Relation proximal point with some dynamical properties

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Abstract :

In this paper we discussed relation proximal points with many of dynamical properties through studied topological transformation group, and it will given necessary condition for proximal relation to be minimal set, and introduce new define replete set and semi-replete set by using concept of the replete set and semi-replete set as well as we introduce that many of new relations and theorem.

Key words: Proximal point, replete proximal point, syndetic set, semi-replete set, minimal set, almost periodic point.

Introduction :

The first has been introduced the proximal point in topological transformation group \((Z,T,\pi)\) with compact hausdorff \(Z\) was Ellis R. and Gottschalk W.[4][5]. We show relation proximal point with some dynamical properties (fixed point-minimal point-almost periodic), also we studied replete proximal and semi-replete proximal independed on replete set and semi-replete set, and it will be given a necessary and sufficient condition for proximal and semi-replete proximal to be syndetic set, and we studied the image of proximal points under epimomorphism, we give some theorem about proximal points. We use symbol \(\Delta\) to indication the end.

1. Preliminaries:

In this section we given important concepts that we needed in this work.

Definition (1-1) [4]:

A topological group is a set \(T\) with two structures:

1. \(T\) is a group
2. \(T\) is a topological space

Definition (1-2)[4]:

A subset \(\rho\) of \(T\) is said to be \(\{left\},\{right\}\) syndetic in \(T\) if and only if there exists a compact subset \(k\) of \(T\) such that \(\rho k = k \rho = T\).
**Definition (1-3) [4]**

Let \((Z, T)\) be topological group:

1. A subset \(w\) of \(T\) is said to be replete if for each compact set \(\beta\) of \(T\) there exist \(t_1, t_2 \in T\) such that \(t_1 \beta t_2 \subset w\).
2. A subset \(w\) of \(T\) is said to be semi-replete if for each compact set \(\beta\) of \(T\) there exist \(t \in T\) such that \(\beta t \subset w\).

**Definition (1-4) [4]:**

A right topological transformation group is a triple \((Z, T, \pi)\) where \(Z\) is a topological space called the phase space, \(T\) is a topological group called the phase group, \(\pi : Z \times T \rightarrow Z, \pi(z, t) \rightarrow xt\) is a continuous mapping such that:

1. \(ze = z\) \((z \in Z)\), where \(e\) is the identity of \(T\)
2. \((zt)b = z(tb)\) \((z \in Z, z, b \in T)\)

**Definition (1-5) [3]:**

Let \((Z, T, \pi)\) be a topological transformation group

1. A subset \(A \subset Z\) is said to be invariant set if \(AT = A\).
2. A non-empty closed invariant set \(A \subset Z\) is said to be minimal set if it contains no non-empty, proper, closed invariant subset.

**Definition (1-6) [4]:**

Let \((Z, T, \pi)\) be a topological transformation group, then the set \(zT = \{zt: t \in T\}\) is called the orbit of \(z\) and the set \(zT\) the orbit closure of \(z\).

**Definition (1-7):**

Let \((Z, T, \pi)\) be a topological transformation group is said to be strongly effective if for each \(z \in Z, t = e\) there exist \(t \in T\) such that \(zt = z\).

**Definition (1-8) [5]:**

Let \((Z, T, \pi)\) be a topological transformation group:

1. Let \(z \in Z\), is said to be Fixed point under \(T\) if \(zT = z\).
2. Let \(z \in Z\), is said to be minimal points if closure orbit is minimal set.
3. Let \( z \in Z \), is said to be almost periodic points if for each invariant neighborhood \( V \) of \( z \) there exist syndetic subset \( A \) of \( T \) such that \( zA \subset V \).

**Definition (1-9) [5]**

Let \((Z, T, \pi)\) and \((H, G, \sigma)\) be a topological transformation group, and \( \tau: Z \rightarrow H \) be continuous, \( \gamma: T \rightarrow G \) be continuous homomorphism then \((\tau, \gamma): (Z, T, \pi) \rightarrow (H, G, \sigma)\) is said to be homomorphism and \((z, t)\pi\tau = ((z)\pi, (t)\gamma)\sigma\). If for each \( \tau, \gamma \) onto then homomorphism is said to be epimorphism.

**Theorem (1-10):**

Let \((Z, T, \pi)\) be a topological transformation group, \(wg\) be replete subset of \( T \) if and only if \( w \) is replete set.

Proof: Let \( wg \) be replete subset of \( T \) then for each compact subset \( \beta \) of \( T \) there exist \( t_1, t_2 \in T \) such that \( t_1\beta t_2 \subset wg \), since \( T \) group there exist \( g^{-1} \in T \) in such that \( t_1\beta t = t_1\beta t_2 g^{-1} \subset w \) for some \( t \in T \) thus \( w \) is replete set. Same method proof part other.

