Mathematical modeling of Fish Resources Harvesting with Predator at Maximum Sustainable Yield

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Abstract

In this study, the population dynamic of fish is considered following Logistic model with the inclusion of harvesting. The prey-predator interaction is also considered with an assumption that the predator population which is completely theoretical and not physically defined has a little effect on the growth of prey population provided that there are no limiting factors other than the predators. This is to say that the prey-predator cycle remains stable as far as other factors are constant in the natural environment. The growth function of the predator population is constructed corresponding to the prey population, and its results showed that the predator population size is either convergent to a finite positive limit, zero or diverges to positive infinity; while the fish population size follows Logistic function and grows to an upper asymptote. Furthermore, the prey-predator interaction is considered with the assumption that the predator population has an effect on the growth of the prey population and the predator population has intra-specific competition for a limited environmental resource. Its result showed that both the prey- predator population size settle down to their corresponding coexistence equilibrium point.In both cases the maximum sustainable yield is obtained, numerical simulation and stability analysis of the model are included.

Keywords: Fish harvesting, Logistic model, Prey-predator, Koya-Goshu, Numerical simulation, Stability analysis, Maximum sustainable yield.

1. Introduction

Fishing has a lot of benefits to human beings. It serves as food, creates job opportunities and generates income. In general, it has great impact on the socioeconomic and infrastructure development of a country. As result, the demand for fish increases from time to time leading to over fishing including the spawning fishes and this may lead to a decrease in their population and finally to extinction, if no remedial measures are put in place [1, 7, 15, 9]. Researchers and scientists devise strategies to prevent the extinction of renewable resources such as fish by harvesting only optimum yield while maintaining the renewable resources above sustainable level [3, 11, 14, 15, 16].

In doing so, researchers and scientists use mathematical models to examine the interactions among populations and to predict the population size in the long run following successive harvests, side by side ensuring maximum sustainability of the population. The interaction of population dynamic in an environment can be modeled by autonomous differential equation or system of autonomous differential equations [3,7-9,11,12,15,16]. Many of differential equations especially, nonlinear differential equations have no analytic solution, but in such cases the qualitative approach together with numerical method insights the behaviors of its solution [4, 5, 6].

A generalized mathematical model is introduced for biological growth in [13] which includes the known functions such as Generalized Logistic, Particular Case of Logistic, Richards, Von Bertalanffy, Brody, Logistic, Gompertz and many other models. Furthermore, some theoretical mathematical aspects of prey predator interaction have been introduced with an assumption that “the interaction of predation leads to a little or no effect on growth of the prey population” in [12]. By considering that the prey population grows following Logistic and Von Bertalanffy the corresponding growth models describing the dynamics of predator population have been constructed and studied.

The interaction of natural communities such as preys and predators is complex and it may lead to various outcomes. Studying how predators affect the prey populations and vice versa and what stabilizes prey-predator interactions and what prevents their extinction is an important and interesting biological phenomenon.

The objective of this study was to extend the prey-predator interaction introduced in [12] by considering the prey as fish population to include harvesting fish at a maximum sustainable level. All the assumptions mentioned in [12] are taken as they are. The additional objective of this study was to extend the extended model by adding an assumption that the predation of prey has an effect on the growth of prey in addition to the harvesting effect and the predators have intra-specific competition for limited food resource.

The study includes two main parts: harvesting of fish with and without predator i.e. section 2 presents fish harvesting model without predator and section 3 presents fish harvesting with predator. The stability analysis and numerical simulation are also included. Finally, the study is completed by presenting conclusions.

2. Fish harvesting model without predator

Let's assume that the dynamics of a fish population in the environment is governed by logistic equation

$$dx/dt = rx(1 - x/K)\ (1)$$

where r and $K$are constants such that r is the linear per capita growth rate or intrinsic growth rate and K is the natural carrying capacity. Assume also that fish harvesting is started in the environment. The modeling problem is how to maximize the sustainability of the yield by determining the population growth dynamics so as to fix the harvesting rate that keeps the population at its maximum growth rate [8]. The model starts from the logistic equation and additionally assumes a level of fishing per unit of time which is proportional to the fish stock.

$$dx/dt = rx(1 - x/K) - Ex(2)$$

where Ex is the harvesting yield per unit time and E is positive constant such that the measure of the effort expended. It is clear that if $E = 0$, then the harvesting fish model of equation 2 reduces to the non-harvesting fish model of equation 1. By rearranging the harvesting fish model of equation 2, it is expressed as $dx/dt = (r - E)x\{1 - x/[K(r - E)/r]\}$. This is also a Logistic model with intrinsic growth rate $r - E$ and an asymptotic value or carrying capacity of the environment is $K(r - E)/r$ provided that

$r > E$.

The equilibrium points or constant solutions are $x = 0$ which is the trivial equilibrium point and

$x = K(r - E)/r$ which is a non-trivial equilibrium point if $r > E$. Clearly for $E > r$, $x = K(r - E)/r < 0$ which shows 0 is the only equilibrium point. The non-trivial equilibrium point is an asymptotic growth value of the harvesting fish model. Since $K(r - E)/r \leq K$ for $r - E > 0$ implies that the asymptotic values of harvesting fish population lower than the non-harvesting fish population see figure 3. If $E < r$ the maximum sustained yield (MSY) denoted by $Y(E)$ is obtained by the product of the effort $E$and the non-trivial equilibrium point $x = K(r - E)/r$. In other words,

$$Y(E) = E\,K(r - E)/r$$

Since $dY(E)/dE = K - 2KE/r = 0$ gives $E = r/2$ and $d^2Y(E)/dE^2 = -2K/r < 0$ which shows $Y(E)$ has maximum value at $E = r/2$. Therefore $Y(E) = rx/2 = rK/4$, because $x = K/2$ from the non-trivial equilibrium point. This shows that the maximum yield is obtained if the equilibrium point x reaches half of its carrying capacity Kor the maximum yield is obtained if the measure of the effort expended $E$is half of the growth rate r.

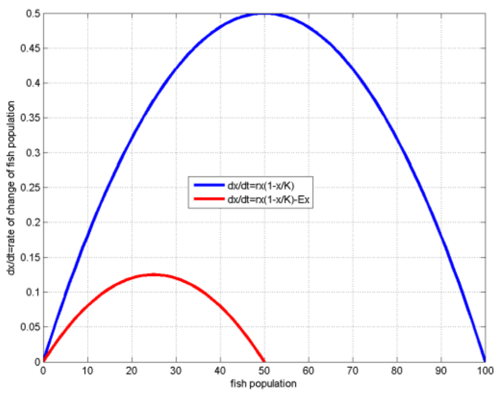

Figure 1 : Graphs of the non-harvesting fish $\boldsymbol{dx/dt = rx(1 - x/K)}$and harvesting fish $\boldsymbol{dx/dt = rx(1 - x/K) - Ex}$, where $\boldsymbol{r = 0.02}$, $\boldsymbol{K = 100}$ and $\boldsymbol{E = 0.01 = r/2}$.

