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A Mathematical Model for Population Density Dynamics of Annual Weeds and its Application to Bush Mint Weed (*Hyptis suaveolens*)

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ABSTRACT

In this paper, a discrete-time stage-structured mathematical model was formulated for the population density dynamics of annual weeds. Biological process was employed to develop the model equations and incorporates density-dependent effects at germination and established seedling stages within the weed life-cycle. Besides, the developed model framework was applied to investigate the population density dynamics of Bush Mint weed (*Hyptis suaveolens*). The analysis revealed that the steady state solution is locally asymptotically stable and conclude that, whenever the steady state population is disturbed through management effort the weeds will always proliferate. Also, the steady state density of *H. suaveolens* is globally asymptotically stable and concludes that its population density may be control or eradicated.

Keywords Discrete-time, density-dependent, stage –structured, steady-state, annual weeds**,** biological-process, *Hyptis suaveolens*

1. INTRODUCTION

Weeds are generally defined as uncultivated plant species that proliferate in agricultural setting thereby, interfering with crop production. In fact, weed is a term applied to any plant that grows naturally in a place it is not wanted. They exist only in natural environments that have been disturbed by human's activities such as agricultural lands, recreational parks, and irrigation dams [1]. However, weeds form an important part of the land ecosystem, providing food and cover for animals and birds which are an important indicator of biodiversity health [2]. Managing weeds to limit both crop yield loss and environmental impacts is a major challenge of agriculture [3] and one of the key elements of most agricultural system. The development of weed's management strategies to limit the deleterious effects of weeds growing with crop plants requires thorough quantitative and qualitative in-sight in the behaviour of weeds in agro-ecosystems and their effects. These effects can be quite variable and involves understanding the dynamics of weed population, crop-weed competition, weed rate of spread (spatial) and invasion as well as environmental interaction.

Population dynamics involve the study of population numerical change in time, composition and spatial dispersion. The objectives are to identify the causes of numerical change in population and to explain how this cause act and interact to produce the observed pattern. Not until recent past population models were concentrated mainly on the use of differential equations. Although, most populations such as weeds and phylogenetically more evolved organisms live in seasonal environments and because of this, have annual rhythms of reproduction and death. Since plant has discrete generations (seasonal reproduction), difference dynamical equation systems are an appropriate mathematical tool to model behaviour of population with no overlapping generations such as weeds. Furthermore, many researchers have paid attention in recent times to discrete- time population models, since the discrete time models governed by discrete systems are more appropriate than the continuous ones when the populations have non over lapping generations [4].

Attempts to describe and predict the population dynamics of plants such as arable weeds have tendered to be compromised by a lack of generality [5]. Most predictive models have been developed almost exclusively from data sets derived from one or only a few years' trial. Besides, most attempts to model population dynamics of arable weeds are compromised by a lack of data. Either data are not available over a long time period or studies are not spatially replicated.

It is well-established that population models can be derived from two different sources, data and biological process. The first relies completely upon data to look at the dynamics of the population [6]. The second is a model defined by biological processes, which do not include any data, but instead attempt to understand the dynamics of populations purely from what is expected to occur. These two types of models require different frame works; one needs a biologically defined state and the other a data defined state.

The aim of this paper is to propose a deterministic discrete-time stage-structured population model using biological processes for the dynamics of weed density interaction. Incorporating density-dependent effects at different stages of recruitment and maturity within the weed's life – cycle and apply the formulated model framework to investigate the population dynamics of Bush mint weed (*H. suaveolens*).

2. MATERIALS AND METHODS

2.1 Formulation of the Model Equations

In general, a model for the dynamics of an arable weed requires component to describe changes in the number of vegetative plants stages. We divided the life-cycle of an annual weed into three stages; seed (S) in the seed bank established seedling (E) and mature weeds (M). Because the life-cycle events of weeds are usually assumed to be synchronous, this requires system of difference equations that relate the numbers of seeds, seedling and mature weeds at time t to the numbers at time (t+1).
On the basis of the classification into three stages, we consider the schematic life-cycle graph of the annual weed model shown in Figure (1) below,

