

FIXED POINT METRICALLY CONVEX METRIC SPACE

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Abstract

In this paper we establish a fixed point theorem for the hybrid pair of multivalued and single valued nonself JSR mapping in metrically convex metric space. Also we give example in support of the result.

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Introduction and Preliminaries

A bulk of literature exist with commuting pairs and its weaker forms such as weakly commuting ,compatible, compatible of type A, D-compatible, semi compatible , etc. new pair termed as JSR mapping which is defined by Shrivastav R et. el.[4] in fuzzy menger space and will prove a fixed point theorem for hybrid pair of multivalued and single valued nonself mapping satisfying the ϕ-contraction in a metrically convex metric space.

Let (X,d) be a metric space. Then, following **Nadler** [2] ,we have

CB(X) = {A:A is a nonempty closed and bounded subset of X},

C(X)= {A:A is a nonempty compact subset of X },

BN(X) = {A:A is a nonempty bounded subset of X}.

For non empty subsets A and B of X and $x \in X$,

$D(A,B) = \inf \{d(a,b): a\in A \text{ and } b\in B\}$,

$H(A,B) \max[\sup\{D(a,B): a\in A\}, \sup\{D(A,b): b\in B\}]$,

$\delta(A,B) = \sup\{d(a,b) : a\in A \text{ and } b\in B\}$,

$d'(x,A) = \inf\{d(x,a): a\in A\}$,

∂K is boundary of K.

A metric space (X,d) is said to be metrically convex if for any $x, y \in X$ $(x\neq y)$ $\partial z \in X$ $(x\neq y\neq z)$ such that $d(x,z) + d(z,y) = d(x,y)$. Further if K is a non empty closed subset of X and $x \in K$, $y \notin K$, then there exists a point z $\in \partial z$ such that d(x,z) + d(z,y) = d(x,y).

The following lemmas are from **Rus[3]** and **Khan [1]**.

Lemma 1 Let $A \subset CB(X)$ and $0 < \theta < 1$ be given. Then for every $x \in A$ there exists a point $a \in A$ such that

$d(x,z) \geq \theta\, \delta(x,A)$ and $d(x,z) \geq \theta\, \delta(x,A)$.

Lemma 2 For any $x \in X$, and any A,B in CB(X),

$$|d'(x,A)-d'(x,B)| \leq H(A,B).$$

Lemma 3 For any $x,y \in X$ and $A \subseteq X$, $|d'(x,A)-d'(y,A)| \leq d(x,y)$

Let K be nonempty closed subset of a metric space (X,d).A mapping

$T:K \to CB(X)$ is said to be continuous at $x_0 \in K$ if for any $\varepsilon>0$, there exists $\delta>0$ such that $H(Tx,Tx_0)<\varepsilon$,whenever $d(x,x_0)<\delta$. If T is continuous at every point of K, we say that T is continuous at K.

Let $\Re^+$ be the set of non-negative real and ψ the set of function $\phi:(\Re^+)^5 \to \Re$

Satisfying the following properties:

(i) ϕ is continuous and increasing in each co-ordinate variable

(ii) $\phi(1,1,1,1,1) = h < 1$ $(h \in \Re^+)$

(iii) Either $u \leq \phi(u,v,u,v,v)$ or $u \leq \phi(v,u,v,u,v)$ or $u \leq \phi(v,u,v,v,u)$ implies $u \leq hv$

Let S and T be two self maps of a metric space (X,d). The pair {S,T} is said to be S-JSR mappings iff

$$\alpha\, d(STx_n,Tx_n) \leq \alpha\, d(SSx_n,Sx_n)$$

where α = lim Sup or lim inf and $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ *for some t in X*.

Example Let X = [0,1] with d(x,y) =|x-y| and S,T are two self mapping on X defined by

$Sx = \frac{2}{x+2}$ *and* $Tx = \frac{1}{x+1}$ for $x \in X$.

Now we have the sequence $\{x_n\}$ in X is defined as $x_n = 1/n$,$n\in N$. Then we have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1$$

$|STx_n-Tx_n| \to 1/3$ and $|SSx_n-Sx_n| \to 2/3$ as $n\to\infty$.

Clearly we have

$|TSx_n-Tx_n| < |SSx_n-Sx_n|$.

Thus pair {S,T} is S-JSR mapping. But This pair is neither compatible nor weakly compatible nor other non commuting mapping. Hence pair of JSR mapping is more general then others.

Let S self map of a metric space (X,d) and T be multivalued map . The pair {S,T} is said to be hybrid T-JSR mappings iff

$$\alpha\, d'(TSx_n,Sx_n) \le \alpha\, H(TTx_n,Tx_n)$$

where α = lim Sup or lim inf and $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t\in X$.