2 *Stability analysis of fish harvesting without predator*

Stability analysis is determined based on the behavior of orbits closed to a constant solution or equilibrium point. An equilibrium point of a given differential equation is stable if all solution curves or orbits of the differential equation by taking the initial value near the equilibrium point attract towards to the equilibrium point otherwise the equilibrium point is unstable.

The differential equation given by equation 2 is an autonomous differential equation of one variable. The Jacobian equationis $J(x) = \partial/\partial x\,(dx/dt) = r(1 - 2\,x/K) - E$. Hence $J(0) = r - E > 0$, which shows that equilibrium point $x = 0$ is not stable, whereas $J(K\,(r - E)/r) = -(r - E) < 0$ because $r - E > 0$, which shows that equilibrium point $x = K\,(r - E)/r$ is stable. In other ward, the solution curve attractive towards the nontrivial equilibrium point and go away from the trivial equilibrium point see figure 2. The vector field visualizes a flow pattern of solution curves for the differential equation.

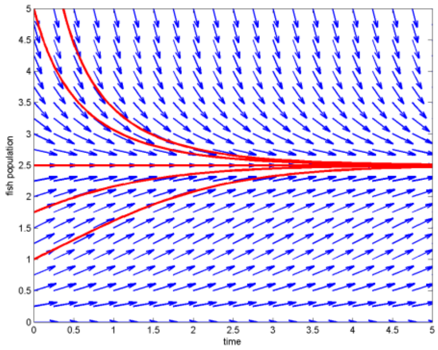

Figure 2: Vector field and some solution curve of $\boldsymbol{dx/dt = rx(1 - x/K) - Ex}$.

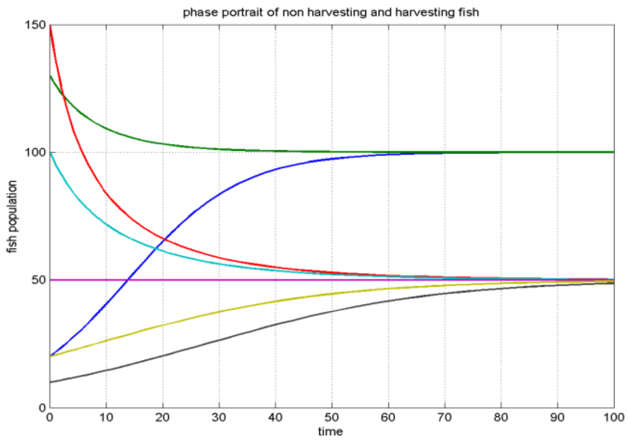

Figure 3: Orbits of fish with no harvesting $\boldsymbol{E = 0}$ and fish harvesting $\boldsymbol{E = r/2}$ with different initial values.

Both the vector field and the orbit indicates that the fish population decreases to the asymptotic growth value or non-trivial equilibrium point if the initial value is above the asymptotic growth value, and the fish population increases to the asymptotic growth value or non-trivial equilibrium point if the initial value is below the asymptotic growth value see figure 2 and 3.

1.2. *Numerical simulation of fish harvesting without predator*

In this section it is considered numerical simulation population dynamic of fish with variable combinations of the parameters r and E to understand the comparison of the asymptotic values the population dynamic of fish with and without harvesting. For more convenient, it is considered four cases on the parameter of harvesting effort E relative to the parameter of intrinsic growth rate r. Those are if $E = 0$, if $0 < E < r/2$, if $E = r/2$ and if $r/2 < E < r$. Clearly the assumption $E = 0$ describes no harvesting. The other parameters are put fixed at as K = 100, r = 0.1, $x0$ = 20. The numerical results are in figure 4.

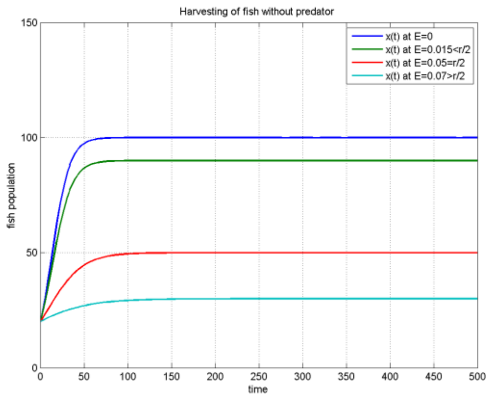

Figure 4: Numerical solution of harvesting model without predator, with initial value 20, growth rate $r = 0.1$, carrying capacity $K = 100$ and the measure of the effort is varied.

The Logistic growth function assumes that the asymptotic growth value of a population is independent of the initial value considered. This fact can be verified through simulation study. For this purpose it is solved the harvesting and non-harvesting models with different initial values and the results are given in Figure 5 and 6. Further, even if the initial population sizes are more than the asymptotic growth values all solution curves of both the harvesting and non-harvesting fishery sizes converge to their respective asymptotic growth values.

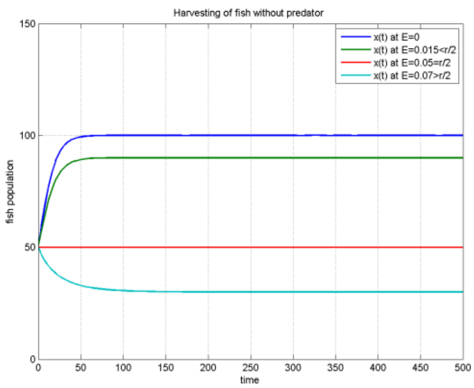

Figure 5: Numerical solution of harvesting model without predator, with initial value 50, growthrate $\boldsymbol{r = 0.1}$, carrying capacity $\boldsymbol{K = 100}$ and the measure of the effort is varied.