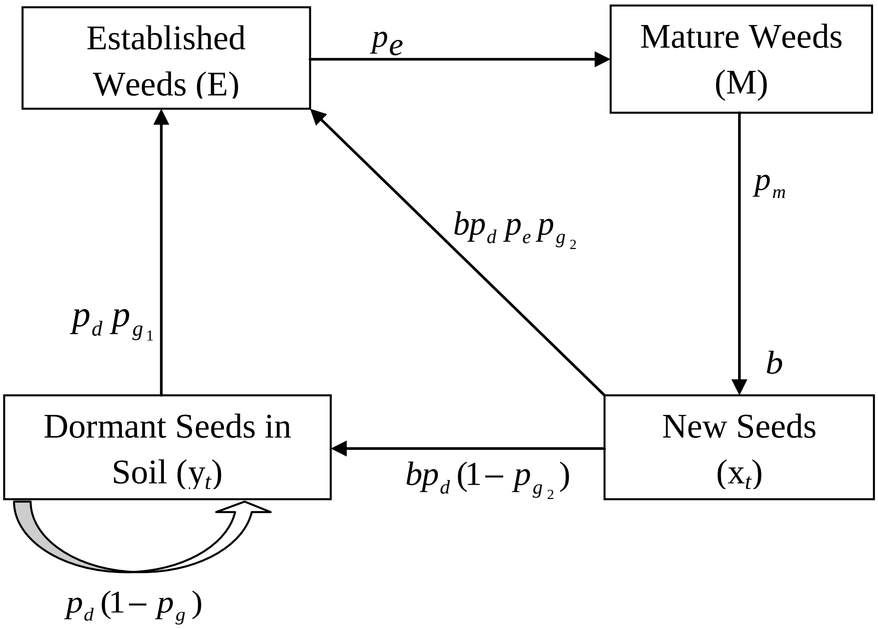

Figure 2.1 Schematic diagram for Modified Three-Stages of Annual Weed Model

The feedback life cycle diagram shows the fate of seeds and above the ground weed densities.
The individuals that are classed as dormant seeds may remain as viable dormant seeds in the seed bank after surviving dry season or they may germinate at some rate to become established seedling. The established seedling may survive and grow at some rate to become reproductive mature weeds those individuals classed as mature weeds has two possible routes after flowering and seeds production:
Newly produced seeds may either germinate late in the season to become establish seedling or become part of the dormant seed bank. In order to track number of individuals in the various life stages, we use the variables below for the three biological stages of the life cycle of our hypothetical annual weed.

Variables	**Biological interpretation**
$\boldsymbol{n_{1,t}}$	Density of seeds in the seed bank
$\boldsymbol{n_{2,t}}$	Density of the established seedling
$\boldsymbol{n_{3,t}}$	Density of mature weeds

Their units are weed density per unit area.

Parameters

p_d	Fraction (possibility) of seeds in the seed bank that survive and viable
p_{g_2}	Fraction of the survive and viable new seeds that germinate within the season
p_{g_1}	Fraction (possibility) of viable seeds older than one year germinate from the seed bank
p_e	Fraction (possibility) of germinated seeds that become established seedlings
p_m	Fraction of the established seedlings that survive to mature weeds

b average number of seeds produced by the mature weed per unit area.

Assumptions

We made the following simple biologically realistic assumptions

i. There is no mortality of dormant seeds once they enter the seed bank, seeds can only leave the seed bank through germination
ii. There is no significant difference in seed germination of newly produced seeds and old seed from the seed bank
iii. Seedling recruitment and the established seedling (growth) become mature are density dependent
iv. There is competition among the population for the available micro site space while there are enough of other resources.
v. The number of established seedling between the time interval increases with the available micro site (space) for seeds to germinate and establish.

Applying the variables, parameters definitions and the above assumptions, the stage structured population model for the abundant densities of seeds ($n_{1,t}$), established seedlings ($n_{2,t}$) and mature-weeds ($n_{3,t}$) is given by the following system of difference equations:

$$n_{1,t+1} = p_d(1-p_{g_1}(n_2))n_{1,t} + bp_d(1-p_{g_1}(n_2))n_{3,t}\,, \tag{1}$$

$$n_{2,t+1} = p_d\, p_e\, p_{g_1}(n_2)n_{1,t} + bp_d\, p_e\, p_{g_2}(n_2)n_{3,t}\,, \tag{2}$$

$$n_{3,t+1} = p_m(n_2)n_{2,t}\,. \tag{3}$$

Biologically, it has been observed that any of the parameters p_{g_i}, p_e, p_m and b may experience density-dependence due to resource limitation (e.g. space, nutrient, water and light). A thorough understanding of the occurrence and effects of density-dependence at different stage has been called for by several authors. However, density-dependent fecundity and survival have been observed most frequently in perennial plant population [7,8]. Density-dependence germination (recruitment), survival and growth of annual weed population are hardly been investigated as far as we know.

We consider density-dependence seed germination (seedling recruitment) $p_{g_i}(n_i)$, and survival of the established seedling to mature weeds $p_m(n_i)$. For the functions $p_{g_i}(n_i)$ and $p_m(n_2)$ we adopted the Beaverton-Holt density-dependence function type (we choose this form because it can be derived by assuming competition among individuals for an available resource) due to the assumption 4. Thus;

$$p_{g_i}(n_2) = \frac{g_i}{1+\alpha n_{2,t}} \tag{4}$$

and

$$p_m(n_2) = \frac{m}{1+\alpha n_{2,t}}\,, \tag{5}$$

where

g_i = the maximum value of p_{g_i} at a low density of established seedling $(n_{2,t})$.
α = the effect of established seedling density on the rate of seed germination (recruitment) and survival to mature weed.

So, substituting (4) and (5) into (1) – (3) we have

$$n_{1,t+1} = p_d\left(1-\frac{g_1}{1+\alpha n_{2,t}}\right)n_{1,t} + bp_d\left(1-\frac{g_2}{1+\alpha n_{2,t}}\right)n_{3,t}\,, \tag{6}$$

$$n_{2,t+1} = p_d\, p_e\frac{g_1 n_{1,t}}{1+\alpha n_{2,t}} + bp_d\, p_e\frac{g_2 n_{3,t}}{1+\alpha n_{2,t}}\,, \tag{7}$$

$$n_{3,t+1} = \frac{m n_{2,t}}{1+\alpha n_{2,t}}\,. \tag{8}$$

In the next section, we carried out the stability analysis of the model equations proposed above.

