Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.4, 2016

Approximate solution of two-dimensional Volterra integral equation by Chebyshev polynomial method and Adomian decomposition method

M.H.Saleh, D.Sh.Mohamed, and R.A.Taher*

Mathematics Department, Faculty of Science,
Zagazig University, Zagazig,Egypt Egypt
** E-mail:raghadadel987@yahoo.com*

Abstract

In this paper we investigate the numerical solution of two dimensional Volterra integral equations by two different methods, Chebyshev polynomial and Adomian decomposion method. Two numerical examples are given to illustrate the methods. Acomparison between the two methods is given.

keywords: Chebyshev polynomials , Adomian decomposion method , two dimensional integral equations , collocation points.

1. Introduction

Many problems can be modeled by two dimensional integral equations from various sciences and engineering applications. furthermore most problems can not be solved analytically , and hence finding good approximate solution , using numerical method.

Recently , many works have been focusing on the development of more advanced and efficients methods for two dimensional integral equations such as collocation method , Chebyshev polynomial method , Successive approximation method , Galerkin method , Variational iteration method , Adomian decompostion method and the Homotopy perturbation method and others see ([1-5], [8-10]).

In this paper we will apply Chebyshev polynomials method and Adomian decomposition method for solving two dimensional Volterra integral equations of the second kind. For this aim we are concerned with the numerical solution of the following two dimensional Volterra integral equation :

$$u(x,t) = f(x,t) + \int_{-1}^{t}\int_{-1}^{x} k(x,t,y,z)\ u(y,z)dydz, \qquad x,t \in [0,1], \tag{1.1}$$

where $u(x,t)$ is the unknown function, $f(x,t)$ and $k(x,t,y,z)$ are given continuous functions defined on $[-1,1]^2$ and $[-1,1]^4$ respectively.

2.Solution of two dimensional integral equations by Chebyshev polynomials method

In this section we introduce some properties of Chebyshev polynomial of first kind which will be help us to construct our main results . This method is based on approximating the unknown function $u(x,t)$ as :

$$u(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}\ T_i(x)\ T_j(t)\ ,\qquad x,t\in[0,1]\ , \tag{2.1}$$

where a_{ij}, are constants to be determined,

$T_i(x)$ is Chebyshev polynomial of the first kind which is defined as:

$$T_i(x)=\cos i\theta\ ,\qquad\qquad x=\cos\theta,$$

and the following recurrence formulas :

$$T_i(x)=2xT_{i-1}(x)-T_{i-2}(x),\qquad i=2,3,\dots,$$

with the initial conditions :

$$T_0(x)=1\ ,\qquad T_1(x)=x.$$

If the infinite series in (2.1) is truncated, then (2.1) can be written as :

$$u(x,t)=u_N(x,t)\approx\sum_{i=0}^{N}\sum_{j=0}^{N}a_{ij}T_i(x)\ T_j(t)\ . \tag{2.2}$$

Substituting from equation (2.2) into equation (1.1) we obtain :

$$\sum_{i=0}^{N}\sum_{j=0}^{N}a_{ij}\ [T_i(x)\ T_j(t)-\int_{-1}^{t}\int_{-1}^{x}k(x,t,y,z)\sum_{i=0}^{N}\sum_{j=0}^{N}a_{ij}\ T_i(y)\ T_j(z)dy\ dz=f(x,t)\ . \tag{2.3}$$

Hence the residual equation is defined as :

$$R_N(x_r,t_s)=\sum_{i=0}^{N}\sum_{j=0}^{N}a_{ij}[T_i(x_r)T_j(t_s)-\int_{-1}^{t_s}\int_{-1}^{x_r}k(x_r,t_s,y,z)\ T_i(y)\ T_j(z)dydz]-f(x_r,t_s)=0\ , \tag{2.4}$$

for Gauss - Chebyshev - lobatto collocation points [6]

$$x_r=\cos(\frac{r\pi}{N})\ ,\qquad t_s=\cos(\frac{s\pi}{N})\qquad r,s=0,1,\dots,N. \tag{2.5}$$

Equation (2.4)can be written as :

$$\sum_{i=0}^{N}\sum_{j=0}^{N}a_{ij}\ [T_i(x_r)\ T_j(t_s)-\int_{-1}^{t_s}\int_{-1}^{x_r}k(x_r,t_s,y,z)\ T_i(y)\ T_j(z)\ dy\ dz\ ]=f(x_r,t_s). \tag{2.6}$$

Clearly, the obtained system of linear algebraic equations contains $(N+1)^2$ equations in the same number as unknowns. Solving this sysyem we obtain the value of the constants a_{ij} such that $i,j=0,\dots,N$.

