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Jordan (σ,τ) - Higher Homomorphisms of a ring R into a ring R'

Fawaz Raad Jarulla[1] Claus Haetinger [2]
Department of Mathematics,college of Education, Al-Mustansirya University , Iraq[1]
Center of Exact and Technological Sciences, UNIVATES University Center, Brazil [2]

Abstract

Let R , R' be two prime rings and σ^n,τ^n be two higher homomorphisms of a ring R for all n ∈ N , in the present paper we show that under certain conditions of R , every Jordan (σ,τ)-higher homomorphism of a ring R into a prime ring R' is either (σ,τ)-higher homomorphism or (σ,τ)-higher anti homomorphism.

Mathematic subject classification : 16 N 60 , 16 U 80 .
Key Words: prime ring , homomorphism , Jordan higher homomorphism.

1.Introduction

A ring R is called a prime if aRb = (0) implies a = 0 or b = 0, where a , b ∈ R, this definition is due to [3] .

A -ring R is called semiprime if aRa = (0) implies = 0, such that a ∈ R, this definition is due to [3] .

Let R be a 2-torsion free semiprime ring and suppose that a , b ∈ R if $a$rb + br$a$ = 0 , for all r ∈ R, then $a$rb = br$a$ = 0 , this definition is due to [3].

Let R be a ring then R is called 2-torsion free if 2a = 0 implies a = 0, for every a ∈ R, this definition is due to [3].

Let θ be an additive mapping of a ring R into a ring R', θ is called a homomorphism if θ(a b) = θ(a) θ(b).

And θ is called a Jordan homomorphism if for all a, b ∈ R

θ(a b + b a) = θ(a) θ(b) + θ(b) θ(a) for all a , b ∈ R , [4] .

Let $\theta=(\phi_i)_{i\in N}$ be a family of additive mappings of aring R into aring R' . Then θ is said to be higher homomorphism if for every n∈N we have $\phi_n(ab)=\sum_{i=1}^{n}\phi_i(a)\phi_i(b)$, for all a,b∈R ,and Let $\theta=(\phi_i)_{i\in N}$ be a family of additive mappings of aring R into aring R' .

Then θ is said to be Jordan higher homomorphism if for every n∈N we have $\phi_n(ab+ba)=\sum_{i=1}^{n}\phi_i(a)\phi_i(b)+\sum_{i=1}^{n}\phi_i(b)\phi_i(a)$, for all a,b ∈ R , [1] .

Also, N.Jacobson and C.E.Rickart [4] proved that every Jordan homomorphism of a ring into an integral domain is either homomorphism or an anti homomorphism.
Later 1956 , I.N.Herstein [2] proved that every Jordan homomorphism of a ring into prime ring of characteristic different from 2 and 3 is either a homomorphism or an anti homomorphism.In 1957, M.F.Smiley [5] simplified the result as:

Every Jordan homomorphism of a ring into a 2-torsion free prime ring is either a homomorphism or an anti homomorphism. After this in 1969 ,I.N.Herstein [3] proved that every Jordan homomorphism of a ring onto prime ring is either a homomorphism or an anti homomorphism. In 2006 A.K.Faraj [1] proved that every Jordan higher homomorphism of a ring R onto a 2-torsion free prime ring R' is either higher homomorphism or higher anti homomorphism .

Now, the main purpose of this paper is that every Jordan (σ,τ)-higher homomorphism of a ring R into a prime ring R' is either (σ,τ)-higher homomorphism or (σ,τ)- higher anti homomorphism and every Jordan(σ,τ)-higher homomorphism from a ring R into a 2-torsion free ring R' such that , such that $,\sigma^{i^2}=\sigma^i,\ \tau^{i^2}=\tau^i,\ \sigma^i\tau^i=\sigma^i\tau^{n-i}$ and $\sigma^i\,\tau^i=\tau^i\sigma^i$ .Then θ is Jordan triple (σ,τ)-higher homomorphism .