2.2 Stability Analysis of the Model Equations

In order to carry out the stability analysis the stage-structured model equations (6) – (8) is re-expressed in matrix form without any loss or gain of mathematical content. Although it reveals more, the structure of the stage population (life cycle) of the weed, Thus;

$$\begin{pmatrix} n_{1,t+1} \\ n_{2,t+1} \\ n_{3,t+1} \end{pmatrix} = \begin{pmatrix} p_d\left(1-\dfrac{g_1}{1+\alpha n_{2,t}}\right) & 0 & bp_d\left(1-\dfrac{g_2}{1+\alpha n_{2,t}}\right) \\ p_d\, p_e\dfrac{g_1}{1+\alpha n_{2,t}} & 0 & bp_d\, p_e\dfrac{g_2}{1+\alpha n_{2,t}} \\ 0 & \dfrac{m}{1+\alpha n_{2,t}} & 0 \end{pmatrix} \begin{pmatrix} n_{1,t} \\ n_{2,t} \\ n_{3,t} \end{pmatrix}. \tag{9}$$

Written (9) in a general matrix equation, we have

$$\mathbf{n_{t+1}} = \mathbf{P}(\theta, n_t)\mathbf{n_t}, \tag{10}$$

Where $\mathbf{n_t} = (n_{1,t}, n_{2,t}, n_{3,t})^T$ is the stage density vector at time t and $\mathbf{n_{t+1}}$, density vector at time (t+1). $\mathbf{P} = \left(p_{ij}\right)$ is the stage density-dependent matrix elements called projection (or transition) matrix with entries p_{ij} which are density dependents. Here;

p_{11} = fraction of a seed bank seeds that remain dormant.
p_{21} = fraction of the germinated seeds from the seed bank that survive to established weed seedling
p_{23} = fraction of newly produced seed that germinate directly during the raining season and become established seedling .
p_{32} =fraction of the established weed seedling that survive to become mature weeds.
p_{13} = fraction of new seeds that enter the seed bank (i.e become dormant).

Applying the idea in [9, 10], we assume that seedling recruitment and established seedling survival to maturity functions, $p_{g_i}(n)$ $i = 1,2$ and $p_m(n)$ as given in (4) and (5) respectively satisfy the following conditions,

Condition C1

a. $p_{g_i}(n), p_m(n) \in C^1[0,\infty)$, $p_{g_i}(0) = g_i$, $p_m(0) = m$, $0 < g_i < 1$.

b. $\dfrac{d}{dn} p_{g_i}(n) < 0$, $\dfrac{d}{dn} p_m(n) < 0$, $\dfrac{d}{dn}(p_{g_i}(n))n > 0$, $\dfrac{d}{dn}(p_m(n))n > 0$.

c. $\lim\limits_{n\to\infty} p_{g_i}(n) = \lim\limits_{n\to\infty} p_m(n) = 0$, $\lim\limits_{n\to\infty}(p_{g_i}(n)) = \overline{g_i}$ and $\lim\limits_{n\to\infty}(p_m(n)) = \overline{m}$.

The analysis of the steady-state solutions are carried out next.

2.2.1 Stability of Zero Steady-State (Extinction of Weed Population)

System (9) has the zero steady-state solution (extinction) $E_0(n_1 \quad n_2 \quad n_3)^T = E_0(0 \quad 0 \quad 0)$. This is the situation liken to clearance of agricultural field before planting season either by fire, tillage, or weeding, that destroys all above – ground weed biomass, and the initial population will consist entirely of seedling germinated (recruited) from surviving viable seeds left in the seed bank. At a very low population density, individuals rarely interfere with each other, there is enough space for seeds to germinate and established so the intra- species competitive pressure does not occurred. In consequence, the density dependent germination (recruitment) and maturity (survival) terms will be negligible. So the weed populations will exponentially grow in constant environment (time-invariant) if all the resources are available. This can be achieved mathematically either by setting $n_2(t) = 0$ in (9) or by its linearization at the trivial steady-state point which gives a density-dependent linear population model, in a constant environment [11, 12, 10] characterized by exponential growth. The rate of its population growth describes the asymptotic dynamics or long-term behaviour and the stability of the zero steady state point. This rate is computed as the dominant eigenvalue (λ) of the population variational (projection) matrix $\mathbf{P}(\theta, n_t)$. However, inherent net reproductive value (***R₀***) idea is employed instead to determined the stability of (9). The value $\boldsymbol{R_0}$ also determines the asymptotic dynamics of the linear systems and the stability of its zero (trivial) steady-state point and the bifurcation of positive steady-state [13, 11, 16, 12,10 and 14).

The stability analysis based on variational principle [15] and inherent basic reproductive value [16, 12, 17, 18, 19] are used. The variational matrix ***D(n)*** which is obtained from the partial derivatives of the right side of equations (6) – (8), thus has the form;

$$D(n)=\begin{pmatrix} d\left(1-\dfrac{g_1}{1+\alpha n_2}\right) & \dfrac{\alpha g_1 n_1+bd\alpha g_2 n_3}{(1+\alpha n_2)^2} & bd\left(1-\dfrac{g_2}{1+\alpha n_2}\right) \\ de\dfrac{g_1}{1+\alpha n_2} & -\dfrac{e^2\alpha g_1 n_1+bde\alpha g_2 n_3}{(1+\alpha n_2)^2} & bde\dfrac{g_2}{1+\alpha n_2} \\ 0 & \dfrac{m}{(1+\alpha n_2)^2} & 0 \end{pmatrix}. \qquad (11)$$

Note: Subscript of the vital parameter is used to represent the parameter value in (11) for convenience of expression.

Evaluating the variational matrix (11) at $E_0(n_1 \quad n_2 \quad n_3)^T=(0 \quad 0 \quad 0)$ gives the inherent variational matrix of the nonlinear system (10) thus

$$D(0)=\begin{pmatrix} d(1-g_1) & 0 & bd(1-g_2) \\ \mathrm{deg}_1 & 0 & b\,\mathrm{deg}_2 \\ 0 & m & 0 \end{pmatrix}=P(0) \qquad (12)$$

As stated in the assumption, not all dormant seeds are viable due to heat and predation by insects so survivorship probability is assumed to be $0<d\le 1$.

