FIXED POINT METRICALLY CONVEX METRIC SPACE

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Abstract
In this paper we establish a fixed point theorem for the hybrid pair of multivalued and single valued nonself JSR mapping in metrically convex metric space. Also we give example in support of the result.

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Introduction and Preliminaries
A bulk of literature exist with commuting pairs and its weaker forms such as weakly commuting , compatible, compatible of type A, D-compatible, semi compatible, etc. new pair termed as JSR mapping which is defined by Shrivastav R et. el.[4] in fuzzy menger space and will prove a fixed point theorem for hybrid pair of multivalued and single valued nonself mapping satisfying the $\phi$-contraction in a metrically convex metric space.

Let $(X,d)$ be a metric space. Then, following Nadler [2], we have

$CB(X) = \{A : A$ is a nonempty closed and bounded subset of $X\}$,

$C(X) = \{A : A$ is a nonempty compact subset of $X$ $\}$,

$BN(X) = \{A : A$ is a nonempty bounded subset of $X\}$.

For non empty subsets $A$ and $B$ of $X$ and $x \in X$,

$D(A,B) = \inf \{d(a,b) : a \in A$ and $b \in B\},$

$H(A,B) = \max[ \sup\{D(a,B) : a \in A\}, \sup\{D(A,b) : b \in B\}],$

$\delta(A,B) = \sup\{d(a,b) : a \in A$ and $b \in B\},$

$d'(x,A) = \inf\{d(x,a) : a \in A\},$

$\partial K$ is boundary of $K$.

A metric space $(X,d)$ is said to be metrically convex if for any $x, y \in X$ $(x \neq y)$ $\partial z \in X$ $(x \neq y \neq z)$ such that $d(x,z) + d(z,y) = d(x,y).$ Further if $K$ is a non
empty closed subset of $X$ and $x \in K$, $y \notin K$, then there exists a point $z \in \partial z$ such that $d(x,z) + d(z,y) = d(x,y)$.

The following lemmas are from Rus[3] and Khan [1].

**Lemma 1** Let $A \subseteq \text{CB}(X)$ and $0 < \theta < 1$ be given. Then for every $x \in A$ there exists a point $a \in A$ such that $d(x,z) \geq \theta \delta(x,A)$ and $d(x,z) \geq \theta \delta(x,A)$.

**Lemma 2** For any $x \in X$, and any $A,B$ in $\text{CB}(X)$, $|d'(x,A) - d'(x,B)| \leq H(A,B)$.

**Lemma 3** For any $x,y \in X$ and $A \subseteq X$, $|d'(x,A) - d'(y,A)| \leq d(x,y)$

Let $K$ be nonempty closed subset of a metric space $(X,d)$, A mapping $T : K \rightarrow \text{CB}(X)$ is said to be continuous at $x_0 \in K$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $H(Tx,Tx_0) < \varepsilon$, whenever $d(x,x_0) < \delta$. If $T$ is continuous at every point of $K$, we say that $T$ is continuous at $K$.

Let $\mathbb{R}^+$ be the set of non-negative real and $\psi$ the set of function $\phi: (\mathbb{R}^+)^5 \rightarrow \mathbb{R}$ satisfying the following properties:

(i) $\phi$ is continuous and increasing in each coordinate variable

(ii) $\phi(1,1,1,1,1) = h < 1$ ($h \in \mathbb{R}^+$)

(iii) Either $u \leq \phi(u,v,u,v,v)$ or $u \leq \phi(v,u,v,u,v)$ implies $u \leq hv$

Let $S$ and $T$ be two self maps of a metric space $(X,d)$. The pair $\{S,T\}$ is said to be $S$-JSR mappings iff

\[ \alpha \ d(STx_n,Tx_n) \leq \alpha \ d(SSx_n,Sx_n) \]

where $\alpha = \lim Sup$ or $\lim inf$ and $\{x_n\}$ is a sequence in $X$ such that

\[ \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some} \ t \ \text{in} \ X. \]

**Example** Let $X = [0,1]$ with $d(x,y) = |x-y|$ and $S,T$ are two self mapping on $X$ defined by

\[ Sx = \frac{2}{x+2} \quad \text{and} \quad Tx = \frac{1}{x+1} \quad \text{for} \ x \in X. \]
Now we have the sequence \( \{x_n\} \) in \( X \) is defined as \( x_n = 1/n \), \( n \in \mathbb{N} \). Then we have

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = 1
\]

\[|STx_n - Tx_n| \to 1/3 \text{ and } |SSx_n - Sx_n| \to 2/3 \text{ as } n \to \infty.\]

Clearly we have

\[|TSx_n - Tx_n| < |SSx_n - Sx_n|.\]

Thus pair \( \{S,T\} \) is S-JSR mapping. But This pair is neither compatible nor weakly compatible nor other non commuting mapping. Hence pair of JSR mapping is more general then others.