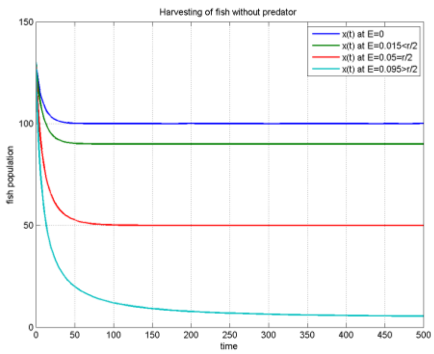

Figure 6: Numerical solution of harvesting model without predator, with initial value 50, growth rate $\boldsymbol{r = 0.1}$, carrying capacity $\boldsymbol{K = 100}$ and the measure of the effort is varied.

In general in this case it is observed that the fish population without harvesting is higher than the fish population with harvesting and grows towards the carrying capacity of the environment regardless of the initial values. The maximum sustainable yield is obtained when the harvesting effort E is half of the intrinsic growth rate r or the population size of the fish reaches half of its carrying capacity and the yield is $rK/4$. As the harvesting rate increases the fish population decreases and it may reach up to extinction.

3. Fish harvesting model with predator

Let's consider the prey-predator model with prey $x(t)$ as fish and $y(t)$ as predator. Assume also, the interaction of the prey and predator population leads to a little or no effect on growth of the prey population. Moreover, it is assumed that the growth of prey population is a simple logistic with the inclusion of a harvesting term as described earlier in equation 2. Since the DE of equation 2 is separable DE its solution is given by

$$x(t) = [K(r-E)/r]/\left[1 + Be^{-(r-E)t}\right] (3)$$

where $B = \{K(r-E)/r\,K_0\} - 1$, $K_0 = x(0)$. The harvesting curve has a single point of inflection [2] at time $a = 1/(r-E)\log[\{K(r-E)/r\,K_0\} - 1]$when the growth reaches half of its asymptotic growth $x(a) = K(r-E)/2r$. It is clear that the given harvesting model is also logistic model with absolute growth $r - E$ and asymptotic growth $K(r-E)\,/r$ provided that $r - E > 0$.

In the case of predator, it is assumed that the predator population declines exponentially in the absence of prey population and grows with a rate proportional to a function of both x and y. That is, the rate of change of predator population with respect to time *t* is given by

$$dy\,/dt = -vy + sxy (4)$$

where$v$and $s$are positive constants of death rate and birth rate of the predator population. After substituting equation 3 in equation 4, the corresponding predator population growth function is derived to be:

$$y(t) = y_0 K_0^{Ks/r} e^{[(Ks(r-E)/r)-v]t}\left([K(r-E)/r]/\left[1 + Be^{-(r-E)t}\right]\right)^{-Ks/r} \qquad (5)$$

The objective is not only to obtain the analytic solution of predator corresponding the prey, but also to study and analysis what happen the predator population depending on the values of the birth parameter s and death parameter v. To study this there are three cases discussed here below.

Case I: If $Ks(r-E)/r - v = 0$, then $y(t) = y_0 K_0^{Ks/r}\left([K(r-E)/r]/\left[1 + Be^{-(r-E)t}\right]\right)^{-Ks/r}$ and

$y(\infty) = y_0(K_0/(K(r-E)/r))^{\frac{Ks}{r}}$. It can be interpreted that the predator population decays to lower asymptotic

or grows toupper asymptotic given by $y(\infty) = y_0(K_0/(K(r-E)/r))^{\frac{Ks}{r}}$, while the prey grows following

Logistic curve and reaches the upper asymptote $K(r-E)/r$ see figure 10. The two populations converge to same size if birth rate of predator is:

$$s = \frac{r}{K}\frac{\log(K(r-E)/r) - \log(y_0)}{\log(K_0) - \log(K(r-E)/r)}$$

Case II: If $Ks(r-E)/r < v$, then the predator population decays and eventually declines down to 0, while the prey population remains to follow logistic growth model with inclusion harvesting and approaches an upper asymptote $K(r-E)/r$ see figure 11.

Case III: If $Ks(r-E)/r > v$, then the predator population declines for a while and grows higher and eventually diverges ∞, while the prey population follows logistic model and grows to an upper asymptote $K(r-E)/r$ see figure 12.

3.1 Stability analysis fish harvesting with predator

The equilibrium point of prey-predator model is obtained from equation 2 and 4 simply by equating the equations to zero and solving simultaneously. The result shows the equilibrium points of the system are given at $(0, 0)$ and $(K\ (r\ - E)/r, 0)$. The Jacobian matrix of the system is

$$J(x, y) = \begin{pmatrix} r(1 - 2x/K) - E & 0 \\ sy & -v + sx \end{pmatrix}$$

The Jacobian matrix at a trivial steady state or equilibrium point is given by $J(0,0) = \begin{pmatrix} r - E & 0 \\ 0 & -v \end{pmatrix}$and its eigenvalues are $\lambda = r - E$ and $\lambda = -v$. Since $r - E > 0$ and v is positive shows the eigenvalues are opposite in sign. Therefore, the trivial steady state or equilibrium point is unstable.The Jacobian matrix at a non-trivial equilibrium point is given by $J\ (K(r - E)/r, 0) = \begin{pmatrix} -(r - E) & 0 \\ 0 & Ks(r - E)/r - v \end{pmatrix}$ and its eigenvalues are $\lambda = -(r - E)$ and$\lambda = Ks(r - E)/r - v$. Since $r - E > 0$, then $-(r - E)$ is non-positive. For $\lambda = Ks(r - E)/r - v$ thereare three cases.

Case I: If $\lambda = Ks(r - E)/r - v = 0$ implies both the eigenvalues are non-positive, then the equilibrium point $(K(r - E)/r, 0)$ is stable see figure 7.

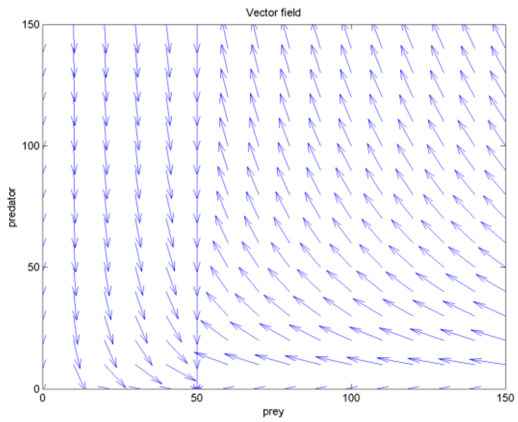

Figure 7(a): Vector field for $\boldsymbol{r = 0.1}$ and $\boldsymbol{E = r/2 = 0.05}$ and $\boldsymbol{s = 0.0084}$ for case I