Using the techniques and framework given in [12, 16, 9, 10, 14], the matrix (12) is non-negative, irreducible and primitive since the first entry in the third column is positive (that is $g_2<1$), therefore, it is additively decomposed into the sum of two non-negative matrices;

$D(0)=F(0)+T(0)$,

Where

$$F(0)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & m & 0 \end{pmatrix} \text{ and } T(0)=\begin{pmatrix} d(1-g_1) & 0 & bd(1-g_2) \\ \mathrm{deg}_1 & 0 & b\,\mathrm{deg}_2 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (13)$$

Then, the inherent net reproductive value ($\boldsymbol{R_0}$) is the positive, simple, and strictly dominant eigenvalue of $F(I-T)^{-1}$. It is simply calculated as

$$(I-T)=\begin{pmatrix} [1-d(1-g_1)] & 0 & -bd(1-g_2) \\ -\mathrm{deg}_1 & 1 & -b\,\mathrm{deg}_2 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (14)$$

$$(I-T)^{-1}=\begin{pmatrix} \dfrac{1}{[1-d(1-g_1)]} & 0 & \dfrac{bd(1-g_2)}{[1-d(1-g_1)]} \\ \dfrac{\mathrm{deg}_1}{[1-d(1-g_1)]} & 1 & \dfrac{b\,\mathrm{deg}_2[1-d(1-g_1)]+\mathrm{deg}_1\,bd(1-g_2)}{[1-d(1-g_1)]} \\ 0 & 0 & 1 \end{pmatrix}. \qquad (15)$$

$(I-T)^{-1}>0$ (exists and greater than zero). So

$$(I-T)^{-1}F = \begin{pmatrix} 0 & \dfrac{mbd(1-g_2)}{[1-d(1-g_1)]} & 0 \\ 0 & \dfrac{m[b\deg_2(1-d(1-g_1))+eg_1bd^2(1-g_2)]}{[1-d(1-g_1)]} & 0 \\ 0 & m & 0 \end{pmatrix}. \qquad (16)$$

The matrix (16) has 0 as double eigenvalues, and $\dfrac{m[b\deg_2(1-d(1-g_1))+eg_1bd^2(1-g_2)]}{[1-d(1-g_1)]}$ which gives the dominant eigenvalue if it is greater than zero and its associated right eigenvector ***w*** > 0. Hence,

$$\mathbf{R}_0 = \frac{mbde[g_2(1-d(1-g_1))+dg_1(1-g_2)]}{[1-d(1-g_1)]}. \qquad (17)$$

The right eigenvector associated with $\mathbf{R_0}$ gives the population stable stage distribution (that is the proportion of individuals in each stage.). Thus, **w** after normalization is

$$w = \begin{pmatrix} \dfrac{bd(1-g_2)}{[1-d(1-g_1)]} \\ \dfrac{bde[g_2(1-d(1-g_1))+g_1d(1-g_2)]}{[1-d(1-g_1)]} \\ 1 \end{pmatrix}. \qquad (18)$$

w is positive and independent of initial size. R_0 as given in (17) may not have the biological interpretation of the expected number of seeds produced per individual weed per season life-time. Hence, may not be the net reproductive value which depends on how variational matrix is additively decomposed, but by means of which stability can be determined.

To determine the stability of the zero steady-state point we apply the following stability Theorem (1) which has been adopted by several researchers

Theorem 1

The zero steady-state point of model (10), $E_0(0 \quad 0 \quad 0)$ is globally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof

Since the stage densities $\mathbf{n}_t \geq 0$, it follows from (3.169) and condition C2 that for any initial densities $\mathbf{n}_0$, we have

$$n_1 = P(n_0)n_0 \leq P(0)n_0, \qquad (19)$$

$$n_2 = P(n_1)n_1 \leq P(0)n_1 \leq P(0)^2 n_0. \qquad (20)$$

Continuing iterating in this manner gives

$$n_t \leq P(0)^t n_0 \qquad (21)$$

Then observed that the inherent projection matrix *P(0)* is non-negative, irreducible, and primitive, it has a positive, simple and strictly dominant eigenvalue λ. So, it follows from Theorem 2.4, since $R_0 < 1$ also $\lambda < 1$ and results in (21), thus

$\lim\limits_{t\to\infty} P(0)^t = 0$ and subsequently $\lim\limits_{t\to\infty} P(0)^t n_0 \to 0$

Therefore, $E_0(0 \quad 0 \quad 0)$ is globally asymptotically stable if $R_0 < 1$.

But, clearly the value of R_0 as given in (17) indicates that $R_0 \geq 1$. That is

$$\mathbf{R}_0 = \frac{mbde[g_2(1-d(1-g_1))+dg_1(1-g_2)]}{[1-d(1-g_1)]} > 1. \tag{22}$$

Since, $0 < g_i < 1$, $0 < m, e < 1$, $0 < d \le 1$ and $1 \le b < \infty$.

So, we conclude that the zero steady-state point $E_0(0 \quad 0 \quad 0)$ is unstable, and there exists a positive steady-state point that bifurcates from the zero steady state at $R_0 = 1$.

This positive steady-state point exist for $R_0 > 1$ and is obtained analytically next.