**Remark (1-11)**

Let \((Z, T, \pi)\) be a topological transformation group, the following statements are valid.

1. if \( K \) compact subset of \( T \) then \( gK \) compact set for some \( g \in T \).
2. if \( A \) syndetic subset of \( T \) then \( gA \) syndetic set for some \( g \in T \).

**Theorem (2-12)**

Let \((Z, T, \pi)\) be a topological transformation group, \((n, m) \in P\), then \((Z, T, \pi)\) strongly effective.

Proof: Let \( n \) and \( m \) are proximal points then for each index \( \varphi \) in \( Z \) there exist a \( t \in T \) such that \((n, m)t \in \varphi\), it is enough to show that \( t = e \), since \( \varphi \) be invariant then \((n, m)t \in \varphi t\) by hypothesis there exist \( t^{-1} \in T \), it follows that \((n, m)e \in \varphi\), and \((n, m)t \cap (n, m)e \neq \emptyset\) this lead to \((n, m)t \subset (n, m)e\) and \((nt, mt) = (ne, me)\).

Clearly \( t = e \) therefore \((Z, T, \pi)\) strongly effective. \( \Delta \)

**2. Main results**

In this section, we introduce proximal point in topological transformation group and show that relation proximal point with dynamical properties.
Definition (2-1) [1,2]

Let \((Z, T, \pi)\) be a topological transformation group, a two points \(n\) and \(m\) of \(Z\) are called proximal proved that for each index \(\varphi\) in \(Z\) there exist a \(t \in T\) such that \((n, m)t \in \varphi\). The set of all proximal pairs are called the proximal relation and are denoted by \(P(Z, T)\) or simply \(P\).

Definition (2-2)

Let \((Z, T, \pi)\) be a topological transformation group, a two points \(n\) and \(m\) of \(Z\) are called replete proximal proved that for each index \(\varphi\) in \(Z\) there exist a replete subset \(w\) of \(T\) such that \((n, m)w \subset \varphi\). The set of all replete proximal pairs are called the replete proximal relation and are denoted by \(RP(Z, T)\) or simply \(RP\).

Definition (2-3)

Let \((Z, T, \pi)\) be a topological transformation group, a two points \(n\) and \(m\) of \(Z\) are called semi-replete proximal proved that for each index \(\varphi\) in \(Z\) there exist a semi-replete subset \(v\) of \(T\) such that \((n, m)v \subset \varphi\). The set of all semi-replete proximal pairs are called the semi-replete proximal relation and are denoted by \(SRP(Z, T)\) or simply \(SRP\).

Remark (2-4)

Let \((Z, T, \pi)\) be a topological transformation group then:

1. \(P(X, T)\) is invariant set.
2. \(P(X, T)\) is close set.

Theorem (2-5)

Let \((Z, T, \pi)\) be a topological transformation abelian group, \((n, m)\) are replete proximal points then \(\varphi\) is invariant.

Proof: We may assume that \(T\) be replete group, let \(n\) and \(m\) are replete proximal points then for each index \(\varphi\) in \(Z\) there exist a replete subset \(w\) of \(T\) such that \((n, m)w \subset \varphi\) and \((n, m)w^T \subset \varphi^T\) so \((n, m)w^T \cap (n, m)w \neq \emptyset\) thus \(\varphi^T \subset (n, m)w \subset \varphi\) since \(\varphi e \subset \varphi^T\), thus \(\varphi\) is invariant under \(T\).

Theorem (2-6)

Let \((Z, T, \pi)\) be a topological transformation abelian group, \((n, m)\) are proximal points if and only if \((n, m)\) are replete proximal points.
Proof: Suppose that $\beta$ be compact subset of $T$, since $n$ and $m$ are proximal points then for each index $\varphi$ in $Z$ there exist a $t \in T$ such that $(n,m)t \in \varphi$, then we have to prove $T$ is semireplete , by hypothesis $\mu t \subset Tt \subset T$ .Then $T$ be semireplete proximal thus and $n$ and $m$ are proximal points. Conversely let $n$ and $m$ are semi-replete proximal points ,then for each index $\varphi$ in $Z$ there exist a semi-replete subset $w$ of $T$ such that $(n,m)w \subset \varphi$, since $w$ semi-replete subset of $T$ then for each compact set $\beta$ of $T$ there exist $t_1, t_2 \in T$ such that $t_1 \beta t_2 \subset T$ , $(n,m)t_1 \beta t_2 \subset (n,m)w \subset \varphi$ so $(n,m)t_1 \beta t_2 T \subset \varphi T$ by theorem (2-5) we obtain $(n,m)t_1 \beta t_2 \subset \varphi$ by hypothesis $(n,m)\beta t_1 t_2 \subset \varphi T$ and $(n,m)\beta T \neq \emptyset$, thus $(n,m)\beta T \subset \varphi$ since $T$ syndetic set and $\beta$ compact set then $(n,m)T \subset \varphi$.\Delta

**Theorem (2-7)**

Let $(Z,T,\pi)$ be a topological transformation group,$(n,m)$ are proximal points uf and only if $(n,m)$ are semi-replete proximal points.