2- Jordan(σ,τ)-Higher Homomorphisms on Rings

Definition (2.1):

Let $\theta=(\phi_i)_{i\in N}$ be a family of additive mappings of a ring R into a ring R' and σ,τ be two endomorphisms of R . θ is called a **(σ,τ)-higher homomorphism** if

$\phi_n(ab)=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\tau^i(b))$, for all *a*, *b* ∈ R and n ∈ N .

Example (2.2):

Let S_1, S_2 be two rings and $\theta=(\theta_i)_{i\in N}$ be a (σ,τ)-higher homomorphism of a ring S_1 into a ring S_2. Let R= $S_1\oplus S_1$ and R'= $S_2\oplus S_2$. Let $\phi=(\phi_i)_{i\in N}$ be a family of additive mappings of a ring R into a ring R', such that

$\phi_n((a,b))=(\theta_n(a),\theta_n(b))$, for all $(a,b)\in$ R.

Let σ_1^n,τ_1^n be two endomorphisms of R, such that

$\sigma_1^n((a,b)) = (\sigma^n(a),\sigma^n(b))$, $\tau_1^n((a,b)) = (\tau^n(a),\tau^n(b))$.

Then ϕ_n is a (σ,τ)-higher homomorphism.

Definition (2.3):

Let $\theta = (\phi_i)_{i\in N}$ be a family of additive mappings of a ring R into a ring R'

and σ,τ be two endomorphisms of R. θ is called **Jordan (σ,τ)-higher homomorphism** if

$\phi_n(ab+ba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(a))$, for all $a, b \in$ R and n $\in$ N .

Remark(2.4):

Clearly every (σ,τ)-higher homomorphism is Jordan (σ,τ)-higher homomorphism but the converse is not true in general, as shown by the following example :

Example (2.5):

Let S be any ring with nontrivial involution $*$ and R = S $\oplus$ S , such that $a \in Z(S)$, $a^2 = a$ and $s_1 a s_2 = 0$, for all $s_1, s_2 \in S$. Let $\theta = (\phi_i)_{i\in N}$ be a family of additive mappings of a ring R into itself defined by:

$$\phi_n((s,t)) = \begin{cases} ((2-n)as,(n-1)t^*) & , n=1,2 \\ 0 & , n \geq 3 \end{cases}$$, for all (s,t) $\in$ R.

Let σ^n, τ^n be two endomorphisms of R, such that $\sigma^n((s,t)) = (ns,t)$, $\tau^n((s,t)) = (n^2 s,t)$. Then θ is a Jordan (σ,τ)-higher homomorphism but not (σ,τ)-higher homomorphism.

Definition (2.6):

Let $\theta = (\phi_i)_{i\in N}$ be a family of additive mappings of a ring R into a ring R' and σ,τ be two endomorphisms of R . θ is called a **Jordan triple (σ,τ)-higher homomorphism** if

$\phi_n(aba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\, \phi_i(\sigma^i \tau^{n-i}(b))\, \phi_i(\tau^i(a))$, for all $a, b \in$ R and n $\in$ N .

Definition (2.7):

Let $\theta = (\phi_i)_{i\in N}$ be a family of additive mappings of a ring R into a ring R into a ring R'

and σ,τ be two endomorphisms of R . θ is called a **(σ,τ)-higher anti homomorphism** if

$\phi_n(ab) = \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(a))$, for all $a, b \in$ R and n $\in$ N .

Lemma (2.3.8):

Let $\theta = (\phi_i)_{i\in N}$ be a Jordan triple (σ,τ)-higher homomorphism of a ring R into a ring R', then for all $a, b, c \in$ R and n $\in$ N

(i) $$\phi_n(abc + cba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))$$

(ii) In particular , if R, R' be two commutative rings and R' is a 2-torsion free ring , then

$$\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\,\phi_i(\sigma^i\tau^{n-i}(b))\,\phi_i(\tau^i(c)).$$

Proof:

(i) Replace $a + c$ for a in Definition (2.6) , we get :

$$\phi_n((a+c)b(a+c)) = \sum_{i=1}^{n} \phi_i(\sigma^i(a+c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a+c))$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a) + \sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a) + \tau^i(c))$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) +$$

$$\sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) \quad \ldots(1)$$

On the other hand:

$$\phi_n((a+c)b(a+c)) = \phi_n(aba + abc + cba + cbc)$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) + \phi_n(abc + cba)$$

…(2)

Comparing (1) and (2), we get:

$$\phi_n(abc + cba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))$$

(ii) By (i) and since R, R' be two commutative rings and R' is a 2-torsion free ring

$$\phi_n(abc + abc) = 2\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))$$

Definition (2.9):

Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism of a ring R into a ring R', then for all $a, b \in$ R and n $\in$ N, we define G_n:R×R ⟶R' by:

$$G_n(a,b) = \phi_n(ab) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b))$$

Lemma (2.10):

If $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism of a ring R into a ring R', then for all $a, b, c \in$ R and n $\in$ N:

(i) $G_n(a,b) - G_n(b,a)$

(ii) $G_n(a + b,c) = G_n(a,c) + G_n(b,c)$

(iii) $G_n(a,b + c) = G_n(a,b) + G_n(a,c)$

Proof:

(i) By Definition (2.3):

$$\phi_n(ab + ba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) + \phi_i(\tau^i(b))\phi_i(\sigma^i(a))$$

$$= \phi_n(ab) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) = -(\phi_n(ba) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(a)))$$

$G_n(a,b) = -G_n(b,a)$

(ii) $$G_n(a+b,c) = \phi_n((a+b)c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a+b))\phi_i(\tau^i(c))$$

$$= \phi_n(ac + bc) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(c)) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(c))$$

$$= \phi_n(ac) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(c)) + \phi_n(bc) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\tau^i(c))$$

$= G_n(a,c) + G_n(b,c)$

(iii) $$G_n(a,b + c) = \phi_n(a(b + c)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b + c))$$

$$= \phi_n(ab + ac) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(c))$$

$$= \phi_n(ab) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) + \phi_n(ac) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(c))$$

$= G_n(a,b) + G_n(a,c)$

Remark (2.11):

Note that $\theta = (\phi_i)_{i\in N}$ is a (σ,τ)-higher homomorphism of a ring R into a ring R' if and only if $G_n(a,b) = 0$ for all $a, b \in R$ and $n \in N$.

Lemma (2.12):

Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism of a ring R into a ring R', such that $\sigma^{n^2} = \sigma^n$, $\tau^n\sigma^n = \sigma^n$, $\sigma^i\tau^{n-i} = \tau^i\sigma^i$ and $\sigma^i\tau^i = \tau^i\sigma^i$, then for all $a, b, m \in R$ and $n \in N$

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) +$$
$$G_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) = 0$$

Proof:

We prove by using the induction, for n = 1

Let $w = abmba + bamab$, since θ is Jordan (σ,τ)-homomorphism

$\theta(w) = \theta(a(bmb)a + b(ama)b)$

$= \theta(\sigma(a))\ \theta(\sigma\tau(bmb))\ \theta(\tau(a)) + \theta(\sigma(b))\ \theta(\sigma\tau(ama))\ \theta(\tau(b))$

$= \theta(\sigma(a))\ \theta(\sigma(\sigma\tau(b)))\ \theta(\sigma\tau(\sigma\tau(m)))\ \theta(\tau(\sigma\tau(b)))\ \theta(\tau(a)) +$

$\theta(\sigma(b))\ \theta(\sigma(\sigma\tau(a)))\ \theta(\sigma\tau(\sigma\tau(m)))\ \theta(\tau(\sigma\tau(a)))\ \theta(\tau(b))$ …(1)

On the other hand

$\theta(w) = \theta((ab)\ m\ (ba) + (ba)\ m\ (ab))$

$= \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ \theta(\tau(ba)) + \theta(\sigma(ba))\ \theta(\sigma\tau(m))\ \theta(\tau(ab))$