2.2.2 Stability of Positive Steady-state Solution

The non-zero Steady–state solution of the non-homogeneous stage-structured model were found by setting $\mathbf{n}_{t+1} = \mathbf{n}_t = \mathbf{n}$ where $\mathbf{n} = (n_1 \quad n_2 \quad n_3)^T$ in equations (6) – (8). The analytical expression for the non-zero steady-state (equilibrium), $\mathbf{n} = (n_1, \quad n_2, \quad n_3) \in \operatorname{int} \Re^3_+$ were obtained. Thus;

$$n_1 = d\left(1-\frac{g_1}{1+\alpha n_2}\right)n_1 + bd\left(1-\frac{g_2}{1+\alpha n_2}\right)n_3, \tag{23}$$

$$n_2 = de\frac{g_1}{1+\alpha n_2}n_1 + bde\frac{g_2}{1+\alpha n_2}n_3, \tag{24}$$

$$n_3 = \frac{m}{1+\alpha n_2}n_2. \tag{25}$$

From equation (25) , $n_2 = \frac{1+\alpha n_2}{m}n_3$ and substituting this into equation (24) we obtained

$$\left(\frac{1+\alpha n_2}{m} - \frac{b\deg_2}{1+\alpha n_2}\right)n_3 = \left(\frac{\deg_1}{1+\alpha n_2}\right)n_1, \tag{26}$$

So,

$$n_3 = \left(\frac{m\deg_1}{(1+\alpha n_2)^2 - mb\deg_2}\right)n_1. \tag{27}$$

Putting (27) into (23) and after some algebraic rearrangement we obtained

$$n_1 = d\left(1-\frac{g_1}{1+\alpha n_2}\right)n_1 + bd\left(1-\frac{g_2}{1+\alpha n_2}\right)\left(\frac{m\deg_1}{(1+\alpha n_2)^2 - mb\deg_2}\right)n_1, \tag{28}$$

$$\frac{1}{d} = \left(\frac{1+\alpha n_2 - g_1}{1+\alpha n_2}\right) + \left(\frac{b(1+\alpha n_2 - g_2)m\deg_1}{(1+\alpha n_2)[(1+\alpha n_2)^2 - mb\deg_2]}\right),$$

Assuming that $d \approx 1$ (i.e all dormant seed are viable), we have

$$(1+\alpha n_2)[(1+\alpha n_2)^2 - mbeg_2] = (1+\alpha n_2 - g_1)[(1+\alpha n_2)^2 - mbeg_2] + b(1+\alpha n_2 - g_2)meg_1$$

$$(1+\alpha n_2)^3 - mbeg_2(1+\alpha n_2) = [(1+\alpha n_2)^3 - mbeg_2(1+\alpha n_2) - g_1(1+\alpha n_2)^2 + mbeg_1 g_2] + bmeg_1(1+\alpha n_2) - bmeg_1 g_2,$$

$$g_1(1+\alpha n_2)^2 = bmeg_1(1+\alpha n_2),$$

$$n_2 = \frac{bme-1}{\alpha}. \tag{29}$$

Putting (29) into (25) and simplified to obtained

$$n_3 = \frac{bme-1}{be\alpha}. \tag{30}$$

Putting (29) and (30) into (23) and then simplify to have

$$n_1 = \frac{(bme-1)(bme-g_2)}{eg_1\alpha}. \tag{31}$$

Thus, the analytical expression for positive Steady-state solution is given by the expression;

$$E_1(n_1,\quad n_2,\quad n_3) = \left(\frac{(bme-1)(bme-g_2)}{eg_1\alpha}, \frac{bme-1}{\alpha}, \frac{bme-1}{be\alpha}\right). \tag{32}$$

They all exist and are positive.

Once the Steady-state solutions for a dynamical system are obtained, then we shift our focus to the local stability of the system about the steady state solution.

2.2.3 The Stability of Positive Steady-State Solution

Suppose $R_0 > 1$ and $E_1(n_1,\quad n_2,\quad n_3)$ in (32) is the unique positive Steady-state solution of (9). A precise local stability result of the positive Steady-state solution E_1 is obtained from the variation matrix given in (11), evaluated at $E_1(n_1,\quad n_2,\quad n_3)$ has the form of

$$D(E_1) = \begin{pmatrix} d\left(1-\frac{g_1}{bme}\right) & \frac{(bme-1)(bme-g_2+dg_2)}{e(bme)^2} & d\left(\frac{bme-g_2}{me}\right) \\ d\frac{g_1}{bm} & -\frac{(bme-1)(bme-g_2+dg_2)}{e(bm)^2} & d\frac{g_2}{m} \\ 0 & \frac{1}{m(be)^2} & 0 \end{pmatrix}. \tag{33}$$

Taken $d=1$ (assuming that all the seeds are viable) yield the following matrix after simplification;

$$D(E_1) = \begin{pmatrix} \left(1-\frac{g_1}{bme}\right) & \frac{bme-1}{bme^2} & \frac{bme-g_2}{me} \\ \frac{g_1}{bm} & \frac{1-bme}{bm} & \frac{g_2}{m} \\ 0 & \frac{1}{m(be)^2} & 0 \end{pmatrix}. \tag{34}$$

Then characteristics polynomial of D(E_1) is given by

$$f(\lambda) = \left(\frac{bme-g_1}{bme}-\lambda\right)\left[-\lambda\left(\frac{1-bme}{bm}-\lambda\right)-\frac{g_2}{(bme)^2}\right]-\frac{bme-1}{bme^2}\left(-\frac{g_1}{bm}\lambda\right)+\frac{bme-g_2}{me}\left(\frac{g_1}{b(bme)^2}\right)$$