3. Solution of two dimensional integeral equations by Adomian decomposition method

In this section will study the numerical solution of two dimensional Volterra integral equation (1.1) by Adomian decomposition method.

Adomian decomposition method defines the solution by series [7]:

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \ , \tag{3.1}$$

where the components $u_0(x,t),\ u_1(x,t),\ u_2(x,t),\dots,\ u_i(x,t),\dots$ are determined recursively by :

$$u_0(x,t) = f(x,t),$$

$$u_1(x,t) = \int_{-1}^{t}\int_{-1}^{x} k(x,t,y,z)u_0(y,z)dy\ dz,$$

$$\vdots$$

$$u_{i+1}(x,t) = \int_{-1}^{t}\int_{-1}^{x} k(x,t,y,z)u_i(y,z)dydz, \tag{3.2}$$

decomposition method component $u_0(x,t)$ defined by the function $f(x,t)$ as described above. The other components namely $u_1(x,t), u_2(x,t),\dots, etc$, are derived recurrently.

4. Numerical Examples

In this section some numerical examples of two dimensional volterra integral equation are presented to illustrate the methods. All results are obtained by using Maple 17 .

Example (1) :

consider the following two dimensional Volterra integral equation :

$$u(x,t) = x^2t^2 - \frac{1}{4}(\frac{1}{4}t^4 - \frac{1}{4})\ (x^4-1) - \frac{1}{9}\ xt(x^3+1)\ (t^3+1) + \int_{-1}^{t}\int_{-1}^{x}\ [xt+yz]u(y,z)dy\ dz\ ,$$
$$x,t \in [0,1], \tag{4.1}$$

with the exact solution is $u(x,t) = x^2t^2$.

1. Applying Chebyshev polynomial of the first kind for equation (4.1) when $N = 2$, and by using the collocation points (2.5) we obtain

$$x_0 = 1,\quad x_1 = 0,\quad x_2 = -1,\quad t_0 = 1,\quad t_1 = 0,\quad t_2 = -1.$$

Substituting into (2.2) when $N = 2$ we obtain:

$$u(x,t) = \sum_{i=0}^{2} \sum_{j=0}^{2} a_{ij} T_i(x) T_j(t). \tag{4.2}$$

Substituting from equation (4.2) into (4.1) we have

$$\sum_{i=0}^{2}\sum_{j=0}^{2} a_{ij}[T_i(x_r)T_j(t_s) - \int_{-1}^{t_s}\int_{-1}^{x_r}(x_r t_s + yz)T_i(y)\ T_j(z)dydz] = x_r^2 t_s^2 - \frac{1}{4}\ (\frac{1}{4}t_s^4 - \frac{1}{4})\ (x_r^4 - 1) - \frac{1}{9}\ x_r t_s (x_r^3 + 1)\ (t_s^3 + 1). \tag{4.3}$$

Applying the collocation points to equation (4.3) we obtain a system of linear algebraic equations contains 9 equations with the same number of constants by solving this system we obtain the values of the constans as follows :

$$a_{00} = \frac{1}{4},\quad a_{01} = 0,\quad a_{02} = \frac{1}{4},\quad a_{10} = 0,\quad a_{11} = 0,\quad a_{12} = 0,\quad a_{20} = \frac{1}{4},\quad a_{21} = 0,\quad a_{22} = \frac{1}{4}.$$

Substituting from these values into equation (4.2) we obtain the approxmiate solution which is the exact solution.