Proof: Suppose that $\mu$ be compact subset of $T$, since $n$ and $m$ are proximal points then for each index $\varphi$ in $Z$ there exist a $t \in T$ such that $(n,m)t \in \varphi$.we need to prove $T$ be semi-replete , by hypothesis $\mu t \subset Tt \subset T$.Then $T$ be semi-replete proximal thus and $n$ and $m$ are proximal points. Conversely let $n$ and $m$ are semi-replete proximal points ,then for each index $\varphi$ in $Z$ there exist a semi-replete subset $w$ of $T$ such that $(n,m)w \subset \varphi$, since $w$ semi-replete subset of $T$ then for each compact set $\mu$ of $T$ there exist $t_1 \in T$ such that $t_1 \mu \subset w$, and $(n,m)t_1 \mu \subset \varphi$ so $(n,m)t_1 \mu T \subset \varphi T$ by theorem (2-5) we obtain $(n,m)t_1 \mu T \subset \varphi$ by hypothesis $(n,m)\mu T \subset \varphi$ since $T$ syndetic set and $\beta$ compact set then $(n,m)T \subset \varphi(n,m)$ are proximal points.\Delta

**Theorem (2-8)**

Let $(Z,T,\pi)$ be a topological transformation abelian group,$(n,m)$ are replete proximal points if and only if $(n,m)$ are semi-replete proximal points.

Proof: let $n$ and $m$ are replete proximal points ,then for each index $\varphi$ in $Z$ there exist a replete subset $w$ of $T$ such that $(n,m)w \subset \varphi$, since $w$ replete subset of $T$ then for each compact set $\beta$ of $T$ there exist $t_1, t_2 \in T$ such that $t_1 \beta t_2 \subset w$, $(n,m)t_1 \beta t_2 \subset (n,m)w \subset \varphi$ by hypothesis we get $\beta t \subset w$, for some $t \in T$ then $w$ be semi-replete proximal subset of $T$ thus$(n,m) \in SRP$. Conversely let $n$ and $m$ are semi-replete proximal points ,then for each index $\varphi$ in $Z$ there exist a semi-replete subset $w$ of $T$ such that $(n,m)w \subset \varphi$, since $w$ semi-replete subset of $T$ then for each compact set $\mu$ of $T$ there exist $t_1 \in T$ such that $t_1 \mu \subset w$ so $t_1 \mu e = t_1 \mu t t^{-1} \subset w$ for some $t^{-1} \in T$ and $t_1 \mu t \subset wt$ then $wt$ replete subset of $T$ from theorem (1-10) we get $w$ be replete set and $(n,m) \in RP.\Delta
**Theorem (2-9)**

Let \((Z, T, \pi)\) be a topological transformation group, \((n, m)\) are proximal points then \((Z, T)\) are pointwise proximal points.

Proof: Let \((n, m) \in P\), it follows that \((n, m)T \subset PT\) clearly by remark (2-4) that \((n, m)T \subset P\) so \((n, m)TT \subset PT\), \((n, m)T^2 \subset P\) again \((n, m)T^2T \subset PT\), \((n, m)T^3 \subset P\) after \(n\)-time we get \((n, m)T^n \subset P\) then we obtain that \((Z \times Z, T)\) are proximal for each point in \(Z \times Z\) therefore \((Z, T)\) are pointwise proximal points. \(\Delta\)

**Theorem (2-10)**

Let \((Z, T, \pi)\) be a topological transformation abelain group , \(F \subset T\) and \((n, m) \in P\) then \(F\) syndetic set .

Proof: Let \(F\) be subset of \(T\), since \((n, m)\) are proximal points then for each index \(\varphi\) in \(Z\) there exist \(t \in T\) such that \((n, m)t \in \varphi\) by theorem (2-5) we have \(\varphi T = \varphi\) and \(\varphi TF = \varphi F\) since \(T\) be syndetic there exist compact set \(\beta\) of \(T\) such that \(\varphi F \beta = \varphi F\ \beta\) since \(T\) be abeline then \(\varphi TF \cap \varphi F \beta \subset \varphi T \cap \varphi F \beta\) and \(\varphi TF \cap \varphi F \beta \neq \emptyset\), Thus \(\varphi T \cap \varphi F \beta \neq \emptyset\) so \(T \subset F\beta\) by hypothesis \(T = F\beta\) then \(F\) syndetic set. \(\Delta\)

**Theorem (2-11)**

Let \((Z, T, \pi)\) be a topological transformation abelain group , \(F \subset T\) and \((n, m) \in RP\) then \(F\) syndetic set .