After simplification, it gives

$$f(\lambda) = \lambda^3 - a_1\lambda^2 - a_2\lambda - a_3, \tag{35}$$

Where

$$a_1 = -\frac{e(1+bm-bme)+g_1}{bme}, \quad a_2 = -\frac{g_2+(bme-1)[g_1+e(bme-g_1)]}{bme^2} \quad \text{and}$$

$$a_3 = -\frac{g_1-g_2}{(bme)^2}.$$

To utilize Schur-Cohn stability criteria (also called Jury Criteria), Let

$$b_1 = a_2 - a_3a_1, \qquad b_2 = a_1 - a_3a_2, \qquad b_3 = 1 - a_3^{\ 2}.$$

and

$$c_1 = b_3b_1 - b_1b_3, \qquad c_2 = b_3b_2 - b_1b_2, \qquad c_1 = b_3^{\ 2} - b_1^{\ 2}.$$

The necessary and sufficient conditions for the characteristic polynomial $f(\lambda)$ to have all its solutions inside the unit circle (less than 1) are

i. $f(1) > 0, \quad (-1)^3 f(-1) > 0, \quad |a_3| < a_0 = 1.$

ii. $|b_3| < |b_1|, \quad |c_3| < |c_2|.$

Now, suppose that the existence conditions for E_1 and condition C2 are satisfied, we have the following local asymptotic stability result for the positive steady-state point (solution) E_1.

Theorem 2

The unique positive steady-state point E_1, is locally asymptotically stable if the condition $g_2 < g_1 + (bme)^2$ is satisfied.

Proof

For $\lambda = 1$, (35) becomes

$$f(1) = 1 - a_1 - a_2 - a_3$$

Substituting for $a_1, a_2,$ and a_3 we have

$$f(1) = 1 + \frac{e(1+bm-bme)+g_1}{bme} + \frac{g_2 + (bme-1)[g_1 + e(bme-g_1)]}{bme^2} + \frac{g_1 - g_2}{(bme)^2} > 0. \tag{36}$$

For $\lambda = -1$, (35) becomes

$$(-1)^3 f(-1) = 1 + a_1 - a_2 + a_3$$
$$= 1 + \frac{e(1+bm-bme)+g_1}{bme} - \frac{g_2 + (bme-1)[g_1 + e(bme-g_1)]}{bme^2} + \frac{g_1 - g_2}{(bme)^2} > 0 \tag{37}$$

Also, for the constant term

$$\left| -\frac{g_1 - g_2}{(bme)^2} \right| = \frac{g_2 - g_1}{(bme)^2} < 1,$$

$$g_2 < g_1 + (bme)^2 . \tag{38}$$

Clearly, this inequality holds. Therefore, the positive steady-state point E_1, is locally asymptotically stable. The proof is complete.

This result implies that whenever the steady state population density is disturbed through management effort by mowing or herbicide application the weeds will always grow (Proliferate) to the new steady state if the control is not sustained.

3.0 RESULTS AND DISCUSSION

3.1 Application of the Model Framework to Bush Mint Weed (*Hyptis suaveolens*)

3.1.1 The Study Species

Hyptis suaveolens is a broad-leaved annual Savanna herb of neotropical origin, tropical and subtropical distribution [20, 21]. It is one of the aromatic and odoriferous annual herb weed of aggressive nature belonging to the *Lamiaceae* family that are highly utilized for medicinal purposes and research in Nigeria and endowed Countries of the world [22]. The plant has been considered as an obnoxious weed distributed in the tropical and subtropical regions, among its common names are Mint-weed, Bush-mint, Bush-tea, Horehound, Pignut and American-mint [23]. Some of its local names are efirin in Yoruba, daddoya-ta-daji in Hausa and nchuanwu in Ibo [24]. It has a major three-stage life cycle; persistent seed bank, Seedling, and Adults (mature flowers and set seed) and subsequently dies in dry season. Although it may also behave as a perennial plant if resources are available. *H. suaveolens* usually covers a large area after the rains and not allows the adjoining species to flourish, so has potential for a successful invader. Besides, *H. suaveolens* also shows strategy for better survival and establishment. It exhibits vigorous growth on the agricultural fields, wastelands and along roadsides.

Besides, *H. suaveolens* has good medicinal value owing to the presence of essential oil, a characteristic feature to the family *Lamiaceae*. Although *H. suaveolens* possess medicinal properties but it is not efficiently utilized in this context. Damage to the biodiversity of the adjoining areas, farmlands is much greater than its utilization as medicinal plants. To avoid its spread, therefore efforts should be made to check its spread so that it may not become a successful invader in near future in the farmlands.

A field observation was performed in order to obtain parameter data to demonstrate the theoretical results and the dynamics of the developed models to investigate the population density dynamics and control of *H. suaveolens*. as far as we know, no data of such nature have been collected for the weed species. The field observation was conducted at the School farm of the Federal Polytechnic Nasarawa, Nasarawa State, Nigeria. It was carried out in 2013 raining (cropping) season from April – September. About 90% of the rainfall occurs during this period that begins in March and ends in October. The vegetation at the study field was dominated by the study species growing with other grasses and weeds such as Broom weed (*Sida acuta).* Besides, the study species (*H. suaveolens*) is the most common weed found in farm lands, along roadsides, open land and waste lands in Nasarawa town of Nasarawa State and its surrounding villages. The most common agricultural productions (practice) in the surrounding farms of the study field are corn (Plate III), groundnut and guinea corn. However, the study site was not cultivated before the observation plots were established. Two study sites were established in an open field where *H. suaveolens* were fairly in abundance. Ten (10) 1x1m square plots were marked using metal pegs and wire for each site. Site 1 (one) was mowed and herbicide applied to the site 2 (two). Mean values obtained for these parameters are presented for each site in Table 1.