Let \( S \) self map of a metric space \((X,d)\) and \( T \) be multivalued map . The pair \( \{S,T\} \) is said to be hybrid T-JSR mappings iff

\[
\alpha d'(TSx_n,Sx_n) \leq \alpha H(Tx_n,Tx_n)
\]

where \( \alpha = \lim \sup \) or \( \lim \inf \) and \( \{x_n\} \) is a sequence in \( X \) such that

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \text{ for some } t \in X.
\]

**Main Result**

**Theorem 1** Let \((X,d)\) be a complete and metrically convex metric space and \( K \) be a nonempty closed subset of \( X \). Let \( T:K \to CB(X) \) and \( S:K \to X \) such that

(i) \( \partial K \subseteq SK, \ TK \subseteq SK; \ Sx \in \partial K \Rightarrow TX \subseteq K, \)

\[
H(Tx,Ty) \leq \phi[d(Sx,Sy), \ d'(Sx,Tx), d'(Sy,Ty)], \ d'(Sx,Ty), d'(Sy,Tx)]
\]

For all \( x,y \in K \)

(ii) \( \{T,S\} \) is hybrid T-JSR pair,

(iii) \( SK \) is closed

then there exists a point \( p \) in \( K \) such that \( p=Sp \in Tp \) i.e. \( p \) is common fixed point.

**Proof**. Construct the sequences \( \{x_n\} \) and \( \{y_n\} \) in the following way:

Let \( x \in \partial K \), then there exists a point \( x_0 \in K \) such that \( x=Sx_0 \) as \( \partial K \subseteq SK \). Form \( Sx_0\in\partial K \) and by the implication of \( Sx \in \partial K \Rightarrow TX \subseteq SK \), we conclude that

\[
d(y_1,y_2) \leq \frac{1}{\sqrt{n}} H(Tx_0,Tx_1)
\]

\[
 \leq \phi[d(Sx_0,Sx_1), \{d' (Sx_0,Tx_0),d' (Sx_1,Tx_1)\},d' (Sx_0,Tx_1),d' (Sx_1,Tx_0)]
\]

Suppose \( y_2 \in K \), then \( y_2 \in K \cap TK \subseteq SK \) which implies that there exists a point \( x_2 \in K \) such that \( y_2 = Sx_2 \). Suppose \( y_2 \notin K \), then there exists a point \( w \in K \) such that \( d(Sx_1,w)+d(w,y_2) \)
=d(Sx₁,y₂). Since w ∈ K ⊆ SK, there exists a point x₂ ∈ K such that w = Sx₂ and so d(Sx₁, Sx₂) + d(Sx₂,y₂) = d(Sx₁,y₂).

Let y₃ ∈ Tx₂ such that

d(y₂,y₃) ≤ 1 h H(Tx₁,Tx₂)

On repeating this process, we obtain two sequences {xₙ} and {yₙ} such that

(a) yₙ₊₁ = Tyₙ
(b) yₙ ∈ K ⇒ Sxₙ or yₙ ∉ K ⇒ Sxₙ ∈ ∂K and d'(Txₙ₋₁,Sxₙ) + d(Sxₙ,yₙ) ≥ d(Sxₙ₋₁,yₙ),
(c) d(yₙ,yₙ₊₁) ≤ 1 h H(Txₙ₋₁,Txₙ)

Let us denote P = [Sxⱼ ∈ {Sxₙ} : Sxⱼ = yⱼ] and Q = [Sxⱼ ∈ {Sxₙ} : Sxⱼ ≠ yⱼ].

Now there arise three cases:

**Case I:** If (Sxₙ,Sxₙ₋₁) ∈ PxP then by (a), we get

d(Sxₙ,Sxₙ₊₁) = d(yₙ,yₙ₊₁) ≤ 1 h H(Txₙ₋₁,Txₙ)

⇒ d(Sxₙ,Sxₙ₊₁) ≤ 1 h d(Sxₙ₋₁,Sxₙ)

⇒ d(Sxₙ,Sxₙ₊₁) ≤ h d(Sxₙ₋₁,Sxₙ).