2. Applying Adomian decomposition method for equation (4.1) when $N = 6$,

$$u_0(x,t) = x^2t^2 - (\frac{1}{16}t^4 - \frac{1}{16})\ (x^4 - 1) - \frac{1}{9}\ xt(x^3 + 1)\ (t^3 + 1)$$

$$u_1(x,t) = [\frac{-25}{5184}x^6 - \frac{17}{2592} + \frac{1}{192}x^2 - \frac{1}{162}x^3](t^6 - 1) + [\frac{-1}{90}xt(x^2 - 1) - \frac{1}{144}xt(x^5 + 1) + \frac{1}{80}xt(x+1)](t^5 + 1) + [\frac{1}{16}x^4 - \frac{1}{16}](t^4 - 1) + [\frac{-1}{162}x^6 + \frac{1}{162} + \frac{1}{3}(\frac{1}{3}xt - \frac{1}{27})(x^3 + 1)]\ (t^3 + 1) + [\frac{1}{2}(\frac{-1}{18}xt - \frac{1}{32})(x^2 - 1) + \frac{1}{192}x^6 - \frac{1}{192} - \frac{1}{90}xt(x^5 + 1)](t^2 - 1) - \frac{1}{16}xt(x+1)(t+1) + \frac{1}{80}xt(x^5 + 1)(t+1)$$

$$\vdots$$

absolute error of Adomian decomposition method is shown in Table 1 and figure 1

Table 1: Numerical results of example 1 by Adomian decomposition method $N = 6$

(x,t)	Exact Sol	Approximate sol	Abs.error
$(0,0)$	0	-4.6450×10^{-9}	4.6×10^{-9}
$(0,0.4)$	0	$-4.197029172 \times 10^{-9}$	4.19×10^{-9}
$(0.2,0.6)$	0.0144	0.0143999826	1.4×10^{-8}
$(0.2,0.8)$	0.0256	0.02559995022	4.9×10^{-8}
$(0.4,0.4)$	0.0256	0.0255999548	4.5×10^{-8}
$(0.4,1)$	0.1600	0.1599994338	5.6×10^{-7}
$(0.6,0.6)$	0.1296	0.1295996192	3.8×10^{-7}
$(0.6,1)$	0.3600	0.3599969649	3.03×10^{-6}
$(0.8,0.2)$	0.0256	0.02559995025	4.9×10^{-8}
$(0.8,0.8)$	0.4096	0.409595887	4.1×10^{-6}
$(1,1)$	1.000	0.999950693	4.9×10^{-5}

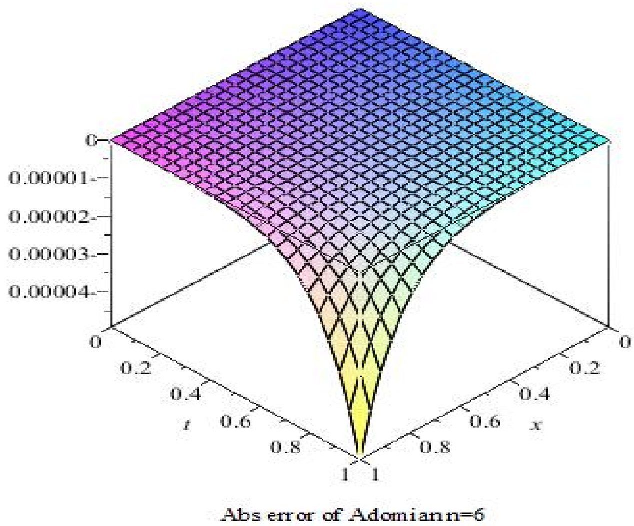

Figure 1: absolute error of example 1 by Adomian decomposition method

Example (2)

consider the following two dimensional Volterra integral equation :

$$u(x,t) = xe^t + \frac{1}{4}e^{-1}x^4 - \frac{1}{4}e^{-1} + \frac{1}{2} - \frac{1}{2}x^2 - \frac{1}{4}x^4e^t + \frac{1}{4}e^t - \frac{1}{2}x^2t + \frac{1}{2}t + \int_{-1}^{t}\int_{-1}^{x}[y^2 + e^{-z}]\, u(y,z)\, dy\, dz,$$
$$x,t \in [0,1], \tag{4.4}$$

with the exact solution $u(x,t) = xe^t$.