$= \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ (\theta(\sigma\tau(a))\ \theta(\tau^2(b)) + \theta(\sigma\tau(b))\ \theta(\tau^2(a)) -$

$\theta(\tau(ab))) + (-\ \theta(\sigma(ab)) + \theta(\sigma^2(a))\ \theta(\tau\sigma(b)) +$

$\theta(\sigma^2(b))\ \theta(\tau\sigma(a)))\ \theta(\sigma\tau(m))\ \theta(\tau(ab))$

$= -\ \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ (\theta(\tau(ab)) - \theta(\sigma\tau(a))\ \theta(\tau^2(b))) -$

$\theta(\sigma(ab))\ \theta(\sigma\tau(m))\ (\theta(\tau(ab)) - \theta(\sigma\tau(b))\ \theta(\tau^2(a))) +$

$\theta(\sigma^2(a))\ \theta(\tau\sigma(b))\ \theta(\sigma\tau(m))\ \theta(\tau(ab)) + \theta(\sigma^2(b))\ \theta(\tau\sigma(a))\ \theta(\sigma\tau(m))\ \theta(\tau(ab))$ …(2)

Compare (1), (2) and since $\sigma\tau = \tau\sigma$

$0 = -\ \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ G(\tau(a),\tau(b)) - \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ G(\tau(b),\tau(a)) +$

$\theta(\sigma^2(a))\ \theta(\tau\sigma(b))\ \theta(\sigma\tau(m))\ \theta(\tau(ab)) + \theta(\sigma^2(b))\ \theta(\tau\sigma(a))\ \theta(\sigma\tau(m))$

$\theta(\tau(ab)) - \theta(\sigma(a))\ \theta(\sigma^2\tau(b))\ \theta(\sigma^2\tau^2(m)\ \theta(\sigma\tau^2(b))\ \theta(\tau(a)) -$

$\theta(\sigma(b))\ \theta(\sigma^2\tau(a))\ \theta(\sigma^2\tau^2(m)\ \theta(\sigma\tau^2(a))\ \theta(\tau(b))$

Since $\sigma^2 = \sigma$ and $\tau^2 = \tau$

$0 = -\ \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ G(\tau(a),\tau(b)) - \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ G(\tau(b),\tau(a)) +$

$\theta(\sigma(a))\ \theta(\tau\sigma(b))\ \theta(\sigma\tau(m))\ \theta(\tau(ab)) + \theta(\sigma(b))\ \theta(\tau\sigma(a))\ \theta(\sigma\tau(m))$

$\theta(\tau(ab)) - \theta(\sigma(a))\ \theta(\tau\sigma(b))\ \theta(\sigma\tau(m)\ \theta(\sigma\tau(b))\ \theta(\tau(a)) -$

$\theta(\sigma(b))\ \theta(\tau\sigma(a))\ \theta(\sigma\tau(m)\ \theta(\sigma\tau(a))\ \theta(\tau(b))$

$0 = -\ \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ G(\tau(a),\tau(b)) - \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ G(\tau(b),\tau(a)) +$

$\theta(\sigma(a))\ \theta(\tau\sigma(b))\ \theta(\sigma\tau(m))\ (\theta(\tau(ab)) - \theta(\sigma\tau(b))\ \theta(\tau(a))) +$

$\theta(\sigma(b))\ \theta(\tau\sigma(a))\ \theta(\sigma\tau(m))\ (\theta(\tau(ab)) - \theta(\sigma\tau(a))\ \theta(\tau(b)))$

$0 = -\ \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ G(\tau(a),\tau(b)) - \theta(\sigma(ab))\ \theta(\sigma\tau(m))\ G(\tau(b),\tau(a)) +$

$\theta(\sigma(a))\ \theta(\tau\sigma(b))\ \theta(\sigma\tau(m))\ G(\tau(b),\tau(a)) + \theta(\sigma(b))\ \theta(\tau\sigma(a))\ \theta(\sigma\tau(m))$
$G(\tau(a),\tau(b))$