**Case I:** If (Sxₙ,Sxₙ₋₁) ∈ PxQ then by (b), we get

d(Sxₙ,Sxₙ₊₁) = d(Sxₙ,yₙ₊₁) ≤ d(yₙ,yₙ₊₁). Proceeding as in case I, we get

⇒ d(Sxₙ,Sxₙ₊₁) ≤ h d(Sxₙ₋₁,Sxₙ).

**Case III:** If (Sxₙ,Sxₙ₋₁) ∈ QxP then Sxₙ₋₁ = yₙ₋₁. Hence

d(Sxₙ,Sxₙ₊₁) = d(Sxₙ,yₙ) + d(yₙ,yₙ₊₁)

≤ d(Sxₙ,yₙ) + 1 h H(Txₙ₋₁,Txₙ)

≤ d(Sxₙ,yₙ) + 1 h [d(Sxₙ₋₁,Sxₙ), d'(Sxₙ₋₁,Txₙ₋₁), d'(Sxₙ,Txₙ)]
\[ \leq d(Sx_n, y_n) + \frac{1}{\sqrt{h}} \phi [d(Sx_{n-1}, Sx_n), d(Sx_{n-1}, y_n), d(Sx_n, Sx_{n+1}), d(Sx_n, y_{n+1})] \]

\[ \leq d(Sx_n, y_n) + \frac{1}{\sqrt{h}} \phi [d(Sx_{n-1}, y_n), d(Sx_{n-1}, y_n), d(Sx_n, y_{n+1}), d(Sx_n, y_{n+1})] \]

\[ \leq d(Sx_n, y_n) + \frac{1}{\sqrt{h}} \phi [d(Sx_{n-1}, y_n), d(Sx_{n-1}, y_n), d(Sx_n, y_{n+1}), +d(y_n, y_{n+1}), d(Sx_n, y_{n+1}), d(Sx_n, y_{n+1})] \]

By using triangular inequality, we obtain

\[ d(Sx_{n-1}, Sx_n) \leq \sqrt{h} d(Sx_{n-1}, Sx_n) \}

Since \( Sx_{n-1} = y_{n-1} \), as in case(II), we obtain

\[ d(Sx_{n-1}, Sx_n) \leq \sqrt{h} d(Sx_{n-2}, Sx_{n-1}) \}

On continuing this process we obtain that \( \{Sx_n\}\) is a cauchy sequence and so it converge to a point \( p \) in \( X \) such that \( p = Su \) for some \( u \) in \( K \).

Thus, there exists a subsequence \( \{x_{nk}\}\), such that \( y_{nk} = Sx_n = Tx_{nk-1} \)

It implies that \( p = Su \in T v \) for some \( v \) in \( X \). Thus by using hybrid T- JSR pair \( \{S,T\}\), we have

\[ Sx_n \in Tx_{n-1} \cap K \] and \( Sx_{n-1} \in K \),

\[ \alpha d(TSx_{n-1}, Sp) \leq \alpha H(TTx_{n-1}, Tp) \]

On letting \( n \to \infty \), we get

\[ \alpha d(TSu, Sp) \leq \alpha H(Tp, Tp) \]

\[ \Rightarrow \alpha d(Tp, Sp) \leq \alpha H(Tp, Tp) \]

\[ \Rightarrow Sp \in Tp \) (as \( Tp \) is closed).

Now consider,

\[ D(Sx_n, Sp) \leq \frac{1}{\sqrt{h}} H(Tx_{n-1}, Tp) \]

\[ \leq \frac{1}{\sqrt{h}} \phi [d(Sx_{n-1}, Sp), d(Sx_{n-1}, Tx_{n-1}), d(Sp, Tp), d(Sx_{n-1}, Tp), d(Sp, Tx_{n-1})] \]

\[ \leq \frac{1}{\sqrt{h}} \phi [d(Sx_{n-1}, Sp), d(Sx_{n-1}, Sx_n), d(Sp, Tp), d(Sx_{n-1}, Tp), d(Sp, Sx_n)] \]

On letting \( n \to \infty \), we get

\[ d(p, Sp) \leq \frac{1}{\sqrt{h}} h d(p, Sp) \}

\[ d(p, Sp) \leq \sqrt{h} d(p, Sp) \}

\[ \Rightarrow p = Sp. \]
Hence \( p = \text{Sp} \in \text{Tp} \).

**Example:** Let \( X = [1, \infty) \) with usual metric. Define \( S:X \to X \) as \( Sx = 2 + x/3 \) and \( T:CB(X) \to X \) as \( Tx = [1, 2 + x] \). Consider the sequence \( \{x_n\} = \{3 + 1/n\} \). Then all conditions are satisfied by the theorem and hence 3 is the common fixed point.

**References**