Proof: Let \((n, m)\) are replete proximal points then for each index \(\varphi\) in \(Z\) there exist replete subset \(E\) of \(T\) such that \((n, m)E \subset \varphi\) since \(E\) replete set then for each compact set \(K\) in \(T\) there exist \(g_1, g_2 \in T\) such that \(g_1, K g_2 \subset E\) since \(T\) by syndetic and \(K\) by remark (1-11 number (1)) we have \(T g_2 \subset E\) and \(T g_2 K \subset E K\) so \(T \subset E K\) by hypothesis we obtain \(T = E K\) therefore \(E\) be syndetic set. \(\Delta\)

**Remark (2-12)**

Let \((Z, T, \pi)\) be a topological transformation abelain group , \(F \subset T\), if \((n, m) \in SRP\) then \(F\) syndetic set .

**Theorem (2-13)**

Let \((Z, T, \pi)\) be a topological transformation group then \(P(X, T)\) is minimal set.

Proof: Let \((n, m) \in P\) and \((n, m)T \subset PT\) by remark (2-4 number (1)) we obtain \((n, m)T \in P\), so \((n, m)T \subset \tilde{P}\) then \((n, m)T \subset P\) by remark (2-4 number (2)) since \((n, m)T\) be least
closed invariant subset \( Z \times Z \) contain \((n, m)\) therefore \( P \subset (n, m)T \) ,thus \( P \subset (n, m)T \). then \( P \) be minimal set and \((n, m)\) be minimal points . \( \Delta \)

**Theorem (2.14)**

Let \((Z, T, \pi)\) be a topological transformation group then \((n, m)\) is fixed points.

**Theorem (2.15)**

Let \((Z, T, \pi)\) be a topological transformation group , \((n, m) \in P\), if and only if \((n, m)\) is almost periodic points.

Proof: Assume that \( V \) be invariant neighborhood of \((n, m)\) and \( A \) subset of \( T \) It is enough to show that \( A \) be syndetic set ,since \((n, m)\) are proximal points then for each index \( \varphi \) in \( Z \) there exist \( t \in T \) such that \((n, m)t \in \varphi \) and \((n, m)A \subset (n, m)T \subset \varphi \). It follows that \( A \) be syndetic set by hypothesis \((n, m)A \subset VA \subset VT \subset V\) ,thus \((n, m)\) is almost periodic points .Conversely assume that \((n, m) \in \varphi \) since \((n, m)\) almost periodic point then for each invariant neighborhood \( V \) of \((n, m)\) there exist syndetic subset \( A \) of \( T \) such that \((n, m)A \subset V\) , \((n, m)Ag \subset Vg = V \) for some \( g \in T\). It is enough to show that \( Ag \) be semi-replete set ,since \( A \) be syndetic set then \( Ag \) be syndetic set by remark (1-11 number (2)) there exist compact subset \( K \) of \( T \) such that \( AgK = T \),for each \( t \in T \) there exist \( a \in A, k \in K \) such that \( agk = t \) so \( agk = tk^{-1}g^{-1} \) for some \( k^{-1}, g^{-1} \in T \), it follows that \( tK \subset Ag \) thus \( Ag \) be semi-replete set by hypothesis it was found \((n, m)Ag \subset \varphi\), then \((n, m)\) be semi-replete proximal points by theorem (2.7) we obtain \((n, m) \in P\). \( \Delta \)

**Theorem (2.16)**

Let\((\tau, \gamma): (Z, T, \pi) \rightarrow (H, G, \sigma)\) epimorphism , \((n, m)\) are proximal points under \( T \) then \(((n)\tau, (m)\tau)\) are proximal points under \( G \).

Proof: Let \((n, m)\) are proximal points then for each index \( \varphi \) in \( Z \) there exist \( t \in T \) such that \((n, m)t \in \varphi\). It follows that from define homomorphism we obtain \(((n)\tau, (m)\tau)T\gamma \in (\varphi)\tau\), since \( \gamma \) be onto then we get \(((n)\tau, (m)\tau)G \in (\varphi)\tau\) therefore \(((n)\tau, (m)\tau)G \subset (\varphi)\tau\) thus \(((n)\tau, (m)\tau)\) are proximal points under \( G\). \( \Delta \).
References