Plate I The established seedling of Bush Mint weed (*H. suaveolens*) growing at high density tends to have fewer stems and compete for survival to mature weed.

Plate II Mature Bush mint weed (*H. suaveolens*) in a dense infestation flowering and producing seeds.

Source Field observation 2013

3.1.2 Parameter Data and Base-line Values

Table 1 Approximate values obtained for some parameters in each population of *H.* *Suaveolens*.

Site	No of Plot 1x1 m^2	Seedling Density(mean)	established seedling Density (mean)	Mature weed Density (mean)	Fraction of establish seedling (P_e)	Fraction of mature weed (P_m)	Average seed production per Plant (b)
1	10	254.00	151,00	78.00	0.5944	0.5166	25 – 128
2	10	175.00	114.00	63.00	0.6514	0.5526	31 - 190

Source: Field Observation (2013)

The parameter base values used to validate the developed models and formulation of specific model equations for the population density dynamics of *H. suaveolens* is presented in Table 2 below

Table 2 Parameters Baseline Value

Parameter	Variation in observed rates	Base Value for the model
p_d	0.60 - 1.00	0.95
p_g	0.10 - 0.20	0.15
p_e	0.59; 0.65	0.62
p_m	0.51 ; 0.55	0.53
B	25 – 128; 31 - 190	30

3.1.3 The Dynamics of Population Density of *H. sauveolens*

The population growth rate of *H. sauveolens* at low density in the absence of intra-specific competition is examined by using (9) to obtain the projection matrix $P(0)$,thus

$$P(0)=\begin{pmatrix} 0.80 & 0 & 24.22 \\ 0.08 & 0 & 2.65 \\ 0 & 0.53 & 0 \end{pmatrix} \tag{39}$$

For the stability of the Steady-state $E_0(0,0,0)$ the characteristic equation is

$$f(\lambda)=\lambda^3-0.8\lambda^2-1.4\lambda+13.96=0 \tag{40}$$

So the three eigenvalues are

$\lambda_1=1.62$, $\lambda_2=0.07$ and $\lambda_3=-0.89$.

The dominant eigenvalue is $\lambda_1=1.62$. Therefore, the annual population growth rate of *H. suaveolens* is 62%. It is noticed that one of the eigenvalues $\lambda_1=1.62>1$ which, implies that the zero steady-state is not stable. That is bush-mint weed will always proliferate to a positive steady state density. Besides, $\mathbf{R}_0=7.07>1$ and the right eigenvector *w* associated with $\mathbf{R_0}$ gives the approximate population stable stage distribution. Thus, for *H. sauveolens* it is

$$w=\begin{pmatrix} 124 \\ 8 \\ 1 \end{pmatrix}>0. \tag{41}$$

In order to obtain its positive steady state, we use the parameter values in Table 2 and expression (32) to obtain the new fixed point (steady state) densities $E(n_1(t),\ n_2(t),\ n_3(t))$. Thus;

$$E(n_1,\ n_2,\ n_3)=\left(\frac{85.84}{0.005},\frac{8.85}{0.05},\frac{8.85}{0.93}\right)\approx(1700,177,10). \tag{42}$$

It exists and positive.

For the stability of the steady-state we use (34) at the steady state to obtain

$$D(E_1)=\begin{pmatrix} 0.93 & 0.15 & 27.93 \\ 0.01 & -0.06 & 0.27 \\ 0 & 0.01 & 0 \end{pmatrix}. \tag{43}$$

The characteristic polynomial of (43) is

$$f(\lambda)=\lambda^3-0.86\lambda^2-0.06\lambda. \tag{44}$$

From the Jury stability criterion given in Table4.1 the solutions of the characteristic polynomial (44) have magnitudes less than one.

Table 3 Jury's stability criterion for Population Density of *H.sauveolens* Model

	Condition	System	Stability Result
1	$f(1) > 0$	$f(1) = 0.08 > 0$	satisfied
2	$(-1)^n f(-1) > 0$	$(-1)^3 f(-1) = 1.80 > 0$	satisfied
3	$\|a_3\| < 1$	$\|0\| < 1$	satisfied

Since all the three conditions are satisfied, the positive steady state solution E_1 is locally asymptotically stable, which indicates that the weed density growth of *H.sauveolen.* is predictable and can be controlled. Besides, this implies that whenever the equilibrium (steady state) population density is perturbed through management effort by mowing or herbicide application the *H. suaveolens* will always grow, if the management effort is not sustained.