Main Result

Theorem 1 Let (X,d) be a complete and metrically convex metric space and K be a nonempty closed subset of X. Let $T:K\to CB(X)$ and $S:K\to X$ such that

(i) $\partial K\subseteq SK$, $TK\subseteq SK$; $Sx \in \partial K \Rightarrow TX \subseteq K$,

$H(Tx,Ty) \le \phi[d(Sx,Sy), d'(Sx,Tx), d'(Sy,Ty)\}, d'(Sx,Ty), d'(Sy,Tx)]$

For all $x,y \in K$

(ii) {T,S} is hybrid T-JSR pair,

(iii) SK is closed

then there exists a point p in K such that $p=Sp \in Tp$ i.e. p is common fixed point.

Proof . Construct the sequences $\{x_n\}$and $\{y_n\}$ in the following way:

Let $x\in\partial K$, then there exists a point $x_0 \in K$ such that $x=Sx_0$ as $\partial K \subseteq SK$. Form $Sx_0\in\partial K$ and by the implication of $Sx\in\partial K \Rightarrow Tx\subseteq SK$, we conclude that

$$d(y_1,y_2)\le \frac{1}{\sqrt{h}} H(Tx_0,Tx_1)$$

$$\le \phi[d(Sx_0,Sx_1),\{d'(Sx_0,Tx_0),d'(Sx_1,Tx_1)\},d'(Sx_0,Tx_1),d'(Sx_1,Tx_0)]$$

Suppose $y_2 \in K$, then $y_2\in K \cap TK \subseteq SK$ which implies that there exists a point $x_2 \in K$ such that $y_2 = Sx_2$. Suppose $y_2 \notin K$, then there exists a point $w\in K$ such that $d(Sx_1,w)+d(w,y_2)$

$=d(Sx_1,y_2)$. Since $w \in K \subseteq SK$, there exists a point $x_2 \in K$ such that $w=Sx_2$ and so $d(Sx_1, Sx_2)+d(Sx_2,y_2) = d(Sx_1,y_2)$.

Let $y_3 \in Tx_2$ such that

$$d(y_2,y_3) \le \frac{1}{\sqrt{h}} H(Tx_1,Tx_2)$$

On repeating this process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(a) $y_{n+1} = Ty_n$

(b) $y_n \in K \Rightarrow Sx_n$ or $y_n \notin K \Rightarrow Sx_n \in \partial K$ and $d'(Tx_{n-1},Sx_n) + d(Sx_n,y_n) \ge d(Sx_{n-1},y_n)$,

(c) $d(y_n,y_{n+1}) \le \frac{1}{\sqrt{h}} H(Tx_{n-1},Tx_n)$

Let us denote $P = [Sx_j \in \{Sx_n\} : Sx_j = y_j]$ and $Q = [Sx_j \in \{Sx_n\} : Sx_j \ne y_j]$.

Now there arise three cases:

Case I: If $(Sx_n,Sx_{n-1}) \in PxP$ then by (a), we get

$$d(Sx_n,Sx_{n+1}) = d(y_n,y_{n+1}) \le \frac{1}{\sqrt{h}} H(Tx_{n-1},Tx_n)$$

$$\le \frac{1}{\sqrt{h}} \phi[d(Sx_{n-1},Sx_n),d'(Sx_{n-1},Tx_{n-1}),d'(Sx_n,Tx_n),d'(Sx_{n-1},Tx_n),d'(Sx_n,Tx_{n-1})]$$

$$\le \frac{1}{\sqrt{h}} \phi[d(Sx_{n-1},Sx_n),\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1}),d(Sx_{n-1},Sx_{n+1}),d(Sx_n,Sx_n)]$$

$$\le \frac{1}{\sqrt{h}} \phi[d(Sx_{n-1},Sx_n), d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1}),d(Sx_{n-1},Sx_n)+ d(Sx_n,Sx_{n+1}), d(Sx_n,Sx_n)]$$

By triangular inequality, we obtain

$$\Rightarrow d(Sx_n,Sx_{n+1}) \le \frac{1}{\sqrt{h}} h.d(Sx_{n-1},Sx_n)\}$$

$\Rightarrow d(Sx_n,Sx_{n+1}) \le \sqrt{h}.d(Sx_{n-1},Sx_n)\}$.

Case I: If $(Sx_n,Sx_{n-1}) \in PxQ$ then by (b), we get

$d(Sx_n,Sx_{n+1}) = d(Sx_n,y_{n+1}) \le d(y_n,y_{n+1})$, Proceeding as in **case I**, we get

$\Rightarrow d(Sx_n,Sx_{n+1}) \le \sqrt{h}.d(Sx_{n-1},Sx_n)\}$.

