Jordan (σ,τ) - Higher Homomorphisms of a ring R into a ring R'

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Abstract

Let R, R' be two prime rings and σ\textsuperscript{n}, τ\textsuperscript{n} be two higher homomorphisms of a ring R for all n \in \mathbb{N}, in the present paper we show that under certain conditions of R, every Jordan (σ,τ)-higher homomorphism of a ring R into a prime ring R' is either (σ,τ)-higher homomorphism or (σ,τ)-higher anti homomorphism.

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Key Words: prime ring, homomorphism, Jordan higher homomorphism.

1. Introduction

A ring R is called a prime if aRb = (0) implies a = 0 or b = 0, where a, b \in R, this definition is due to [3].

A -ring R is called semiprime if aRa = (0) implies = 0, such that a \in R, this definition is due to [3].

Let R be a 2-torsion free semiprime ring and suppose that a, b \in R if arb + bra = 0, for all r \in R, then arb = bra = 0 , this definition is due to [3].

Let R be a ring then R is called 2-torsion free if 2a = 0 implies a = 0, for every a \in R, this definition is due to [3].

Let 0 be an additive mapping of a ring R into a ring R', 0 is called a homomorphism if 0(a b) = 0(a) 0(b).

And 0 is called a Jordan homomorphism if for all a, b \in R

0(a b + b a) = 0(a) 0(b) + 0(b) 0(a) for all a, b \in R , [4].

Let \(\theta=\langle\phi_i\rangle_{i\in\mathbb{N}}\) be a family of additive mappings of aring R into aring R'. Then \(\theta\) is said to be higher homomorphism if for every n \in \mathbb{N} we have \(\phi_n(ab) = \sum_{i=1}^{n} \phi_i(a)\phi_i(b)\), for all a,b \in R, and Let \(\theta=\langle\phi_i\rangle_{i\in\mathbb{N}}\) be a family of additive mappings of aring R into aring R'.

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Then $\theta$ is said to be Jordan higher homomorphism if for every $n \in \mathbb{N}$ we have

$$\phi_n(ab + ba) = \sum_{i=1}^{n} \phi_i(a)\phi_i(b) + \sum_{i=1}^{n} \phi_i(b)\phi_i(a),$$

for all $a, b \in R$, [1].

Also, N.Jacobson and C.E.Rickart [4] proved that every Jordan homomorphism of a ring into an integral domain is either homomorphism or an anti homomorphism.

Later 1956, I.N.Herstein [2] proved that every Jordan homomorphism of a ring into prime ring of characteristic different from 2 and 3 is either a homomorphism or an anti homomorphism. In 1957, M.F.Smiley [5] simplified the result as:

Every Jordan homomorphism of a ring into a 2-torsion free prime ring is either a homomorphism or an anti homomorphism. After this in 1969, I.N.Herstein [3] proved that every Jordan homomorphism of a ring onto prime ring is either a homomorphism or an anti homomorphism. In 2006 A.K.Faraj [1] proved that every Jordan higher homomorphism of a ring $R$ onto a 2-torsion free prime ring $R'$ is either higher homomorphism or higher anti homomorphism.

Now, the main purpose of this paper is that every Jordan $(\sigma, \tau)$-higher homomorphism of a ring $R$ into a prime ring $R'$ is either $(\sigma, \tau)$-higher homomorphism or $(\sigma, \tau)$-higher anti homomorphism and every Jordan$(\sigma, \tau)$-higher homomorphism from a ring $R$ into a 2-torsion free ring $R'$ such that, such that $\sigma^{i^2} = \sigma^i$, $\tau^{i^2} = \tau^i$, $\sigma^i \tau^i = \sigma^i \tau^{n-i}$ and $\sigma^i \tau^i = \tau^i \sigma^i$. Then $\theta$ is Jordan triple $(\sigma, \tau)$-higher homomorphism.

**2- Jordan$(\sigma, \tau)$-Higher Homomorphisms on Rings**

**Definition (2.1):**

Let $\theta = (\phi_i)_{i \in \mathbb{N}}$ be a family of additive mappings of a ring $R$ into a ring $R'$ and $\sigma, \tau$ be two endomorphisms of $R$. $\theta$ is called a $(\sigma, \tau)$-higher homomorphism if

$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)),$$

for all $a, b \in R$ and $n \in \mathbb{N}$.