3.1.4 Graphical Profiles of the Population Density Dynamics of *H. sauveolens*

The parameters base value in the table 2 were employed in equations (6) – (8) to obtained the stage-structured model for non-homogeneous population density of *H. suaveolens*. The graphical profiles of the resulting system were obtained as shown in figures 2 and 3 below

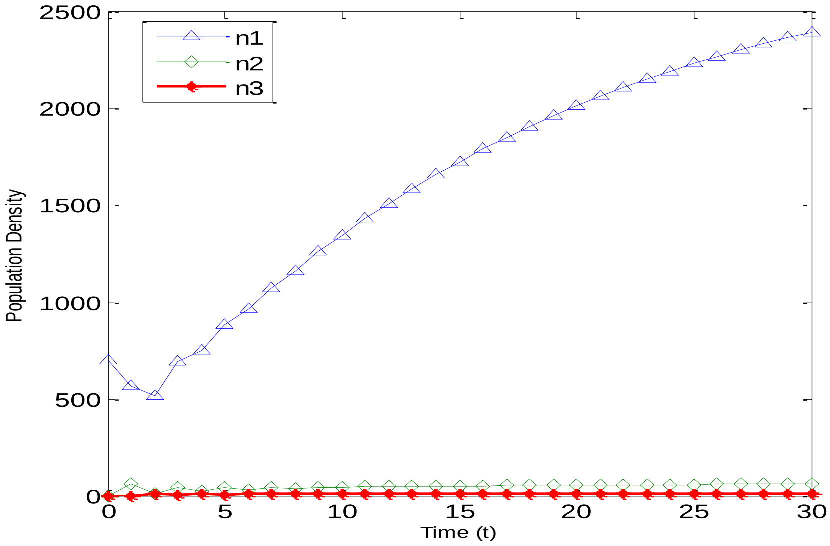

Figure 2 Dynamics of non-homogeneous population density of *H. suaveolens* with initial densities

$$(n_{1,t} \quad n_{2,t} \quad n_{3,t}) = (700 \quad 0 \quad 0)$$

The seed bank seed population were depleted due to the germination of seedling that becomes established within the first two time steps, when there were no mature weeds, which produces seed to replenish the seed bank. However, the abundance of seeds in the seed bank rises due to constant seed production by the matured *H. suaveolens* from one time step to another.

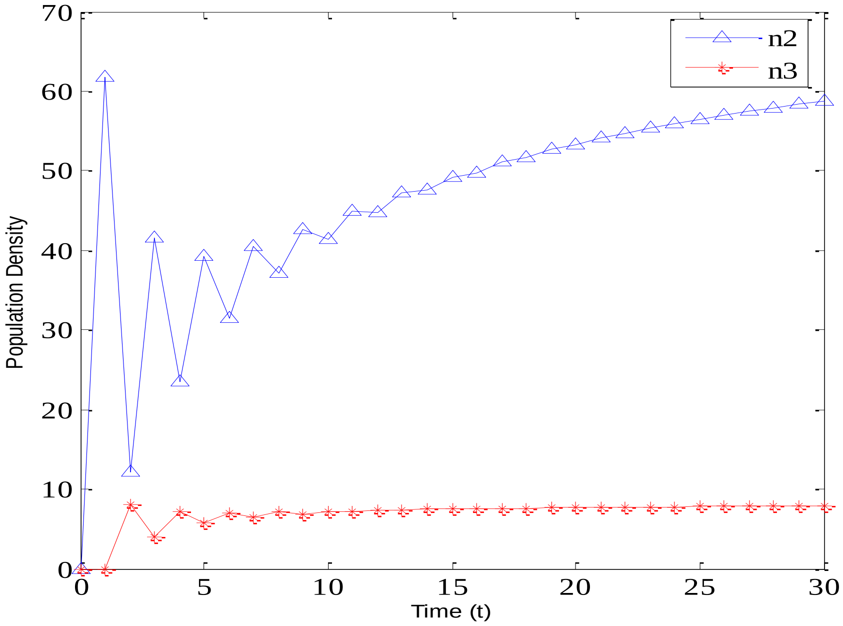

Figure 3 Dynamics of the Effective Population Density of *H. suaveolens* with initial densities

$$(n_{2,t} \quad n_{3,t}) = (0 \quad 0)$$

As observed in the figure 3, the effective densities of *H. suaveolens* tend to reach a stable stage distribution at about 30 time steps. But it's matured density become stable after 15 time steps. This may be due to its extinction after seed production, which usually create micro-site space for established seedling to become mature and more seedlings to be recruited.

4. CONCLUSION

In this paper a discrete-time staged-structured model was formulated to investigate the population density dynamics of weed proliferation. The models equation was analyzed for stability based on variational and basic reproductive value principles. The model frame work was applied to investigate the population density dynamics of Bush mint (*H. suaveolens)*, an annual weed. From the analysis, the following findings / conclusion were made.

i. The zero steady-state solution of non-homogeneous model equation always exists and is unstable. The positive steady state exists and is locally asymptotically stable. From the biological point of view we conclude that whenever the steady state population is disturbed through management effort the weeds will always proliferate

ii. Steady state density of *H. suaveolens* is globally asymptotically stable and concludes that its population density may be control or eradicated.

Competing Interests

There are no competing interests among the authors.

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Appendix

Matlab Code for Density-Dependent Stage-structured Population Density dynamics Model of Bush Mint Weed (*H. sauveolens*)

```
set(0,'DefaultAxesFontSize',12);%size of fonts;
time=30;
d=0.95;
g1=0.15;
g2=0.15;
a=0.05;
b=30;
e=0.62;
m=0.53;
n1(1)=700;
n2(1)=0;
n3(1)=0;
for t=1:time;
n1(t+1)=d*(1-g/(1+a*n2(t)))*n1(t)+b*d*(1+h/(1+a*n2(t)))*n3(t);
n2(t+1)=d*e*g*n1(t)/(1+a*n2(t))+b*d*e*h*n3(t)/(1+a*n2(t));
n3(t+1)=m*n2(t)/(1+a*n2(t));
end
%--------The graph-----------
plot(0 : time, n1, 0 : time, n2, 0 : time, n3);
leg1 = legend(‘n1’ , ‘n2’ , ‘n3’) ;
plottools
```