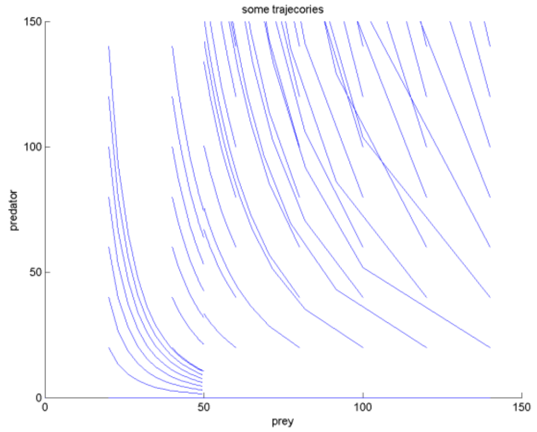

Figure 7(b): Trajectories with $\boldsymbol{r = 0.1}$ and $\boldsymbol{E = r/2 = 0.05}$ and $\boldsymbol{s = 0.0084}$ for case I

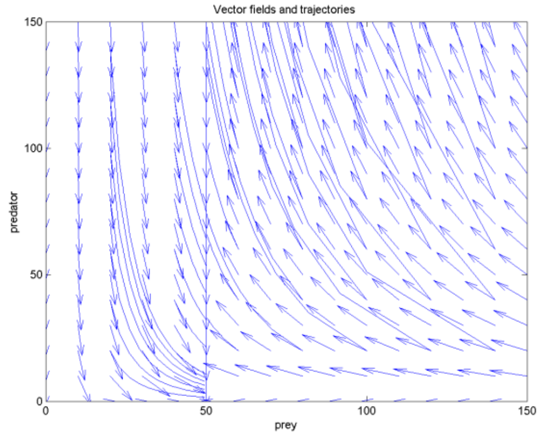

Figure 7(c): Vector field and trajectories for case I

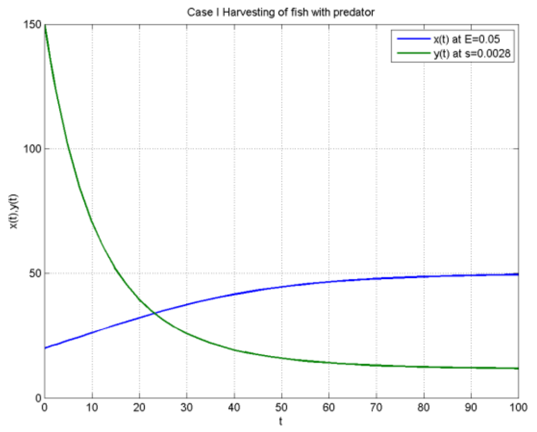

Figure 7(d): Solution curve with $\boldsymbol{K = 100}$, $\boldsymbol{x(0) = 20}$, $\boldsymbol{y(0) = 150}$, $\boldsymbol{r = 0.1}$, $\boldsymbol{E = r/2 = 0.05}$ and $\boldsymbol{s = 0.0084}$ for case I

Case II: If $\boldsymbol{\lambda = Ks(r - E)/r - v < 0}$ implies both the eigenvalues are non-positive, then the equilibrium point $(\boldsymbol{K(r - E)/r, 0})$ is stable see figure 8.

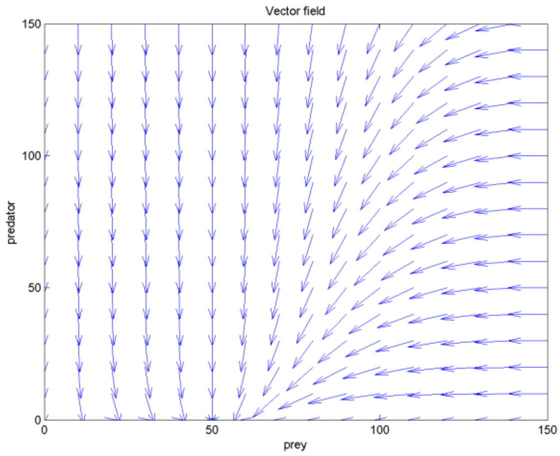

Figure8(a): Vector field with $r = 0.1$and $E = r/2 = 0.05$, $s = 0.002$ and $v = 0.3$ for case II

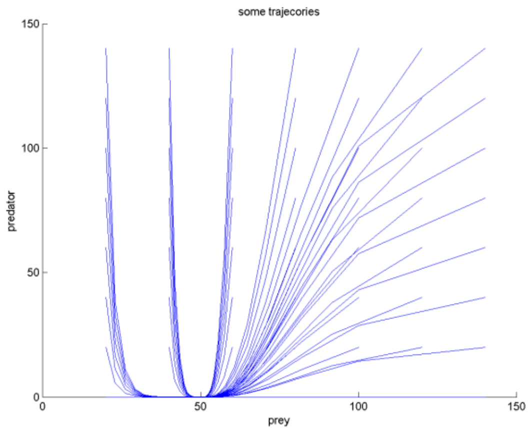

Figure8(b): Trajectories with $r = 0.1$and $E = r/2 = 0.05$, $s = 0.002$ and $v = 0.3$ for case II

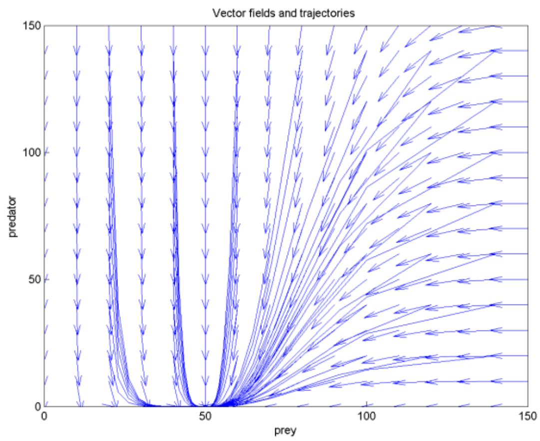

Figure 8(c): Vector field and trajectories for case II

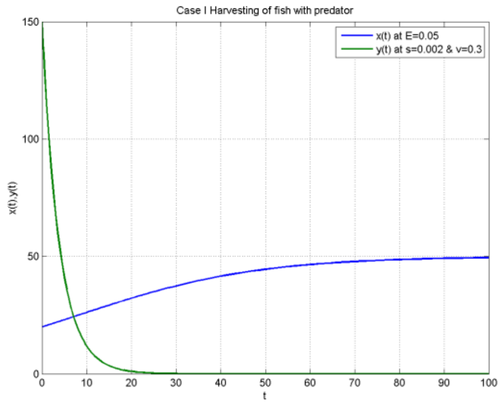

Figure 8(d): Solution curve with $\boldsymbol{K = 100}$, $\boldsymbol{x(0) = 20}$, $\boldsymbol{y(0) = 150}$, $\boldsymbol{r = 0.1}$, $\boldsymbol{E = r/2 = 0.05}$ and $\boldsymbol{s = 0.002}$&$\boldsymbol{v = 0.3}$ for case II

Case III: If $\lambda = Ks(r - E)/r - v > 0$ implies both the eigenvalues are positive, then the equilibrium point $(K(r - E)/r, 0)$ is unstable see figure 9.