**Example (2.2):**

Let $S_1, S_2$ be two rings and $\theta = (\theta_i)_{i \in \mathbb{N}}$ be a $(\sigma, \tau)$-higher homomorphism of a ring $S_1$ into a ring $S_2$. Let $R = S_1 \oplus S_1$ and $R' = S_2 \oplus S_2$. Let $\phi = (\phi_i)_{i \in \mathbb{N}}$ be a family of additive mappings of a ring $R$ into a ring $R'$, such that $\phi_n((a,b)) = (\theta_n(a), \theta_n(b))$, for all $(a,b) \in R$.

Let $\sigma_1^i, \tau_1^i$ be two endomorphisms of $R$, such that
\[ \sigma_i^n ((a,b)) = (\sigma^0(a), \sigma^0(b)), \quad \tau_i^n ((a,b)) = (\tau^0(a), \tau^0(b)). \]

Then \( \phi_n \) is a \((\sigma, \tau)\)-higher homomorphism.

**Definition (2.3):**

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a family of additive mappings of a ring \( R \) into a ring \( R' \) and \( \sigma, \tau \) be two endomorphisms of \( R \). \( \theta \) is called **Jordan \((\sigma, \tau)\)-higher homomorphism** if

\[
\phi_i(ab + b'a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(a)), \quad \text{for all } a, b \in R \text{ and } n \in \mathbb{N}.
\]

**Remark (2.4):**

Clearly every \((\sigma, \tau)\)-higher homomorphism is Jordan \((\sigma, \tau)\)-higher homomorphism but the converse is not true in general, as shown by the following example:

**Example (2.5):**

Let \( S \) be any ring with nontrivial involution * and \( R = S \oplus S \), such that \( a \in Z(S), a^2 = a \) and \( s_1as_2 = 0 \), for all \( s_1, s_2 \in S \). Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a family of additive mappings of a ring \( R \) into itself defined by:

\[
\phi_i((s,t)) = \begin{cases} 
(2 - n)as, (n-1)t \ast, & n = 1, 2 \\
0, & n \geq 3
\end{cases}, \quad \text{for all } (s,t) \in R.
\]

Let \( \sigma^n, \tau^n \) be two endomorphisms of \( R \), such that \( \sigma^n((s,t)) = (ns,t), \tau^n((s,t)) = (n^2s,t) \). Then \( \theta \) is a Jordan \((\sigma, \tau)\)-higher homomorphism but not \((\sigma, \tau)\)-higher homomorphism.

**Definition (2.6):**

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a family of additive mappings of a ring \( R \) into a ring \( R' \) and \( \sigma, \tau \) be two endomorphisms of \( R \). \( \theta \) is called a **Jordan triple \((\sigma, \tau)\)-higher homomorphism** if

\[
\phi_i(aba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b))\phi_i(\tau^i(a)), \quad \text{for all } a, b \in R \text{ and } n \in \mathbb{N}.
\]

**Definition (2.7):**

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a family of additive mappings of a ring \( R \) into a ring \( R' \) and \( \sigma, \tau \) be two endomorphisms of \( R \). \( \theta \) is called a **\((\sigma, \tau)\)-higher anti homomorphism** if

\[
\phi_i(ab) = \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(a)), \quad \text{for all } a, b \in R \text{ and } n \in \mathbb{N}.
\]

**Lemma (2.3.8):**

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple \((\sigma, \tau)\)-higher homomorphism of a ring \( R \) into a ring \( R' \), then for all \( a, b, c \in R \) and \( n \in \mathbb{N} \),
(i) $\phi_n(ab \cdot c + cba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) +$

$\sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))$

(ii) In particular, if $R, R'$ be two commutative rings and $R'$ is a 2-torsion free ring, then

$\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))$. 