$0 = -\ (\theta(\sigma(ab)) - \theta(\sigma(b))\ \theta(\tau\sigma(a))\ \theta(\sigma\tau(m))\ G(\tau(a),\tau(b)) -$

$(\theta(\sigma(ab) - \theta(\sigma(a))\ \theta(\tau\sigma(b))\ \theta(\sigma\tau(m))\ G(\tau(b),\tau(a))$

Thus, we have:

$G(\sigma(a),\sigma(b))\ \theta(\sigma\tau(m))\ G(\tau(b),\tau(a)) + G(\sigma(b),\sigma(a))\ \theta(\sigma\tau(m))\ G(\tau(a),\tau(b)) = 0.$

Now, we can assume that:

$G_s(\sigma^s(a),\sigma^s(b))\phi_s(\sigma^s(m))G_s(\tau^s(b),\tau^s(a)) +$

$G_s(\sigma^s(b),\sigma^s(a))\phi_s(\sigma^s(m))G_s(\tau^s(a),\tau^s(b)) = 0$, for all $a, b, m \in R$, and $s, n \in N$, $s < n$.

Let w = *abmba+bamab*

Since θ is a Jordan (σ,τ)-higher homomorphism , then

$\phi_n(w) = \phi_n(a(bmb)a + b(ama)b)$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(bmb))\phi_i(\tau^i(a)) + \sum_{i=1}^{n}\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(ama))\phi_i(\tau^i(b))$$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\left(\sum_{j=1}^{i}\phi_j(\sigma^j(\sigma^j\tau^{n-j}(b)))\phi_j(\sigma^j\tau^{n-j}(\sigma^j\tau^{n-j}(m)))\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\right)\phi_i(\tau^i(a)) +$$

$$\sum_{i=1}^{n}\phi_i(\sigma^i(b))\left(\sum_{j=1}^{i}\phi_j(\sigma^j(\sigma^j\tau^{n-j}(a)))\phi_j(\sigma^j\tau^{n-j}(\sigma^j\tau^{n-j}(m)))\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\right)\phi_i(\tau^i(b))$$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(b)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\phi_i(\tau^i(\sigma^i\tau^{n-i}(b)))\phi_i(\tau^i(a)) +$$

$$\sum_{i=1}^{n}\phi_i(\sigma^i(b))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(a)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\phi_i(\tau^i(\sigma^i\tau^{n-i}(a)))\phi_i(\tau^i(b))$$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(b)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\phi_j(\tau^j(a)) +$$

$$\sum_{i=1}^{n}\phi_i(\sigma^i(b))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(a)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\phi_j(\tau^j(b))$$

$$=\phi_n(\sigma^n(a))\phi_n(\sigma^n(\sigma^n(b)))\phi_n(\sigma^n(\sigma^n(m)))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\phi_j(\tau^j(a)) +$$

$$\sum_{i=1}^{n-1}\phi_i(\sigma^i(a))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(b)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\phi_j(\tau^j(a)) +$$

$$\phi_n(\sigma^n(b))\phi_n(\sigma^n(\sigma^n(a)))\phi_n(\sigma^n(\sigma^n(m)))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\phi_j(\tau^j(b)) +$$

$$\sum_{i=1}^{n-1}\phi_i(\sigma^i(b))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(a)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\phi_j(\tau^j(b))\ldots(3)$$

On the other hand:

$\phi_n(w) = \phi_n((ab)m(ba) + (ba)m(ab))$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ba)) + \sum_{i=1}^{n}\phi_i(\sigma^i(ba))\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab))$$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(ab))\phi_i((\sigma^i\tau^{n-i}(m))\left(\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\phi_j(\tau^{j^2}(b)) + \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\phi_j(\tau^{j^2}(a)) - \phi_i(\tau^i(ab))\right) +$$