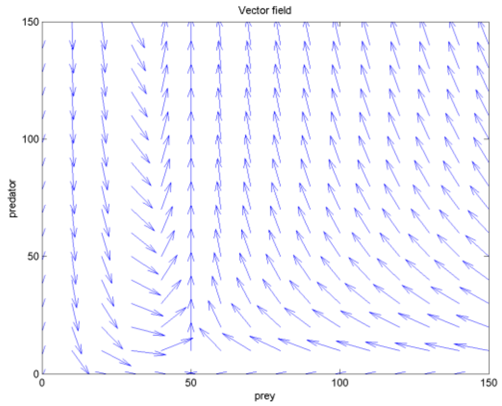

Figure 9(a): Vector field with $\boldsymbol{r = 0.1}$and $\boldsymbol{E = r/2 = 0.05}$, $\boldsymbol{s = 0.003}$ and $\boldsymbol{v = 0.1}$ for case III

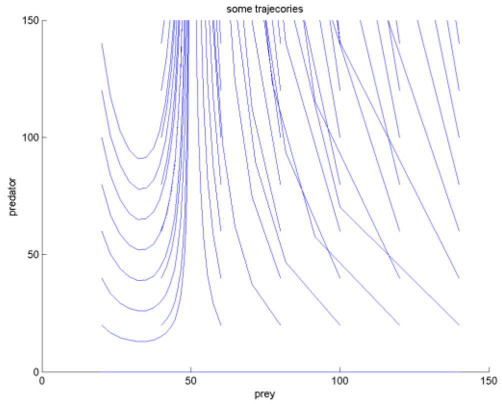

Figure 9(b): Trajectories with $\boldsymbol{r = 0.1}$and $\boldsymbol{E = r/2 = 0.05}$, $\boldsymbol{s = 0.003}$ and $\boldsymbol{v = 0.1}$ for case III

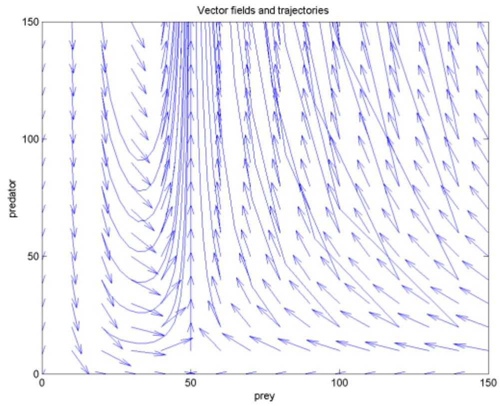

Figure 9(c): Vector field and trajectories for case III

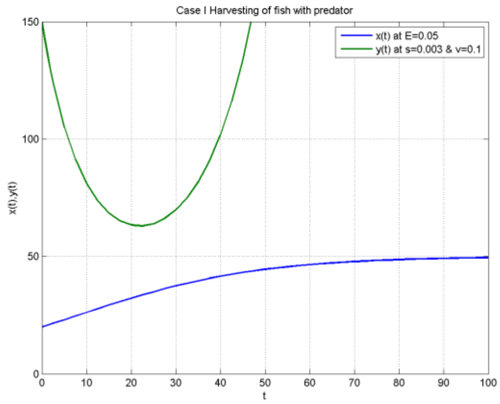

Figure 9(a): Solution curve with $\boldsymbol{K = 100}$, $\boldsymbol{x(0) = 20}$, $\boldsymbol{y(0) = 150}$, $\boldsymbol{r = 0.1}$, $\boldsymbol{E = r/2 = 0.05}$ and $\boldsymbol{s = 0.003 \& v = 0.1}$

All the vector fields and the trajectories shown in figure 7 - 9 are obtained by specifying the intrinsic growth rate $r = 0.1$, the natural carrying capacity $K = 100$, the initial value of prey $x(0) = 20$, the harvesting effort of prey $E = 0.05 = r/2$ and the initial value of predator $y(0) = 150$ with the case $Ks(r - E)/r - v = 0$, $Ks(r - E)/r - v < 0$ and $Ks(r - E)/r - v > 0$ respectively. Hence the non-trivial equilibrium point is$(K(r - E)/r, 0) = (50,0)$. For case I and case II the trajectory that starts near to the equilibrium point stays close to the equilibrium point implies the equilibrium point is stable see figure 7 and 8. For case III the trajectory that starts near to the equilibrium point go away from the equilibrium point implies the equilibrium point is unstable see figure 9. The vector field also outlines what looks like the trajectories near the equilibrium point and out of the equilibrium point.

3.2 Numerical simulation of harvesting model with predator

The advantage of numerical simulation to this study is to analyzed what happens a solution of the system based on the three cases $Ks(r - E)/r - v = 0$, $Ks(r - E)/r - v < 0$ and $Ks(r - E)/r - v > 0$ and on each case by adding four cases ($E = 0$, $0 < E < r/2$, $E = r/2$ and $r/2 < E < r$). In each case the results are obtained by specifying the remaining parameters of the prey and predator as follows.

1. prey's parameter: $K = 100, r = 0.1, K0 = x(0) = 20$
2. predator's parameter: $y0 = 1.5K$

whereas the parameter E, s and v are varied as follows.

Case I: $Ks(r - E)/r - v = 0$

i. $E = 0$: $s = 10-8$ &$v = 10-6$, $s = 0.0001$ &$v = 0.01$, $s = 0.00025$ &$v = 0.025$,$s = 0.01$ &$v = 1$
ii. $E = 0.015 < r/2$: $s = 10-8$ &$v = 8 \times 10-7$, $s = 0.00024$ &$v = 0.0204$, $s = 0.00039$ &$v = 0.03315$, $s = 0.012$ &$v = 1.02$
iii. $E = 0.05 = r/2$:$s = 10-8$ &$v = 5 \times 10-7$,$s = 0.0005$ &$v = 0.025$, $s = 0.0012$ &$v = 0.0599$, $s = 0.02$ &$v = 1$

iv. $E = 0.07 > r/2$: s = 10−8 &v = 3 × 10−7, s = 0.0009 &v = 0.027, s = 0.004 &v = 0.12, s = 0.033 &= 0.99

v.