1. Applying Chebyshev polynomial of the first kind for equation (4.4) when $N = 3$, we obtain the collocation points as:

$$x_0 = 1, \quad x_1 = \frac{1}{2}, \quad x_2 = \frac{-1}{2}, \quad x_3 = -1,$$
$$t_0 = 1, \quad t_1 = \frac{1}{2}, \quad t_2 = \frac{-1}{2}, \quad t_3 = -1,$$

we obtain the approximate solution as :

$$u(x,t) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij}\ T_i(x)T_j(t) \tag{4.5}$$

substituting from equation (4.5) into (4.4) we have

$$\sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij}\ [T_i\ (x_r)T_j(t_s) - \int_{-1}^{t_s} \int_{-1}^{x_r} (y^2 + e^{-z})T_i(y)T_j(z)\ dy\ dz] = x_r e^{t_s} + \frac{1}{4}e^{-1}x_r^4 - \frac{1}{4}e^{-1} + \frac{1}{2} - \frac{1}{2}x_r^2 - \frac{1}{4}x_r^4 e^{t_s} + \frac{1}{4}e^{t_s} - \frac{1}{2}x_r^2 t_s + \frac{1}{2}t_s. \tag{4.6}$$

Applying the collocation points to equation (4.6) we obtain a system of linear algebraic equations contains 16 equations with the same number of constants by solving this system we obtain the values of the constans as follows :

$a_{00} = -0.003236383773,\ a_{01} = -0.002104093053,\ a_{02} = -3.741978941\times 10^{-8},\ a_{03} = -0.001132328140,$

$a_{10} = 1.261005104,\quad a_{11} = 1.125451518,\quad a_{12} = 0.2759499335,\quad a_{13} = 0.04362407872,$

$a_{20} = -0.002543934384,\quad a_{21} = -0.004543675630,\quad a_{22} = -0.001999741245,\quad a_{23} = 0,$

$a_{30} = -0.0006745672043, a_{31} = -0.001234953683, a_{32} = -0.0009799326411, a_{33} = -0.0004195461626.$

when $N = 4$,we obtain the collocation points as follows:

$$x_0 = 1, \quad x_1 = \frac{1}{\sqrt{2}}, \quad x_2 = 0, \quad x_3 = \frac{-1}{\sqrt{2}}, \quad x_4 = -1,$$
$$t_0 = 1, \quad t_1 = \frac{1}{\sqrt{2}}, \quad t_2 = 0, \quad t_3 = \frac{-1}{\sqrt{2}}, \quad t_4 = -1,$$

we obtain the approximate solution as :

$$u(x,t) = \sum_{i=0}^{4} \sum_{j=0}^{4} a_{ij}\ T_i(x)T_j(t), \tag{4.7}$$

substituting from equation (4.7) into (4.4) we have

$$\sum_{i=0}^{4}\sum_{j=0}^{4} a_{ij}\ [T_i(x_r)T_j(t_s)-\int_{-1}^{t_s}\int_{-1}^{x_r}(y^2+e^{-z})T_i(y)\ T_j(z)\ dy\ dz]=x_r e^{t_s}+\frac{1}{4}e^{-1}x_r^4-\frac{1}{4}e^{-1}+$$
$$+\frac{1}{2}-\frac{1}{2}x_r^2-\frac{1}{4}x_r^4e^{t_s}+\frac{1}{4}e^{t_s}-\frac{1}{2}x_r^2t_s+\frac{1}{2}t_s. \quad (4.8)$$

Applying the collocation points to equation (4.8) we obtain a system of linear algebraic equations contains 25 equations with the same number of constants by solving this system we obtain the values of the constans as follows:

$a_{00}=-0.00002742778578, a_{01}=-0.00009688327429, a_{02}=0.00003065836526, a_{03}=0.00004261011838,$

$a_{04}=-0.0000570373539, a_{10}=1.266088925, a_{11}=1.130262372, a_{12}=0.2714859747, a_{13}=0.04490932725,$

$a_{14}=0.005476240223, a_{20}=0.00005840645027, a_{21}=0.00005218918826, a_{22}=-0.00007920261201,$

$a_{23}=-0.00001691334898, a_{24}=0.000056072000102, a_{30}=0.00001164612589, a_{31}=0.00001643440591,$

$a_{32}=3.805361185\times10^{-8}, a_{33}=-0.000003881730473, a_{34}=8.685404949\times10^{-7}, a_{40}=0.000003516182088,$

$a_{41}=0.000002083285025, a_{42}=-0.000005760308908, a_{43}=-2.743862176\times10^{-8}, a_{44}=0.000004299973224.$
Substituting from these constants into (4.7) we obtain the approximate solution.

