estimates are studied in detail in Section 2. In Section 3 the detailed discussion of the results is provided. As far as authors know, this approach to construct the approximations of the energetics has not been considered before.

2. Governing Equations and the energy estimates

By the application of the Hamilton's principle (Maitlo *et al*), the governing equations can easily be formulated. Based on this principle, dimensionless form of the governing equations is taken into account in following manner,

$$\left(y_{tt} + 2Vy_{xt} + \dot{V}y_x + V^2 y_{xx}\right) - y_{xx} + \delta(y_t + Vy_x) = 0;\ \ 0 < x < 1, t \geq 0. \tag{1}$$

Associated nine physical BCs are given as,

$$\begin{aligned}(i)\ y(0,t) = 0, \text{and}\ y(1,t) = 0;\ \ (ii)\ y_x(0,t) = 0, \text{and}\ y_x(1,t) = 0;\ \ (iii)\ y(0,t) \\ = 0, \text{and}\ y_x(1,t)\ = 0;\ \ (iv)\ y_x(0,t) = 0, \text{and}\ y(1,t) \\ = 0;\ \ (v)\ y_x(0,t) + ky(0,t) = 0, \text{and}\ y(1,t) = 0; (vi)\ y(0,t) \\ = 0, \text{and}\ y_x(1,t) + ky(1,t) = 0; (vii)\ y_x(0,t) = 0, \text{and}\ y_x(1,t) + ky(1,t) \\ = 0; (viii)\ y_x(0,t) + ky(0,t) = 0, \text{and}\ y_x(1,t) = 0; (ix)\ y_x(0,t) + ky(0,t) \\ = 0, \text{and}\ y_x(1,t) + ky(1,t) = 0.\end{aligned} \tag{2}$$

If we consider (1) and (2), y models vertical displacement. First four terms in Eq. (1) contained in braces are the acceleration terms often known as local, Coriolis, tangential and centripetal accelerations. The δ is related to external viscous damping coefficient. k, k_1, and k_2 are the spring constants assumed to be positive and constant. y_{xx} represents the elastic force.

Multiplying both the sides of Eq. (1) with material velocity $(y_t + Vy_x)$ and then by performing long mathematical calculations, it turn out

$$\left(\frac{1}{2}(y_t + Vy_x)^2 + \frac{1}{2}y_x^2\right)_t + \left(\frac{1}{2}V(y_t + Vy_x)^2 - y_x\left(y_t + \frac{1}{2}Vy_x\right)\right)_x = -\delta(y_t + Vy_x)^2. \tag{3}$$

In following equation is the sum of kinetic and potential energy for the particle at any position as given by

$$\widehat{\mathrm{E}}(t) = \frac{1}{2}(y_t + Vy_x)^2 + \frac{1}{2}y_x^2, \tag{4}$$

Combining Eqs. (3) and (4), we arrive at

$$\frac{\partial \widehat{\mathrm{E}}(t)}{\partial t} = -\delta(y_t + Vy_x)^2 - \left(\frac{1}{2}V(y_t + Vy_x)^2 - y_x\left(y_t + \frac{1}{2}Vy_x\right)\right)_x. \tag{5}$$

Total mechanical energy E*(t)* of a moving belt for the given spatial domain is as follows

$$\mathrm{E}(t) = \int_0^1 \widehat{\mathrm{E}}(t)dx. \tag{6}$$

Now, we shall use the Reynold's theorem of transportation. By this theorem, the total time derivative rate of mechanical energy within space domain (0,1) yields,

$$\frac{d\mathrm{E}}{dt} = \int_0^1 \left(\hat{E}(t)\right)_t + V\,\widehat{\mathrm{E}}(t)\Big|_0^1. \tag{7}$$

On the right hand side of Eq. (7), first term shows local change whereas second term is for the energy flux through boundaries.

Now, from Eq. (7) it turns out to be

$$\frac{d\mathrm{E}}{dt} = -\delta\int_0^1 (y_t + Vy_x)^2 dx - \left(\frac{1}{2}V(y_t + Vy_x)^2 - y_x\left(y_t + \frac{1}{2}Vy_x\right)\right)\Bigg|_0^1 + \left(\frac{1}{2}V(y_t + Vy_x)^2 + \frac{1}{2}y_x^2\right)\Bigg|_0^1. \tag{8}$$

Eq. (8) can be simplified and finally reaches at

$$\frac{d\mathrm{E}}{dt} = -\delta\int_0^1 (y_t + Vy_x)^2 dx + y_t(1,t)y_x(1,t) - y_t(0,t)y_x(0,t) + V\left(y_x^2(1,t) - y_x^2(0,t)\right). \tag{9}$$

2.1 Case Studies: Nine Physical Boundary Conditions (BCs)

(i) $y(0,t) = 0,$ and $y(1,t) = 0,$ (pulleys of belt are kept fixed)

$y(0,t) = 0 \Rightarrow y_t(0,t) = 0$ and $y(1,t) = 0 \Rightarrow y_t(1,t) = 0$. Now, Eq. (9) can yield

$$\frac{d\mathrm{E}}{dt} = -\delta\int_0^1 (y_t + Vy_x)^2 dx + y_x(1,t)\left(Vy_x(1,t)\right) - y_x(0,t)\left(Vy_x(0,t)\right). \tag{10}$$