**Proof:**

(i) Replace $a + c$ for $a$ in Definition (2.6), we get:

$\phi_i((a+c)b(a+c)) = \sum_{i=1}^{n} \phi_i(\sigma^i(a+c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a+c))$

$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a) + \tau^i(c))$

$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) +$ 

$\sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) \cdots (1)$

On the other hand:

$\phi_i((a+c)b(a+c)) = \phi_n(aba + abc + cba + cbc)$

$= \sum_{i=1}^{n} \phi_i(\sigma^i(aba))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(abc))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) + \phi_n(abc + cba)$

Comparing (1) and (2), we get:

$\phi_n(ab \cdot c + cba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))$

(ii) By (i) and since $R, R'$ be two commutative rings and $R'$ is a 2-torsion free ring

$\phi_n(abc + abc) = 2\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))$

**Definition (2.9):**

Let $\theta = (\phi_i)_{i \in \mathbb{N}}$ be a Jordan $(\sigma, \tau)$-higher homomorphism of a ring $R$ into a ring $R'$, then for all $a, b \in R$ and $n \in \mathbb{N}$, we define $G_n: R \times R \rightarrow R'$ by:

$G_n(a, b) = \phi_n(ab) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b))$

**Lemma (2.10):**

If $\theta = (\phi_i)_{i \in \mathbb{N}}$ be a Jordan $(\sigma, \tau)$-higher homomorphism of a ring $R$ into a ring $R'$, then for all $a, b, c \in R$ and $n \in \mathbb{N}$:

(i) $G_n(a, b) - G_n(b, a)$
\(G_n(a + b, c) = G_n(a, c) + G_n(b, c)\)

(ii) \(G_n(a, b + c) = G_n(a, b) + G_n(a, c)\)

**Proof:**

(i) By Definition (2.3):

\[
\phi_n(ab + ba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) + \phi_i(\tau^i(b))\phi_i(\sigma^i(a))
\]

\[
= \phi_n(ab) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) = -\phi_n(ba) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(a))
\]

\(G_n(a, b) = -G_n(b, a)\)

(ii) \(G_n(a + b, c) = \phi_n((a + b)c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a + b))\phi_i(\tau^i(c))\)

\[
= \phi_n(ac + bc) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(c)) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(c))
\]

\(= G_n(a, c) + G_n(b, c)\)

(iii) \(G_n(a, b + c) = \phi_n((a + c)b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a + c))\phi_i(\tau^i(b))\)

\[
= \phi_n(ab + ac) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) = -\phi_n(ba + ac) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(c))
\]

\(= G_n(a, b) + G_n(a, c)\)

**Remark (2.11):**

Note that \(\theta = (\phi_i)_{i \in \mathbb{N}}\) is a \((\sigma, \tau)\)-higher homomorphism of a ring \(R\) into a ring \(R'\) if and only if \(G_n(a, b) = 0\) for all \(a, b \in R\) and \(n \in \mathbb{N}\).

**Lemma (2.12):**

Let \(\theta = (\phi_i)_{i \in \mathbb{N}}\) be a Jordan \((\sigma, \tau)\)-higher homomorphism of a ring \(R\) into a ring \(R'\), such that \(\sigma^{\tau^i} = \tau^i\sigma^n\), \(\sigma^i\tau^i = \tau^i\sigma^i\) and \(\sigma^i\tau^i = \tau^i\sigma^i\), then for all \(a, b, m \in R\) and \(n \in \mathbb{N}\)

\(G_n(\sigma^n(a), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(a)) = G_n(\sigma^n(b), \sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a), \tau^n(b)) = 0\)

**Proof:**

We prove by using the induction, for \(n = 1\)

Let \(w = abmba + bamab\), since \(\theta\) is Jordan \((\sigma, \tau)\)-homomorphism

\(\theta(w) = \theta(a(bmb)a + b(ama)b)\)

\(= \theta(\sigma(a)) \theta(\sigma(\tau(bmb))) \theta(\tau(a)) + \theta(\sigma(b)) \theta(\sigma(\tau(ama))) \theta(\tau(b))\)

\(= \theta(\sigma(a)) \theta(\sigma(\tau(b))) \theta(\sigma(\tau(m))) \theta(\tau(\sigma(a))) \theta(\tau(b))\)


\[
\theta(\sigma(b)) \theta(\sigma(\tau(a))) \theta(\sigma(\tau(m))) \theta(\tau(\sigma(a))) \theta(\tau(b)) \quad \ldots(1)
\]