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{i}\phi_j(\sigma^{j^2}(a))\phi_j(\tau^j\sigma^j(b)) + \sum_{j=1}^{i}\phi_j(\sigma^{j^2}(b))\phi_j(\tau^j\sigma^j(a)) - \phi_i(\sigma^i(ab))\right)\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab))$$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\phi_j(\tau^{j^2}(b)) + \sum_{i=1}^{n}\phi_i(\sigma^i(ab))\phi_i((\sigma^i\tau^{n-i}(m))\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\phi_j(\tau^{j^2}(a)) -$$

$$\sum_{i=1}^{n}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab)) + \sum_{i=1}^{n}\phi_i(\sigma^{i^2}(a))\phi_i(\tau^i\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab)) + \sum_{i=1}^{n}\phi_i(\sigma^{i^2}(b))\phi_i(\tau^i\sigma^i(a))$$

$$\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab)) - \sum_{i=1}^{n}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab))$$

$$=-\sum_{i=1}^{n}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))(\phi_i(\tau^i(ab)) - \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\phi_j(\tau^{j^2}(b))) -$$

$$\sum_{i=1}^{n}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))(\phi_i(\tau^i(ab)) - \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\phi_j(\tau^{j^2}(a))) + \sum_{i=1}^{n}\phi_i(\sigma^{i^2}(a))\phi_i(\tau^i\sigma^i(b))$$

$$\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab)) + \sum_{i=1}^{n}\phi_i(\sigma^{i^2}(b))\phi_i(\tau^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab))$$

$$= -\phi_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) - \sum_{i=1}^{n-1}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b)) -$$
$$\phi_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) - \sum_{i=1}^{n-1}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b)) +$$
$$\phi_n(\sigma^{n^2}(a))\phi_n(\tau^n\sigma^n(b))\phi_n(\sigma^n(m))\phi_n(\tau^n(ab)) + \sum_{i=1}^{n-1}\phi_i(\sigma^{i^2}(a))\phi_i(\tau^i\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab)) +$$
$$\phi_n(\sigma^{n^2}(b))\phi_n(\tau^n\sigma^n(a))\phi_n(\sigma^n(m))\phi_n(\tau^n(ab)) + \sum_{i=1}^{n-1}\phi_i(\sigma^{i^2}(b))\phi_i(\tau^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab))\ldots(4)$$

Compare (3), (4) and since $\sigma^{n^2} = \sigma^n$, $\tau^n\sigma^n = \sigma^n$, $\sigma^i\tau^{n-i} = \tau^i\sigma^i$, $\sigma^i\tau^i = \tau^i\sigma^i$

$$0 = -\phi_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) - \phi_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) +$$
$$\phi_n(\sigma^n(a))\phi_n(\sigma^n(b))\phi_n(\sigma^n(m))(\phi_n(\tau^n(ab)) - \sum_{i=1}^{n}\phi_i(\tau^i(\sigma^i\tau^{n-i}(b)))\phi_i(\tau^i(a))) +$$
$$\phi_n(\sigma^n(b))\phi_n(\sigma^n(a))\phi_n(\sigma^n(m))(\phi_n(\tau^n(ab)) - \sum_{i=1}^{n}\phi_i(\tau^i(\sigma^i\tau^{n-i}(a)))\phi_i(\tau^i(b))) -$$
$$\sum_{i=1}^{n-1}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b)) - \sum_{i=1}^{n-1}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))$$
$$G_i(\tau^i(b),\tau^i(a)) + \sum_{i=1}^{n-1}\phi_i(\sigma^i(a))\phi_i(\tau^i\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(m))(\phi_i(\tau^i(ab)) -$$
$$\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\phi_j(\tau^j(a))) + \sum_{i=1}^{n-1}\phi_i(\sigma^i(b))\phi_i(\tau^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))$$
$$(\phi_i(\tau^i(ab)) - \sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\phi_j(\tau^j(b)))$$