(ii) $y_x(0,t) = 0,$ and $y_x(1,t) = 0,$ (both pulleys have freedom of movement in transversal direction)

So that Eq. (9) arrives at

$$\frac{d\mathrm{E}}{dt} = -\delta\int_0^1 (y_t + Vy_x)^2 dx. \tag{11}$$

(iii) $y(0,t) = 0,$ and $y_x(1,t) = 0,$ (fixed support-free support)

$$\frac{dE}{dt} = -\delta \int_0^1 (y_t + Vy_x)^2 dx - y_x(0,t)(Vy_x(0,t)). \tag{12}$$

(iv) $y_x(0,t) = 0$, and $y(1,t) = 0$, (free support-fixed support)

$$\frac{dE}{dt} = -\delta \int_0^1 (y_t + Vy_x)^2 dx + y_x(1,t)\big(Vy_x(1,t)\big). \tag{13}$$

(v) $y_x(0,t) + ky(0,t) = 0$, and $y(1,t) = 0$, (elastic support-fixed support)

$$\frac{dE}{dt} = -\delta \int_0^1 (y_t + Vy_x)^2 dx + ky(0,t)y_t(0,t) + y_x(1,t)\big(Vy_x(1,t)\big) - y_x(0,t)\big(Vy_x(0,t)\big). \tag{14}$$

(vi) $y(0,t) = 0$, and $y_x(1,t) + ky(1,t) = 0$, (support kept fixed-elastic support)

$$\frac{dE}{dt} = -\delta \int_0^1 (y_t + Vy_x)^2 dx - ky(1,t)y_t(1,t) + y_x(1,t)\big(Vy_x(1,t)\big) - y_x(0,t)\big(Vy_x(0,t)\big). \tag{15}$$

(vii) $y_x(0,t) = 0$, and $y_x(1,t) + ky(1,t) = 0$, (support free-elastic support)

$$\frac{dE}{dt} = -\delta \int_0^1 (y_t + Vy_x)^2 dx - ky(0,t)y_t(0,t) + y_x(1,t)\big(Vy_x(1,t)\big). \tag{16}$$

(viii) $y_x(0,t) + ky(0,t) = 0$, and $y_x(1,t) = 0$, (elastic boundary-free boundary)

$$\frac{dE}{dt} = -\delta \int_0^1 (y_t + Vy_x)^2 dx + ky(0,t)y_t(0,t) - y_x(0,t)\big(Vy_x(0,t)\big). \tag{17}$$

(ix) $y_x(0,t) + k_1 y(0,t) = 0$, and $y_x(1,t) + k_2 y(1,t) = 0$, (elastic support-elastic support)

$$\frac{dE}{dt} = -\delta \int_0^1 (y_t + Vy_x)^2 dx + k_1 y(0,t)y_t(0,t) - k_2 y(1,t)y_t(1,t) + y_x(1,t)\big(Vy_x(1,t)\big) - y_x(0,t)\big(Vy_x(0,t)\big). \tag{18}$$

3. Results and Discussion

Detailed discussion about obtained results in previous section will be devoted in this Section. In Eqs. (10)-(18) the time derivative of a total energy of axially transporting belt equals the net rate of work done. Mathematical terms expressed into Eqs. (10)-(18) have simple physical definitions. In all terms the time derivative of energy the vibratory energy is suppressed by the constant parameter δ times an integral of vertical velocities along space domain (0,1). In case of BCs (free-free) and (fixed-free) as given in cases (ii) and (iii) the system is completely stable. The restoring force u_x does work on the string through the transversal velocity component Vy_x. So in total the term $y_x(1,t)\big(Vy_x(1,t)\big) - y_x(0,t)\big(Vy_x(0,t)\big)$ is the energy exchange between the boundary supports and in magnitude this term is smaller than the term involving δ. For boundary conditions (fixed-fixed) and (free-fixed) as given in Cases (i) and (iv), mechanical system is obviously bounded. Spring force ky, does mechanical work by the term kyy_t through the local velocity component u_t, in case of elastic support. As we know that spring constant k is positive, but y, y_t can turn out to be either both positive or both negative at a time, so the term kyy_t will always remain positive term. On basis of this technical explanation, it can be concluded that belt system remains stable for boundary conditions (fixed-elastic) and (free-elastic) in cases (vi) and (vii), and it is not possible to conclude whether the belt energy decreases or increases for BCs (elastic-fixed), (elastic-free) and (elastic-elastic) in cases (v), (viii) and (ix). Finally, note that if the magnitude of the term kyy_t becomes larger than the first term related to damping, the system will be destabilized in cases (v), (viii) and (ix).

3. Conclusions

In this paper, an initial-boundary value problem for the stretched and tensioned beam equation under several different boundary conditions have been studied in detail. The studied problem is used as a mathematical model for describing lateral vibrations of a class of axially translating systems. The energy estimates provided are based upon the Reynold's transport theorem where the energy inflow and outflow are also considered at boundaries. Such energy inflow and outflow plays very significant role for the dynamics of the problem. It has been shown that for certain boundary conditions system is always stable and in some other cases the system is unstable. This shows how geometry of BCs of an engineering system changes the physics of the system. In some cases, it is not possible to conclude whether the energy decreases or increases. This happens due to fact that parameters are changing in signs.

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