On the other hand

\[
\theta(w) = \theta((ab) \ m (ba) + (ba) \ m (ab))
\]

\[
= \theta(\sigma(ab)) \theta(\sigma(m)) \theta(\tau(ba)) + \theta(\sigma(ba)) \theta(\sigma(m)) \theta(\tau(ab))
\]

\[
= \theta(\sigma(ab)) \theta(\sigma(m)) (\theta(\sigma(a)) \theta(\tau^2(b)) + \theta(\sigma(b)) \theta(\tau^2(a)) - \\
\theta(\tau(ab))) + ( - \theta(\sigma(ab)) + \theta(\sigma^2(a)) \theta(\tau\sigma(b)) + \\
\theta(\tau^2(b)) \theta(\tau\sigma(a)) \theta(\sigma(m)) \theta(\tau(ab))
\]

\[
= - \theta(\sigma(ab)) \theta(\sigma(m)) (\theta(\tau(ab))) - \theta(\sigma(\sigma(a))) \theta(\tau^2(b)) - \\
\theta(\sigma(\sigma(\sigma(a)))) \theta(\tau^2(b)) \theta(\tau(a)) - \\
\theta(\sigma(b)) \theta(\sigma^2(\tau(a)) \theta(\sigma^2(\tau^2(m)) \theta(\tau^2(a))) \theta(\tau(b))
\]

Since \( \sigma^2 = \sigma \) and \( \tau^2 = \tau \)

\[
0 = - \theta(\sigma(ab)) \theta(\sigma(m)) G(\tau(a), \tau(b)) - \theta(\sigma(ab)) \theta(\sigma(m)) G(\tau(b), \tau(a)) + \\
\theta(\sigma(a)) \theta(\tau(\tau(b)) \theta(\sigma(m)) \theta(\tau(ab)) + \theta(\sigma(b)) \theta(\tau(\tau(a)) \theta(\sigma(m)) \\
\theta(\tau(ab)) - \theta(\sigma(a)) \theta(\tau\sigma(b)) \theta(\sigma(m) \theta(\tau(\sigma(b))) \theta(\tau(a)) - \\
\theta(\sigma(b)) \theta(\tau\sigma(a)) \theta(\sigma(m) \theta(\tau\sigma(a)) \theta(\tau(b))
\]

\[
0 = - \theta(\sigma(ab)) \theta(\sigma(m)) G(\tau(a), \tau(b)) - \theta(\sigma(ab)) \theta(\sigma(m)) G(\tau(b), \tau(a)) + \\
\theta(\sigma(a)) \theta(\tau(\tau(b)) \theta(\sigma(m) \theta(\tau(ab)) + \theta(\sigma(b)) \theta(\tau(\tau(a)) \theta(\sigma(m)) \\
\theta(\tau(ab)) - \theta(\tau(\tau(b))) \theta(\sigma(m)) \theta(\tau(ab)) - \theta(\sigma(\tau(b))) \theta(\tau(b))
\]

\[
0 = - \theta(\sigma(ab)) \theta(\sigma(m)) G(\tau(a), \tau(b)) - \theta(\sigma(ab)) \theta(\sigma(m)) G(\tau(b), \tau(a)) + \\
\theta(\sigma(a)) \theta(\tau\sigma(b)) \theta(\sigma(m)) \theta(\tau(\sigma(b))) \theta(\tau(a)) + \\
\theta(\sigma(b)) \theta(\tau\sigma(a)) \theta(\sigma(m)) \theta(\tau(\sigma(a))) \theta(\sigma(m)) \theta(\tau(\sigma(b))) \theta(\tau(a))
\]

\[
G(\tau(a), \tau(b)) = 0 \quad \text{for all} \ a, b, m \in \mathbb{R}, \text{and} \ s, \ n \in \mathbb{N}, \ s < n.
\]
Let \( w = abmba+bamab \)