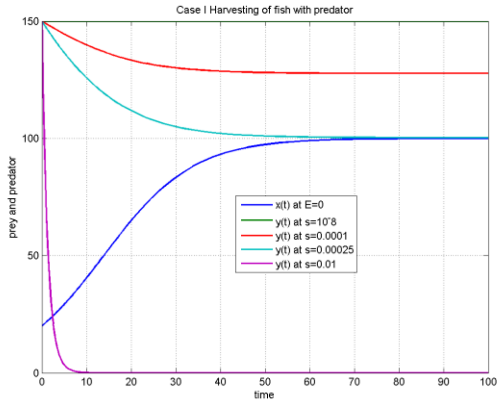

Figure 10(a): Numerical simulation of fish harvesting model in presence of predator, the case I with E=0

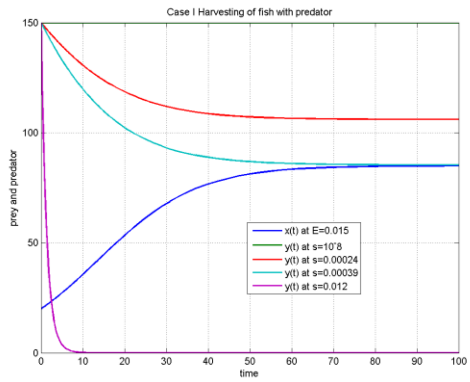

Figure 10(b): Numerical simulation of fish harvesting model in presence of predator, the case I with E=0.015

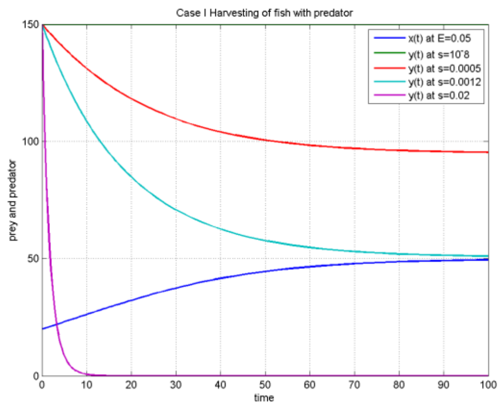

Figure 10(c): Numerical simulation of fish harvesting model in presence of predator, the case I with E=0.05

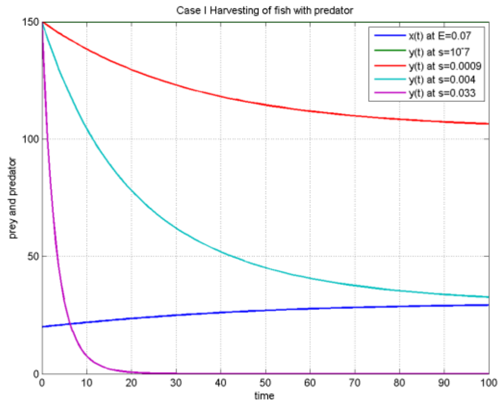

Figure 10(d): Numerical simulation of fish harvesting model in presence of predator, the case I with E=0.07

Figure 10 shows the case $Ks(r-E)/r-v=0$ and $E=0, 0<E<r/2, E=r/2$ and $r/2<E<r$. From its result it can be conclude that the prey population decreases as the harvesting rate increases and the predator population decreases as s increases. There is a situation at which both the prey populations and predator populations converge to the same size.

Case II: $Ks(r-E)/r-v<0$

i. E = 0: s = 0.001 &v = 0.15,
s = 0.001 &v = 0.2, s = 0.002 &v = 0.3, s = 0.005 &v = 0.7

ii. $E = 0.015 < r/2$: s = 0.001 &v = 0.1, s = 0.001 &v = 0.2, s = 0.002 &
v = 0.3, s = 0.005 &v = 0.7

iii. $E = 0.05 = r/2$: s = 0.001 &v = 0.1, s = 0.001 &v = 0.15, s = 0.002 &v = 0.3,
s = 0.005 &v = 0.5

iv. $0.1 = r > E = 0.07 > r/2$: s = 0.001 &v = 0.1, s = 0.001 &v =0.15,
s = 0.002 &v = 0.2, s = 0.005 &v = 0.4

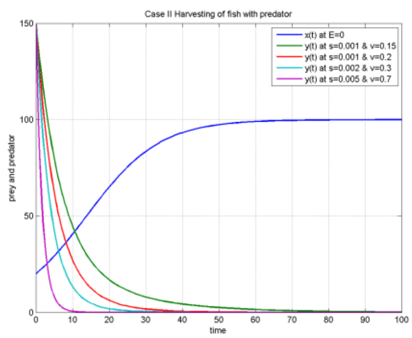

Figure 11(a): Numerical simulation of fish harvesting model in presence of predator, the case II with E=0

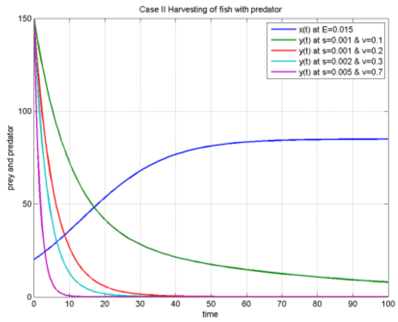

Figure 11(b): Numerical simulation of fish harvesting model in presence of predator, the case II with E=0.015

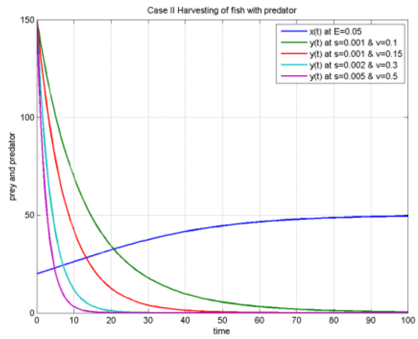

Figure 11(c): Numerical simulation of fish harvesting model in presence of predator, the case II with E=0.05

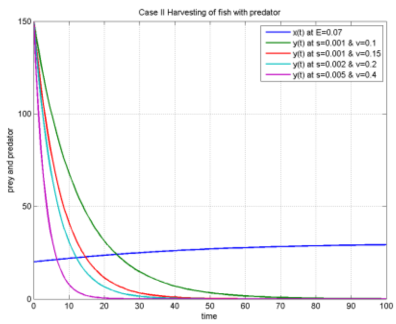

Figure 11(d): Numerical simulation of fish harvesting model in presence of predator, the case II with E=0.07

Figure 10 shows the case $Ks(r-E)/r-v<0$ and $E=0, 0<E<r/2, E=r/2$ and $r/2<E<r$. From its result it can be conclude that the prey population decreases as the harvesting rate increases and converges to lower positive values while the predator population converges to zero faster.