$$0 = -\phi_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) - \phi_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) +$$
$$\phi_n(\sigma^n(a))\phi_n(\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) + \phi_n(\sigma^n(b))\phi_n(\sigma^n(a))\phi_n(\sigma^n(m))$$
$$G_n(\tau^n(a),\tau^n(b)) - \sum_{i=1}^{n-1}\phi_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b)) - \sum_{i=1}^{n-1}\phi_i(\sigma^i(ab))$$
$$\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(b),\tau^i(a)) + \sum_{i=1}^{n-1}\phi_i(\sigma^i(a))\phi_i(\tau^i\sigma^i(b))\ \phi_i(\sigma^i\tau^{n-i}(m))$$
$$G_i(\tau^i(b),\tau^i(a)) + \sum_{i=1}^{n-1}\phi_i(\sigma^i(b))\phi_i(\tau^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b))$$

$$0 = -G_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) -$$
$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) -$$
$$\sum_{i=1}^{n-1}G_i(\sigma^i(b),\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b)) -$$
$$\sum_{i=1}^{n-1}G_i(\sigma^i(a),\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(b),\tau^i(a))$$

$$0 = -(G_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) +$$
$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))) -$$
$$(\sum_{i=1}^{n-1}G_i(\sigma^i(b),\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b)) +$$
$$\sum_{i=1}^{n-1}G_i(\sigma^i(a),\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(b),\tau^i(a)))$$

By our hypothesis, we have:

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) +$$
$$G_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) = 0\cdot$$

Lemma (2.13):

Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism from a ring R into a 2- torsion freer prime ring R', then for all *a*, *b*, m $\in$ R and n $\in$ N

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) =$$

$$G_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) = 0$$

Proof:

By Lemma (2.12), we have:

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) +$$

$$G_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) = 0$$

And by Lemma (Let R be a 2-torsion free semiprime ring and suppose that *a* , b $\in$ R if *a*rb + br*a* = 0 , for all r $\in$ R, then *a*rb = br*a* = 0), we get:

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) =$$

$$G_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) = 0$$

Lemma (2.14):

Let $\theta = (\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism from a ring R into a prime ring R', then for all *a*, *b*, *c*, *d*, m $\in$ R and n $\in$ N

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(c)) = 0$$

Proof:

Replacing $a + c$ for a in Lemma (2.13) , we get:

$$G_n(\sigma^n(a+c),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a+c)) = 0$$

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) +$$
$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c)) +$$
$$G_n(\sigma^n(c),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) +$$
$$G_n(\sigma^n(c),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c)) = 0$$

By Lemma (2.13), we get:

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c)) +$$
$$G_n(\sigma^n(c),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) = 0$$

Therefore, we get:

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))\phi_n(\sigma^n(m))$$
$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c)) = 0$$

$$= -G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))\phi_n(\sigma^n(m))$$
$$G_n(\sigma^n(c),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) = 0$$

Hence, by the primness of R':

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c)) = 0 \qquad \ldots(1)$$

Now, replacing $b+d$ for b in Lemma (2.13) , we get:

$$G_n(\sigma^n(a),\sigma^n(b+d))\phi_n(\sigma^n(m))G_n(\tau^n(b+d),\tau^n(a))=0$$

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))+$$
$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))+$$
$$G_n(\sigma^n(a),\sigma^n(d))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))+$$
$$G_n(\sigma^n(a),\sigma^n(d))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))=0$$

By Lemma (2.13), we get:

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))+$$
$$G_n(\sigma^n(a),\sigma^n(d))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))=0$$

Therefore, we get:

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))\phi_n(\sigma^n(m))$$
$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))=0$$

$$=-G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))\phi_n(\sigma^n(m))$$
$$G_n(\sigma^n(a),\sigma^n(d))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))=0$$

Since M' is a prime ring, then:

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))=0 \quad \ldots(2)$$

Thus, $G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b+d),\tau^n(a+c))=0$

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))+$$
$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))+$$
$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))+$$
$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(c))=0$$

By (1), (2) and Lemma (2.13), we get:

$$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(c))=0\cdot$$

3- The main result

Theorem (3.1):

Every Jordan (σ,τ)-higher homomorphism from a ring R into a prime ring R' is either (σ,τ)-higher homomorphism or (σ,τ)-higher anti homomorphism.