Since 0 is a Jordan \((\sigma, \tau)-\)higher homomorphism, then

\[
\phi_n(w) = \phi_n(abmba + b(ama)b)
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(\tau^{-1}(bmba)))\phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\sigma^i(\tau^{-1}(ama)))\phi_i(\tau^i(b))
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a))(\sum_{j=1}^{i} \phi_j(\sigma^j(\tau^{-1}(b)))\phi_j(\sigma^j(\tau^{-1}(m)))\phi_j(\tau^j(\tau^{-1}(b))))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))(\sum_{j=1}^{i} \phi_j(\sigma^j(\tau^{-1}(a)))\phi_j(\sigma^j(\tau^{-1}(m)))\phi_j(\tau^j(\tau^{-1}(a))))\phi_i(\tau^i(b))
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(\tau^{-1}(b)))\phi_i(\tau^i(\tau^{-1}(b)))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\sigma^i(\tau^{-1}(a)))\phi_i(\tau^i(\tau^{-1}(a)))\phi_i(\tau^i(b))
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(\tau^{-1}(b)))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\sigma^i(\tau^{-1}(a)))\phi_i(\tau^i(b))
\]

\[
= \phi_n(\sigma^0(a))\phi_n(\sigma^0(\tau^{-1}(b)))\phi_n(\sigma^0(\tau^{-1}(m)))\phi_n(\tau^0(a)) + \sum_{i=1}^{n} \phi_i(\tau^i(a))\phi_i(\tau^i(b)) + \phi_n(\sigma^0(b))\phi_n(\sigma^0(\tau^{-1}(a)))\phi_n(\sigma^0(\tau^{-1}(m)))\phi_n(\tau^0(b)) + \sum_{i=1}^{n} \phi_i(\tau^i(\tau^{-1}(a)))\phi_i(\tau^i(b))
\]

On the other hand:

\[
\phi_n(w) = \phi_n((ab)m(ba) + (ba)m(ab))
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(ab))\phi_i(\sigma^i(\tau^{-1}(b)))\phi_i(\tau^i(ba)) + \sum_{i=1}^{n} \phi_i(\sigma^i(ba))\phi_i(\sigma^i(\tau^{-1}(m)))\phi_i(\tau^i(ab))
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(ab))\phi_i(\sigma^i(\tau^{-1}(b)))\phi_i(\tau^i(ba)) + \sum_{i=1}^{n} \phi_i(\sigma^i(ba))\phi_i(\sigma^i(\tau^{-1}(m)))\phi_i(\tau^i(ab)) + \sum_{i=1}^{n} \phi_i(\sigma^i(ab))\phi_i(\sigma^i(\tau^{-1}(m)))\phi_i(\tau^i(ab)) + \sum_{i=1}^{n} \phi_i(\sigma^i(ba))\phi_i(\sigma^i(\tau^{-1}(m)))\phi_i(\tau^i(ab))
\]

\[
= -\sum_{i=1}^{n} \phi_i(\sigma^i(ab))\phi_i(\sigma^i(\tau^{-1}(b)))\phi_i(\tau^i(ab)) - \sum_{i=1}^{n} \phi_i(\sigma^i(ba))\phi_i(\sigma^i(\tau^{-1}(b)))\phi_i(\tau^i(ab)) + \sum_{i=1}^{n} \phi_i(\sigma^i(ab))\phi_i(\sigma^i(\tau^{-1}(m)))\phi_i(\tau^i(ab)) + \sum_{i=1}^{n} \phi_i(\sigma^i(ba))\phi_i(\sigma^i(\tau^{-1}(m)))\phi_i(\tau^i(ab))
\]

\[
\phi_i(\sigma^i\tau^{-1}(m))\phi_i(\tau^i(ab)) + \sum_{i=1}^{n} \phi_i(\sigma^i(ba))\phi_i(\sigma^i(\tau^i(ab))\phi_i(\sigma^i\tau^{-1}(m))\phi_i(\tau^i(ab))
\]
By our hypothesis, we have:

\[ G_n(\sigma^n(a), \sigma^n(b))G_n(\tau^n(a), \tau^n(b)) = 0. \]
Lemma (2.13):