Case III: $Ks(r-E)/r-v>0$

i. E = 0: s = 0.001 &v = 0.075, s = 0.001 &v = 0.07, s = 0.002 &v = 0.15, s = 0.005 &v =0.25

ii. E = 0.015 $<r/2$: s = 0.002 &v = 0.12, s = 0.003 &v = 0.16, s = 0.007 &v = 0.3, s = 0.009 &v = 0.4

iii. E = 0.05 = $r/2$: s = 0.001 &v = 0.034, s = 0.002 &v = 0.07, s = 0.003 &v = 0.1, s = 0.006 &v = 0.2

iv. E = 0.07 $>r/2$: s = 0.001 &v = 0.025, s = 0.002 &v = 0.05, s = .004 &v = 0.1, $s=0.005$ & $v=0.12$

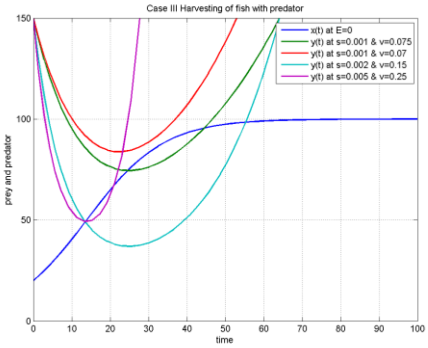

Figure 12(a): Numerical simulation of fish harvesting model in presence of predator, the case III with E=0

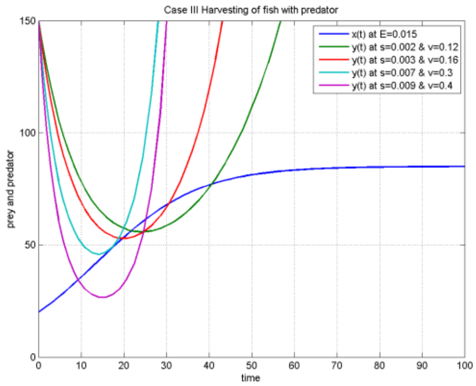

Figure 12(b): Numerical simulation of fish harvesting model in presence of predator, the case III with E=0.015

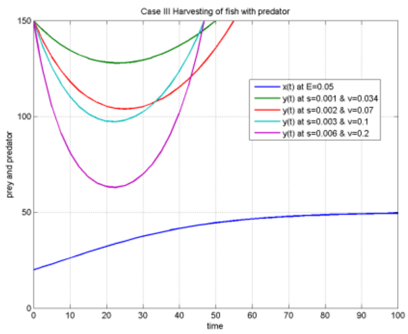

Figure 12(c): Numerical simulation of fish harvesting model in presence of predator, the case III with E=0.05

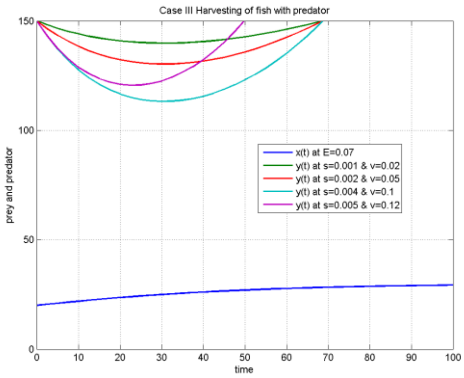

Figure 12(d): Numerical simulation of fish harvesting model in presence of predator, the case III with E=0.07

Figure 10 shows the case $Ks(r-E)/r > v$ and $E = 0, 0 < E < r/2, E = r/2$ and $r/2 < E < r$. From its results it can conclude that the prey population decreases as the harvesting rate increases and converges to lower positive values while the predator population declines or decreases for some time and then increases to infinity. The minimum point at which the curve turns to increase is donated by $t_{min}$and is found

$$t_{min} = \frac{1}{r-E}\log\left[\frac{\{K(r-E)/r\}-1}{\{Ks(r-E)/rv\}-1}\right]$$

In this case the dynamic of fish population with the inclusion of harvesting with predator was studied where fish were considered as prey and the predator is unknown under the assumption of the interaction of prey and predator populations leads to a little effect on the growth of prey population. Its result shows that in both case the prey population grows following Logistic curve and reaches the upper asymptote $K(r-E)/r$. The maximum yield is obtained if the measure of the effort expended E is half of the intrinsic growth rate r and the yield is $rK/4$. As the harvesting rate increases the prey population decreases and converges to lower positive value. The study also shows for the case of predator population either converges to a positive finite limit or zero in the first two cases. But, in case III see Figure 12 it is observed that the predator population goes to ∞ without bound which is not realized where in the case having limited food, space and other factors which helps them to growth. The task in this case is to realize by redefine the assumption of the problem on both the prey and the predator population. In the case of the prey assume that the interaction of prey and predator populations have an effect on the growth of prey population. In the case of the predator, assume the predator population have intra specific competition enhances the death rate of the species. Then the fish harvesting model of equation 2 and the predator model of equation 4 with this assumption is given by

$$\begin{cases} dx/dt = rx(1-x/K) - Ex - \beta xy \\ dy/dt = -vy + sxy - \alpha y^2 \end{cases} \quad (6)$$

whereβ be the measure of the rate of consumption of prey by predator and αbe the intra specific competition rate of predator.