2. Applying Adomian decomposition method for equation (4.4) when $N=5$ we obtain:

$$u_0(x,t)=xe^t+\frac{1}{4}e^{-1}x^4-\frac{1}{4}e^{-1}+\frac{1}{2}-\frac{1}{2}x^2-\frac{1}{4}x^4e^t+\frac{1}{4}e^t-\frac{1}{2}x^2t+\frac{1}{2}\ t$$

$$u_1(x,t)=-[x+\frac{2}{3}+\frac{1}{20}x^5t^2e^t-\frac{1}{4}xte^t-\frac{1}{28}e^{-1+t}x^7t-\frac{1}{12}x^3t^2e^t-\frac{1}{2}tx^2e^t-\frac{1}{6}x^3te^t+$$
$$\frac{3}{20}x^5te^t+\frac{1}{12}e^{-1+t}x^3t-\frac{13}{84}e^{-1+t}-\frac{1}{3}e^{1+t}+\frac{17}{84}e^{2t}-\frac{1}{3}x^3+\frac{1}{28}x^7e^{2t}-\frac{1}{4}e^{2t}x^4$$
$$-\frac{1}{12}x^3e^{2t}-\frac{1}{14}e^{-1+t}x^7+\frac{1}{4}x^4e^{-1+t}+\frac{1}{6}x^3e^{1+t}+\frac{1}{6}e^{-1+t}x^3-\frac{1}{2}xe^{1+t}+\frac{1}{21}te^{-1+t}-$$
$$\frac{1}{4}xe^{-1}+\frac{1}{20}x^5e^{-1}-\frac{1}{6}x^3t+\frac{1}{2}xt+\frac{7}{30}te^t-\frac{1}{30}t^2e^t+\frac{1}{20}x^5e^t-\frac{1}{12}x^3e^t-$$
$$\frac{1}{2}x^2e^t+\frac{1}{3}t-\frac{1}{5}e^{-1}+\frac{7}{15}e^t]e^{-t}$$
$$\vdots$$

absolute error example 2 is shown in Table 2 and figures 2, 3 .

Table 2.Numerical results of example 2 by Chebyshev polynomials method and Adomian decomposition method for different values of N

(x,t)	Abs.error of Chebyshev $N=3$	Abs. error of Chebyshev $N=4$	Abs.error of Adomian $N=5$
$(0,0)$	2.6×10^{-3}	2.9×10^{-4}	4.5×10^{-5}
$(0,0.8)$	2.2×10^{-3}	9.9×10^{-5}	3.02×10^{-4}
$(0.2,0.6)$	5.3×10^{-4}	2.7×10^{-4}	5.2×10^{-4}
$(0.2,0.8)$	2.2×10^{-3}	4.1×10^{-5}	7.1×10^{-4}
$(0.4,0.4)$	4.6×10^{-3}	6.6×10^{-4}	8.1×10^{-4}
$(0.4,0.8)$	8.4×10^{-4}	2.7×10^{-5}	1.6×10^{-3}
$(0.6,0)$	1.2×10^{-2}	1.4×10^{-4}	5.7×10^{-4}
$(0.8,0)$	1.3×10^{-5}	1.3×10^{-5}	1.2×10^{-3}
$(0.8,0.8)$	7.5×10^{-3}	2.1×10^{-4}	9.2×10^{-3}
$(1,1)$	3.1×10^{-2}	6.7×10^{-5}	3.1×10^{-2}
$(1,0.2)$	1.8×10^{-2}	6.6×10^{-4}	5.7×10^{-3}