Let $\theta = (\phi_i)_{i \in \mathbb{N}}$ be a Jordan $(\sigma, \tau)$-higher homomorphism from a ring $R$ into a 2-torsion freer prime ring $R'$, then for all $a, b, m \in R$ and $n \in \mathbb{N}$

$$G_n(\sigma^n(a), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(a)) =$$

$$G_n(\sigma^n(b), \sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a), \tau^n(b)) = 0$$

Proof:

By Lemma (2.12), we have:

$$G_n(\sigma^n(a), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(a)) +$$

$$G_n(\sigma^n(b), \sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a), \tau^n(b)) = 0$$

And by Lemma (Let R be a 2-torsion free semiprime ring and suppose that $a, b \in R$ if $arb + bra = 0$, for all $r \in R$, then $arb = bra = 0$), we get:

$$G_n(\sigma^n(a), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(a)) =$$

$$G_n(\sigma^n(b), \sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a), \tau^n(b)) = 0$$

Lemma (2.14):

Let $\theta = (\phi_i)_{i \in \mathbb{N}}$ be a Jordan $(\sigma, \tau)$-higher homomorphism from a ring $R$ into a prime ring $R'$, then for all $a, b, c, d, m \in R$ and $n \in \mathbb{N}$

$$G_n(\sigma^n(a), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d), \tau^n(c)) = 0$$

Proof:

Replacing $a + c$ for $a$ in Lemma (2.13), we get:

$$G_n(\sigma^n(a + c), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(a + c)) =$$

$$G_n(\sigma^n(b), \sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a), \tau^n(b)) = 0$$

By Lemma (2.12), we get:

$$G_n(\sigma^n(a), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(c)) +$$

$$G_n(\sigma^n(c), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(a)) +$$

$$G_n(\sigma^n(c), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(c)) = 0$$

Therefore, we get:

$$G_n(\sigma^n(a), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(c)) +$$

$$G_n(\sigma^n(c), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(a)) = 0$$

Hence, by the primness of $R'$:

$$G_n(\sigma^n(a), \sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b), \tau^n(c)) = 0$$

... (1)
Now, replacing $b+d$ for $b$ in Lemma (2.13), we get:

\[ G_n(\sigma^n(a), \sigma^n(b + d))\phi_h(\sigma^n(m))G_n(\tau^n(b + d), \tau^n(a)) = 0 \]
\[ G_n(\sigma^n(a), \sigma^n(b))\phi_h(\sigma^n(m))G_n(\tau^n(b), \tau^n(a)) + \]
\[ G_n(\sigma^n(a), \sigma^n(b))\phi_h(\sigma^n(m))G_n(\tau^n(d), \tau^n(a)) + \]
\[ G_n(\sigma^n(a), \sigma^n(d))\phi_h(\sigma^n(m))G_n(\tau^n(b), \tau^n(a)) + \]
\[ G_n(\sigma^n(a), \sigma^n(d))\phi_h(\sigma^n(m))G_n(\tau^n(d), \tau^n(a)) = 0 \]

By Lemma (2.13), we get:

\[ G_n(\sigma^n(a), \sigma^n(b))\phi_h(\sigma^n(m))G_n(\tau^n(d), \tau^n(a)) = 0 \]

Therefore, we get:

\[ G_n(\sigma^n(a), \sigma^n(b))\phi_h(\sigma^n(m))G_n(\tau^n(d), \tau^n(a)) = 0 \]

Since $M'$ is a prime ring, then:

\[ -G_n(\sigma^n(a), \sigma^n(b))\phi_h(\sigma^n(m))G_n(\tau^n(d), \tau^n(a))\phi_h(\sigma^n(m)) = 0 \]

By (1), (2) and Lemma (2.13), we get:

\[ G_n(\sigma^n(a), \sigma^n(b))\phi_h(\sigma^n(m))G_n(\tau^n(d), \tau^n(c)) = 0 \]

3- The main result

**Theorem (3.1):**

Every Jordan $(\sigma, \tau)$-higher homomorphism from a ring $R$ into a prime ring $R'$ is either $(\sigma, \tau)$-higher homomorphism or $(\sigma, \tau)$-higher anti homomorphism.

**Proof:**

Let $\theta = (\phi_i)_{i \in \mathbb{N}}$ be a Jordan $(\sigma, \tau)$-higher homomorphism of a ring $R$ into a prime ring $R'$.