The equilibrium points of equation 6 are $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (K(r-E)/r, 0)$ and $(x_3, y_3) = \left(K\frac{(v\beta + r\alpha - \alpha E)}{Ks\beta + \delta r}, \frac{r(Ks-v) - KsE}{Ks\beta + \alpha r}\right)$ provided that $v\beta + r\alpha > \alpha E$, $rKs > rv + KsE$ and $r > E$. The MSY is obtained at the coexistence equilibrium point and given by,

$$Y(E) = Ex_3 = EK\frac{(v\beta + r\alpha - \alpha E)}{Ks\beta + \delta r}\text{where } E = (\beta v + \alpha r)/2\alpha.$$

Hence the fish harvesting model with predator of equation 6 is not easy to solve analytically, the study considers only the qualitative analysis near to the equilibrium point to study the behavior of its solution with aid of numerical method. The stability analysis of the non-coexistence equilibrium point is already considered in the previous;consider only the stability analysis of the coexistence equilibrium point. The Jacobian matrix of the system is given as

$J(x,y) = \begin{bmatrix} r(1-2x/K) - E - \beta y & -\beta x \\ sy & -2\alpha y + sx \end{bmatrix}$. Then $J(x_3, y_3) = \begin{bmatrix} -\frac{r}{K}x_3 & -\beta x_3 \\ sy_3 & -\alpha y_3 \end{bmatrix}$.The characteristic equation of the Jacobian matrix at the coexistence equilibrium point is

$$\lambda^2 + p\lambda + q = 0$$

where $p = \frac{r}{K}x_3 + \alpha y_3$and $q = \left(\frac{\alpha}{K}r + s\beta\right)x_3 y_3$. Since all the parameters are positive constants and ($x3$, $y3$) are the coexistence equilibrium point implies both p and q are positive. Its consequence is the eigenvalues are either negative distinct real numbers or complex conjugate numbers with negative real part or equal negative real number. In both cases the equilibrium point is stable see figure 13.

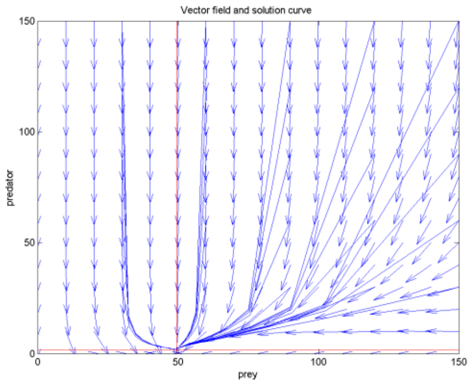

Figure 13: Vector fields and same trajectories of equation 6 with r = 0.1, k = 100, E = 0.0502, v = 0.034, s = 0.001, β = 0.0001, $\boldsymbol{\alpha} = \mathbf{0.01}$ towards the coexistence equilibrium point (x_3, y_3)

The trajectories that starts near to the coexistence equilibrium point stays close to the equilibrium point implies the equilibrium point is stable see figure 13. The vector field also outlines what looks like the trajectories near as well as out of the equilibrium point. Furthermore, the solution curve using numerical method is shown in figure 14. The fish population or prey population grows to their asymptotic growth$K\frac{(v\beta+r\alpha-\alpha E)}{Ks\beta+\delta r}$, and the predator population decreases and closes to $\frac{r(Ks-v)-KsE}{Ks\beta+\alpha r}$. The fish population decreases as the harvesting rate increases and the maximum yield is obtained if the harvesting $E = (\beta v + \alpha r)/2\alpha$.

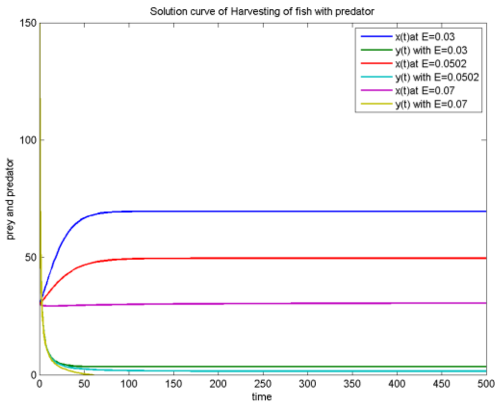

Figure 14: Numerical solution of fish harvesting model with predator with prey initial value $x_0 = 30$, growth rate r = 0.1 for prey, carrying capacity K = 100 for prey, the measure of the effort varied and predator initial value $y0$ = 1.5K, v =0 .034,s =0.001, $\beta = 0.0001$, $\alpha = 0.01$.

4. Conclusions

In this study, the population dynamic of fish with and without harvesting, and with and without predator was considered. The fish population without harvesting is assumed to follow Logistic model with constant intrinsic growth rate r, asymptotic growth K and the harvesting rate is constant. As a result, the corresponding harvesting model is also Logistic model with constant intrinsic growth rate $r - E$ and asymptotic growth $K(r-E)/r$provided that $r > E$. The modeling problem is how to maximize the sustainability of the yield by determining the population growth dynamics so as to fix the harvesting rate which keeps the population at its maximum growth rate. The maximum yield is obtained if the equilibrium point or constant solution of the fish population reaches half of its carrying capacity, K. In other words, the maximum yield is obtained if the measure of the effort expended E is half of the intrinsic growth rate r and the yield is $rK/4$.

The fish population dynamic with constant harvesting rate or prey is also considered with predators based on the assumption that the interaction of a predation leads to a little effect on growth of the prey or fish population. The predator model corresponding to fish harvesting is derived analytically and further analyzed analytically and numerically. The simulation study and the extended analysis of the models show that the predator population either converges to a positive finite limit, zero or diverges to positive infinity, while the prey population grows following Logistic curve and reaches the upper asymptote $K(r-E)/r$. There is a situation at which both the prey populations and predator populations converge to the same size. There is also a situation where the predator population declines for some time and then starts to increase and diverges to infinity without bound which is not realizable when there is limited food, space and other factors necessary for their growth. This follows intra-specific competition among the predator population and enhances the death rate of the species. In addition, if the rate of consumption of prey by predator is not near to zero, the interaction of prey and predator has an effect on the growth of prey population other than the harvesting effect. Besides, the dynamic population of the prey-predator was also considered with the assumption that the interaction of prey and predator has an effect on the growth of prey population in addition to harvesting, and the predators have intra-specific competition among themselves. Its results pointed out that the fish population or prey population grows to their asymptotic growth $K\frac{(v\beta+r\alpha-\alpha E)}{Ks\beta+\delta r}$, and the predator population decreases and comes closer to $\frac{r(Ks-v)-KsE}{Ks\beta+\alpha r}$. The fish population decreases as the harvesting rate increases and the maximum yield is obtained if the harvesting rate is $E=(\beta v+\alpha r)/2\alpha$.

Moreover, in both cases of fish harvesting with and without predator equilibrium points are identified, which are stable only under some specific conditions. In general, the analytic, qualitative and numerical simulation studies have revealed some insights to the problem addressed in this paper so that the models obtained can be applied to the real-world situations, where they may help to get maximum sustainable harvest without extremely affecting the fish population in the environment and also to keep the prey-predator relationship in balanced condition.

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