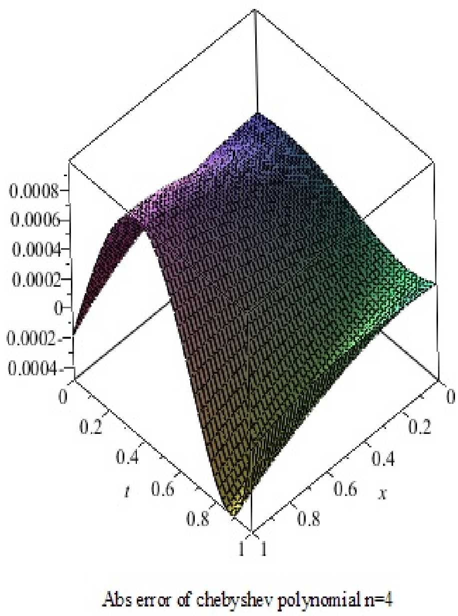

Figure 2: absolute error of example 2 by chebyshev polynomial method

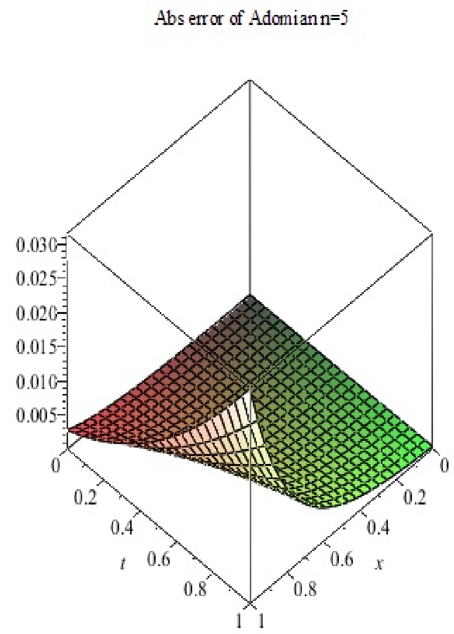

Figure 3: absolute error of example 2 by Adomian decomposition method

Conclusion

This paper concerns the numerical solutions of two dimensional volterra integral equations by using Chebyshev polynomial method and Adomian decomposition method, by comparing the results we find that Chebyshev polynomial method is better than the results of Adomian decomposition method.

References

[1] A. H. Borzabadi and M. Heidari; A successive iterative approach for two dimensional nonlinear Volterra-Fredholm integral equations. Iranian Journal of Numerical Analysis and Optimization. Vol 4, No. 1, (2014), pp 95-104.

[2] A. Fallahzadeh; Solution of Two-dimensional Fredholm Integral Equation via RBF-triangular Method. Journal of Interpolation and Approximation in Scientific Computing. Volume 2012, (2012), 1-5.

[3] F. Ziyaee , A. Tari; Regularization Method for the Two-dimensional Fredholm Integral Equations of the First Kind . International Journal of Nonlinear Science. Vol.18(2014) No.3,pp.189-194.

[4] G.A. Afroozi, J. Vahidi, M. Saeidy; Solving a Class of Two-dimensional Linear and Nonlinear Volterra Integral.Equations by Means of the Homotopy Analysis Method.International Journal of Nonlinear Science. Vol.9(2010) No.2,pp.213-219.

[5] J. Saberi Nadjafi, O. Reza Navid Samadi2 and E. Tohidi; Numerical Solution of Two-Dimensional Volterra Integral Equations by Spectral Galerkin Method. Journal of Applied Mathematics , Bioinformatics, vol.1, no.2, (2011), 159-174.

[6] J. Mason and D. Handscomb; Chebyshev polynomials. CRC Press LLC, 2003.

[7] M. Rahman.; Integral Equations and their Applications. Britain Athenaeum Press Ltd, (2007).

[8] Sh. Sadigh Behzadi; The Use of Iterative Methods to Solve Two-Dimensional Nonlinear Volterra-Fredholm Integro-Differential Equations.Communications in Numerical Analysis. Volume 2012, (2012) 1- 20.

[9] S. Nemati, Y. Ordokhani; Numerical solution of two-dimensional nonlinear Volterra integral equations by the Legendre polynomials. J. Sci. Tarbiat Moallem University. Vol. 11, No. 2, (2012), 195-210.

[10] Z. Avazzadeha, M. Heydarib and G. B. Loghmania; A Comparison between Solving Two Dimensional Integral Equations by the Traditional Collocation Method and Radial Basis Functions. Applied Mathematical Sciences, Vol. 5, (2011), no. 23, 1145 - 1152