Proof:

Let $\theta=(\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism of a ring R into a prime ring R'. Then by Lemma (2.14) :

$G_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(c))=0$

Since R' is a prime ring therefore either $G_n(\sigma^n(a),\sigma^n(b))=0$ or

$G_n(\tau^n(d),\tau^n(c))=0$, for all *a*, *b*, *c*, *d* ∈R and n∈N.

If $G_n(\tau^n(d),\tau^n(c))\neq 0$ for all *c*, *d* ∈ R and n ∈ N then $G_n(\sigma^n(a),\sigma^n(b))=0$

for all *a*, *b* ∈ R and n ∈ N, hence, we get θ is a (σ,τ)-higher homomorphism.

But if $G_n(\tau^n(d),\tau^n(c))=0$ for all *c*, *d* ∈ R and n ∈ N then θ is a (σ,τ)-higher anti homomorphism.

Proposition (3.2):

Let $\theta=(\phi_i)_{i\in N}$ be a Jordan (σ,τ)-higher homomorphism from a ring R into 2-torsion free ring R', such that , $\sigma^{i^2}=\sigma^i$, $\tau^{i^2}=\tau^i$, $\sigma^i\tau^i=\sigma^i\tau^{n-i}$ and $\sigma^i\tau^i=\tau^i\sigma^i$ .Then θ is a Jordan triple (σ,τ)-higher homomorphism.

Proof:

Replace *b* by *ab* + *ba* in Definition (2.3) , we get :

$\phi n(a(ab+ba)+(ab+ba)a)$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\tau^i(ab+ba))+\sum_{i=1}^{n}\phi_i(\sigma^i(ab+ba))\phi_i(\tau^i(a))$$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\tau^i(a)\tau^i(b)+\tau^i(b)\tau^i(a))+\sum_{i=1}^{n}\phi_i(\sigma^i(a)\sigma^i(b)+\sigma^i(b)\sigma^i(a))\phi_i(\tau^i(a))$$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\left(\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\phi_j(\tau^{j^2}(b))+\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\phi_j(\tau^{j^2}(a))\right)+$$
$$\sum_{i=1}^{n}\left(\sum_{j=1}^{i}\phi_j(\sigma^{j^2}(a))\phi_j(\tau^j\sigma^j(b))+\sum_{j=1}^{i}\phi_j(\sigma^{j^2}(b))\phi_j(\tau^j\sigma^j(a))\right)\phi_i(\tau^i(a))$$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^i(a))\phi_i(\tau^{i^2}(b))+\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^i(b))\phi_i(\tau^{i^2}(a))+$$
$$\sum_{i=1}^{n}\phi_i(\sigma^{i^2}(a))\phi_i(\tau^i\sigma^i(b))\phi_i(\tau^i(a))+\sum_{i=1}^{n}\phi_i(\sigma^{i^2}(b))\phi_i(\sigma^i\tau^i(a))\phi_i(\tau^i(a))$$

$\sigma^{i^2}=\sigma^i$, $\tau^{i^2}=\tau^i$, $\sigma^i\tau^i=\sigma^i\tau^{n-i}$ and $\sigma^i\tau^i=\tau^i\sigma^i$, we get

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(a))\phi_i(\tau^i(b))+2\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))+$$
$$\sum_{i=1}^{n}\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(a))\phi_i(\tau^i(a))\qquad ...(1)$$

On the other hand:

$$\phi_n(a(ab+ba)+(ab+ba)a)=\phi_n(aab+aba+aba+baa)$$

$$=\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(a))\phi_i(\tau^i(b))+\sum_{i=1}^{n}\phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(a))\phi_i(\tau^i(a))+2\phi_n(aba)...(2)$$

Compare (1) and (2), we get:

$$2\phi_n(aba)=2\sum_{i=1}^{n}\phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)).$$

Since R' is a 2-torsion free ring, we obtain that θ is a Jordan triple (σ,τ)-higher homomorphism.

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