CaseIII: If $(Sx_n,Sx_{n-1}) \in QxP$ then $Sx_{n-1} = y_{n-1}$. Hence

$d(Sx_n,Sx_{n+1}) = d(Sx_n,y_n)+d(y_n,y_{n+1})$

$$\le d(Sx_n,y_n) + \frac{1}{\sqrt{h}} H(Tx_{n-1},Tx_n)$$

$$\le d(Sx_n,y_n) + \frac{1}{\sqrt{h}} \phi[d(Sx_{n-1},Sx_n),d'(Sx_{n-1},Tx_{n-1}),d'(Sx_n,Tx_n\}, d'(Sx_{n-1},Tx_n), d'(Sx_n,Tx_n)]$$

$\leq d(Sx_n,y_n) + \frac{1}{\sqrt{h}}\phi[d(Sx_{n-1},Sx_n),d(Sx_{n-1},y_n),d(Sx_n,Sx_{n+1}), d(Sx_{n-1},Sx_{n+1}), d(Sx_n,Sx_{n+1})]$

$\leq d(Sx_n,y_n) + \frac{1}{\sqrt{h}}\phi[d(Sx_{n-1},y_n),d(Sx_{n-1},y_n),d(Sx_n,y_{n+1}), d(Sx_{n-1},y_{n+1}),d(Sx_n,y_{n+1})\}]$

$\leq d(Sx_n,y_n) + \frac{1}{\sqrt{h}}\phi[d(Sx_{n-1},y_n),d(Sx_{n-1},y_n),d(Sx_n,y_n),+d(y_n, y_{n+1}), d(Sx_{n-1},y_n) + d(y_n,y_{n+1}),$

$d(Sx_n,y_{n+1})\}]$

By using triangular inequality, we obtain

$\Rightarrow d(Sx_n,Sx_{n+1}) \leq \sqrt{h}.d(Sx_{n-1},Sx_n)\}.$

Since $Sx_{n-1} = y_{n-1}$, as in case(II),we obtain

$d(Sx_{n-1},Sx_n) \leq \sqrt{h}.d(Sx_{n-2},Sx_{n-1})\}$

On continuing this process we obtain that $\{Sx_n\}$ is a cauchy sequence and so it converge to a point p in X such that p = Su for some u in K.

Thus, there exists a subsequence $\{x_{nk}\}$, such that $y_{nk} = Sx_n=Tx_{nk-1}$

It implies that $p = Su \in Tv$ for some v in X. Thus by using hybrid T- JSR pair {S,T}, we have

$Sx_n\in Tx_{n-1}\cap K$ and $Sx_{n-1}\in K$,

$\alpha d(TSx_{n-1},Sp) \leq \alpha H(TTx_{n-1},Tp)$

On letting $n\rightarrow\infty$, we get

$\alpha d(TSu,Sp) \leq \alpha H(Tp,Tp)$

$\Rightarrow \alpha d(Tp,Sp) \leq \alpha H(Tp,Tp)$

$\Rightarrow Sp\in Tp$(as Tp is closed).

Now consider,

$D(Sx_n,Sp) \leq \frac{1}{\sqrt{h}}H(Tx_{n-1},Tp)$

$\leq \frac{1}{\sqrt{h}}\phi[d(Sx_{n-1},Sp),d(Sx_{n-1},Tx_{n-1}),d(Sp,Tp),d(Sx_{n-1},Tp),d(Sp,Tx_{n-1})]$

$\leq \frac{1}{\sqrt{h}}\phi[d(Sx_{n-1},Sp),d(Sx_{n-1},Sx_n),d(Sp,Tp),d(Sx_{n-1},Tp),d(Sp,Sx_n)]$

On letting $n\rightarrow\infty$, we get

$d(p,Sp)\leq \frac{1}{\sqrt{h}}h.d(p,Sp)\}$

$d(p,Sp)\leq \sqrt{h}.d(p,Sp)\}$

$\Rightarrow p = Sp.$

Hence p = Sp $\in$Tp.

Example: Let X = [1,∞) with usual metric .Define S:X$\rightarrow$X as Sx = 2+x/3 and T:CB(X)$\rightarrow$X as Tx = [1,2+x]. Consider the sequence $\{x_n\}$ ={3+1/n}. Then all conditions are satisfies of the theorem and hence 3 is the common fixed point.

References

[1] Khan M.S.: Common fixed point theorems for multi-valued mappings. Pacific J. Math. 95(1981),337-347.

[2] Nadler S.B.: Multivalued contraction mappings, Pacific J. Math. 30(1069), 475-488.

[3] Rus I.A.: Teria punctulie fix, vol.II Univ.Babes Boloyi, Cluj,1973.

[4] Shrivastav Rajesh, Patel Vivek and DhagatVanita Ben , “Fixed Point Result in Fuzzy Menger Space with EA Propert”y, Int. J. Contemp. Math. Sciences, Vol. 8, 2013, no. 2, 53 - 60