Then by Lemma (2.14):
\[ G_n(\sigma^n(a),\sigma^n(b))\phi_i(\sigma^n(m))G_n(\tau^n(d),\tau^n(c)) = 0 \]

Since R' is a prime ring therefore either \[ G_n(\sigma^n(a),\sigma^n(b)) = 0 \] or \[ G_n(\tau^n(d),\tau^n(c)) = 0 \], for all \( a, b, c, d \in R \) and \( n \in N \).

If \[ G_n(\tau^n(d),\tau^n(c)) \neq 0 \] for all \( c, d \in R \) and \( n \in N \) then \[ G_n(\sigma^n(a),\sigma^n(b)) = 0 \] for all \( a, b \in R \) and \( n \in N \), hence, we get \( 0 \) is a \((\sigma,\tau)\)-higher homomorphism.

But if \[ G_n(\tau^n(d),\tau^n(c)) = 0 \] for all \( c, d \in R \) and \( n \in N \) then \( 0 \) is a \((\sigma,\tau)\)-higher anti homomorphism.

**Proposition (3.2):**

Let \( \theta = (\phi_i)_{i \in N} \) be a Jordan \((\sigma,\tau)\)-higher homomorphism from a ring \( R \) into 2-torsion free ring \( R' \), such that \( \sigma^{i^2} = \sigma^i, \tau^{i^2} = \tau^i, \sigma^i \tau^i = \sigma^i \tau^{n-1} \) and \( \sigma^i \tau^i = \tau^i \sigma^i \). Then \( \theta \) is a Jordan triple \((\sigma,\tau)\)-higher homomorphism.

**Proof:**

Replace \( b \) by \( ab + ba \) in Definition (2.3), we get:

\[ \phi(n(a(ab + ba)) + (ab + ba)a) \]

\[ = \sum_{i=1}^{n} \phi(\sigma^i(a))\phi_i(\tau^i(ab + ba)) + \sum_{i=1}^{n} \phi(\sigma^i(ab + ba))\phi_i(\tau^i(a)) \]

\[ = \sum_{i=1}^{n} \phi(\sigma^i(a))\phi_i(\tau^i(a)b + \tau^i(b)a)) + \sum_{i=1}^{n} \phi(\sigma^i(ab + ba))\phi_i(\tau^i(a)) \]

\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\left( \sum_{j=1}^{n} \phi(\sigma^j\tau^i(a))\phi_i(\tau^j(b)) + \sum_{j=1}^{n} \phi(\sigma^j\tau^i(b))\phi_i(\tau^j(a)) \right) + \]

\[ \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \phi(\sigma^j\tau^i(a))\phi_i(\tau^j\sigma^i(b)) + \sum_{j=1}^{n} \phi(\sigma^j\tau^i(b))\phi_i(\tau^j\sigma^i(a)) + \right) \phi_i(\tau^i(a)) \]

\[ \sigma^{i^2} = \sigma^i, \tau^{i^2} = \tau^i, \sigma^i \tau^i = \sigma^i \tau^{n-1} \text{ and } \sigma^i \tau^i = \tau^i \sigma^i, \text{ we get} \]

\[ \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i \tau^{n-1}(a))\phi_i(\tau^i(b)) + 2 \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i \tau^{n-1}(b))\phi_i(\tau^i(a)) + \]

\[ \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\sigma^i \tau^{n-1}(a))\phi_i(\tau^i(a)) \]

\[ \ldots (1) \]

On the other hand:

\[ \phi_i(a(ab + ba) + (ab + ba)a) = \phi_i(a(ab + ba) + aba + bca) \]

\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i \tau^{n-1}(a))\phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\sigma^i \tau^{n-1}(a))\phi_i(\tau^i(a)) + 2 \phi_i(aba) \ldots (2) \]

Compare (1) and (2), we get:

\[ 2\phi_i(ab) = 2 \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i \tau^{n-1}(b))\phi_i(\tau^i(a)). \]
Since $R'$ is a 2-torsion free ring, we obtain that $\theta$ is a Jordan triple $(\sigma, \tau)$-higher homomorphism.

References:


