SOME RESULTS ON SEMI-REFLEXIVITY AND REFLEXIVITY IN LOCALLY CONVEX SPACES

G. C. Dubey¹, S.S. Rajpui² and Atar Singh Meena³
¹²Dept. of Mathematics, Govt. M.G.M. Post-graduate college, Itarsi (M.P.), India.
³Research scholar, Govt. M.G.M. Post-graduate college, Itarsi (M.P.), India.

Abstract
In this paper we discuss that if E[τ] is a barreled space such that every bounded subset of E is relatively compact, then E[τ] is reflexive, and that a barreled space E[τ] in which there is a denumerable system of convex compact subsets is reflexive. We also discuss Some hereditary-type properties of reflexive locally convex spaces.

Keywords: Bornological, barreled, hereditarily-reflexive, M-space, quasi-complete, reflexive, strong dual.


1. Introduction:
For a locally convex space E[τ], which we always consider Hausdorff, the dual is denoted as E′, strong dual is E′[τ(E)] and the bidual of E[τ] is E′′=(E′[τ(E)])′. Locally convex space E[τ] is called semi-reflexive if E = E′′. A semi-reflexive locally convex space E[τ] is called reflexive provided τ = τ(E′). The so called b-reflexivity of Kye, in [1], is nothing but reflexivity. Reflexivity is also discussed in [2] and [3] in terms of V-reflexivity and G-reflexivity, respectively.

We follow the notion of Köthe [3] for notations and terminology, unless specifically mentioned. Recall that a locally convex space E[τ] is said to be quasi-complete if every bounded closed subset of E[τ] is complete. If E is quasi-complete for the weak topology τₐ(E′), then we say that it is weakly quasi-complete. E[τ] is called sequentially complete if every τ-Cauchy sequence has a limit point in E. A closed absorbent absolutely convex set in a locally convex space E[τ] is called a barrel. A locally convex space E[τ] is said to be barreled if the barrels form a base of neighborhoods of origin 0. Equivalently, E[τ] is barreled if and only if topology τ coincides with the strong topology τₐ(E′). E[τ] is called quasi-barreled if τ coincides with τₐ(E′), the topology of uniform convergence on strongly bounded sets in E′. Equivalently, E[τ] is quasi-barreled if and only if every barrel which absorbs all bounded sets of E[τ] is a neighborhood of 0. A barreled space E[τ] is called an (M)-space (Montel space) if every bounded subset of E is relatively compact. A Banach space cannot be an (M)-space unless it is finite-dimensional ([5], §7). A locally convex space is called metrizable if and only if it has a countable base of neighborhoods of 0. A complete metrizable locally convex space is called an (F) space. E[τ] is called bornological if every absolutely convex set which absorbs all the bounded subsets of E is a neighborhood of 0.

Let us mention the following theorem, from [6], of characterizations of semi-reflexivity for locally convex spaces.

1.1 Theorem: For a locally convex space E[τ], the following assertions are equivalent:
(i) E[τ] is semi-reflexive,
(ii) E is weakly quasi-complete,
(iii) Every bounded subset of E[τ] is relatively weakly compact,
(iv) E′[ τₐ(E)] is barreled.

A characterization of reflexivity is given in the following theorem, which due to [4]:

1.2 Theorem: A locally convex space is reflexive if only if it is semi-reflexive and quasi-barreled if and only if it is weakly quasi-complete and quasi-barreled.

We note that a semi-reflexive locally convex space E[τ] is a Banach-Mackey space i.e. τₐ(E)-bounded sets are τₐ(E)-bounded in E′ (see [7]). We also note that, like a reflexive Banach space, a semi-reflexive locally convex space contains no copy of (Co, ∥∥₂) (see [8]).

2. Semi-Reflexivity And Reflexivity In Some Classes Of Locally Convex Spaces
Semi-reflexivity in locally convex spaces is inherited by closed subspaces ([4], §23, 3(5)), but not by quotient spaces, in general. For reflexivity is concerned, in general, it is inherited neither by closed subspaces nor by

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quotient spaces (see example-2.7). However it is a fact that every closed subspace of a reflexive (F)-space is reflexive ([4], §23, (5(10)). We generalize it in the following:

2.1 Theorem: Let F be a closed subspace of a reflexive locally convex space E[τ]. Then F with the induced topology is reflexive if and only if it is quasi-barreled.

Proof: If F is reflexive, then it is always quasi-barreled. Conversely, if F is quasi-barreled then, since E[τ] is semi-reflexive and so the closed subspace F is also semi-reflexive, we obtain, by theorem-1.2, that F is reflexive.

Note that a separated quotient of a reflexive locally convex space is not semi-reflexive, in general. However, we have the following:

2.2 Theorem: Let F be a closed subspace of a reflexive locally convex space E[τ]. Then the quotient space E/F is reflexive if and only if it is quasi-complete for the weak topology.

Proof: If E/F is reflexive (and so semi-reflexive), then it is quasi-complete for the weak topology. Conversely, if E/F is quasi-complete for the weak topology, then it is semi-reflexive (by theorem-1.1). Further, the reflexive locally convex space E[τ] is always quasi-barreled. So its separated quotient space E/F is also quasi-barreled. Hence E/F is reflexive.

An (F)-space is reflexive if and only if it is semi-reflexive (([4], §23, (5(4))). We have generalized this fact to metrizable locally convex spaces in [9], in corollary-2.7. Here, we discuss it with an independent proof.

2.3 Theorem: A metrizable locally convex space E[τ] is reflexive if and only if it is reflexive.

Proof: If E[τ] is metrizable, then, by [4], §21, (5(3)), we have we have τ = τ(E′) and so it is quasi-barreled. Therefore, if E[τ] is semi-reflexive, then it becomes reflexive. Conversely, reflexive locally convex space is always semi-reflexive by definition.

2.4 Theorem: If E[τ] is a barreled space such that every bounded subset of E is relatively compact, then E[τ] is reflexive.

Proof: Let E[τ] be a barreled space. If every bounded subset of E is relatively compact, then every closed, bounded subest B of E is compact. So B is also weakly compact, and hence it is complete for the weak topology. So E is weakly quasi-complete and so (by theorem-1.1) E[τ] is semi-reflexive. Consequently, E[τ] is reflexive.

2.5 Corollary: An (M)-space is always reflexive.

Proof: It is direct from definition of (M)-space and the theorem-2.4.

2.6 Theorem: A barreled space E[τ] in which there is a denumerable system of convex compact subsets is reflexive.

Proof: From hypothesis, E[τ] becomes a strong dual of an (M)-space (by [10], proposition-1). So E[τ] itself is an (M)-space (see [6], 5.9). Hence E[τ] is reflexive (by corollary-2.5).

2.7 Example: This example is borrowed from[4]. Consider the product spaces φω and ωφ (see [4], §13.5.). They contain the vectors of the form x = \{x₁₁, x₁₂, ..., x₂₁, x₂₂, ....\} and the vectors lying in both the products φω and ωφ has only finitely many nonzero xᵢ and therefore we can write φω∩ ωφ = φ. Consider the direct sums ϕφ ⊕ ωφ and ωφ ⊕ ϕφ. Let H₁ be the linear subspace of ϕφ ⊕ ωφ consisting of all (x , -x) , x ∈ ϕ. Let H₂ be the orthogonal space of H₁. It is easy to obtain that H₁┴ = H₂ and H₂┴ = H₁. Note that ϕφ ⊕ ωφ and ωφ ⊕ ϕφ are both locally convex spaces having closed subspaces H₁ and H₂, respectively. Since ϕφ ⊕ ωφ and ωφ ⊕ ϕφ are constructed from K by repeatedly forming locally convex direct sums and topological products, both ϕφ ⊕ ωφ and ωφ ⊕ ϕφ are reflexive and the dual of ϕφ ⊕ ωφ is ϓω ⊕ ϕφ. The quotient space (ϕφ ⊕ ϓω) / H₁ is not semi-reflexive. Now, H₂ is a closed subspace of the reflexive space ϕφ ⊕ ϕφ therefore it is semi-reflexive ([4], §23,5(5)). But H₂┴ = H₁, so the dual of H₂ coincides with (ϕφ ⊕ ϓω) / H₁. Hence H₂ cannot be reflexive. Thus we obtain that the reflexive locally convex space ϕφ ⊕ ϓω whose separated quotient (ϕφ ⊕ ϓω) / H₁ is not semi-reflexive. We also obtain that the closed subspace H₂ of the reflexive locally convex space ϕφ ⊕ ϕφ is not reflexive (Note that, here, H₂ is also an example of a semi-reflexive locally convex space which is not reflexive).

In this example, we note that the space ϓ is an (F)-space as well as an (M)-space. So ϕφ ⊕ ϓω and ϓφ ⊕ ϕφ are (M)-spaces ([4], §27, 2(4)). Now the closed subspace H₂ of an (M)-space ϕφ ⊕ ϕφ in not reflexive. Further, the separated quotient (ϕφ ⊕ ϓω) / H₁ of the (M)-space ϕφ ⊕ ϓω is not semi-reflexive. Therefore, we have the following:

2.8 Theorem: Closed subspace of an (M)-space need not be reflexive.
2.9 Theorem: Separated quotient of an (M)-space need not be semi-reflexive.
If a locally convex space $E[\tau]$ is quasi-complete for the weak topology $\tau_s(E')$, it is also quasi-complete for the finer topology $\tau$. Therefore, a semi-reflexive locally convex space is always quasi-complete. But its converse is not true, in general. Any nonreflexive Banach space is an example for this. However we have the following result due to [4]:

2.10 Theorem: If $E[\tau]$ is quasi-complete and that $\tau$ is the Mackey topology and if the strong dual $E'[\tau_b(E)]$ is semi-reflexive, then $E[\tau]$ is reflexive.

Criteria for semi-reflexivity in sequentially complete locally convex spaces are also discussed in [11], [12], and [13].

We generalize these results for sequentially complete locally convex spaces. First, we introduce the following definition: A locally convex space $E[\tau]$ is said to be boundedly-reflexive if it contains all the linear functional on the dual $E'$ which are bounded on weakly bounded subsets of $E'$.

We, further, recall that a linear functional $f$ on a locally convex space $E[\tau]$ is called locally bounded if its values remains bounded on any bounded subset of $E$. We note as well, that locally bounded linear functionals on $E[\tau]$ are continuous if and only if $E'[\tau_s(E')]$ is bornological ([4],28, 1(3)). Now we have:

2.11 Theorem: A locally convex space $E[\tau]$ is boundedly-reflexive if and only if its Mackey dual is bornological.

Proof: Let $E[\tau]$ be boundedly-reflexive. We prove that $E'[\tau_s(E)]$ is bornological. Let $u$ be a locally bounded linear functional on $E'[\tau_s(E)]$. Then, $f$ is bounded on $\tau_s(E')$-bounded sets and so also bounded on $\tau_s(E)$-bounded sets. Since $E[\tau]$ is boundedly-reflexive, $u \in E$. Now $E = (E'[\tau_s(E)])'$ implies that $u$ is continuous on $E'[\tau_s(E)]$. Hence, by the remark above, $E'[\tau_s(E)]$ is bornological.

Conversely, let the Mackey dual $E'[\tau_s(E')]$ be bornological. Let $u$ be a linear functional on $E'$ which is bounded on weakly bounded subsets of $E'$. Since the Mackey topology $\tau_s(E)$ and the weak topology $\tau(E)$ defines same bounded sets, so the linear functional $u$ is bounded on bounded subsets of $E'[\tau_s(E)]$. That is $u$ is a locally bounded linear functional on $E'[\tau_s(E)]$. Since $E'[\tau_s(E)]$ is bornological, so $u$ is continuous and so $u \in (E'[\tau_s(E)])' = E$. It means $E$ is boundedly-reflexive.

We establish a result on semi-reflexivity in sequentially complete locally convex spaces:

2.12 Theorem: A boundedly-reflexive locally convex space which is also sequentially complete is semi-reflexive.

Proof: Let $E[\tau]$ be a locally convex space which is boundedly-reflexive and sequentially complete. If $u \in E'' = (E'[\tau_s(E)])'$, then $u$ is $\tau_s(E)$-continuous and so $u$ is bounded on $\tau_b(E)$-bounded sets. Since $E[\tau]$ is sequentially complete, weakly bounded subsets of $E'$ are strongly bounded. So $u$ is also bounded on weakly bounded subsets of $E'$. Therefore, since $E[\tau]$ is boundedly-reflexive, we have $u \in E$. Hence $E[\tau]$ is semi-reflexive.

Using theorem-1.2, we have:

2.13 Corollary: A boundedly-reflexive quasi-barreled space is reflexive.

We also have an independent proof of the following result of Hampson and Wilansky [14]:

2.14 Corollary: If $E[\tau]$ is a Banach space such that its Mackey dual is bornological, then it is reflexive.

Proof: Since, the Mackey dual $E'[\tau_s(E)]$ is bornological, $E[\tau]$ is boundedly-reflexive by theorem-2.11. Further, since Banach space $E[\tau]$ is always sequentially complete. So, by theorem-2.12, $E[\tau]$ is semi-reflexive and therefore, reflexive.

3. Hereditary- Reflexivity in Locally Convex Spaces

Let $E[\tau]$ be a locally convex space. Following [15], we define:

Hereditary-reflexivity: $E[\tau]$ is called hereditarily-reflexive if its closed subspaces and separated quotients are all reflexive.
Dually Hereditary-reflexivity: $E[\tau]$ is called dually hereditarily-reflexive if each of its closed subspaces and each of the closed subspaces of its strong dual are reflexive.

Further, following [16], we define:

Strongly hereditary-reflexivity: $E[\tau]$ is called strongly hereditarily-reflexive if its closed subspaces and separated quotients are hereditarily-reflexive.

Dually strongly hereditary-reflexivity: $E[\tau]$ is called dually strongly hereditarily-reflexive if it is strongly hereditarily-reflexive and its strong dual is also strongly hereditarily-reflexive.

Now we have:

3.1 Theorem: A reflexive locally convex space $E[\tau]$ is hereditarily-reflexive if and only if its closed subspace is quasi-barreled and each of its separated quotient is weakly quasi-complete.

Proof: If $E[\tau]$ is hereditarily-reflexive, each of its closed subspaces $F$ and each of its separated quotients $E/F$ are reflexive. Therefore, $F$ is quasi-barreled and $E/F$ is weakly quasi-complete. Conversely, let $F$ be a closed subspace of $E[\tau]$ such that $F$ is quasi-barreled. Since $E[\tau]$ is reflexive, the closed subspace $F$ is also reflexive by theorem-2.1. Let $E/F$ be a separated quotient of $E[\tau]$. It is (given) weakly quasi-complete, Again, $E[\tau]$ is reflexive and so, by theorem-2.2, $E/F$ is reflexive. Hence proved.

3.2 Theorem: Hereditarily-reflexive locally convex space is reflexive.

Proof: If a locally convex space $E[\tau]$ is hereditarily-reflexive, each of its closed subspace is reflexive. In particular, the space $E$, being a closed subspace of it-self, is reflexive.

3.3 Theorem: If $E[\tau]$ is a reflexive Banach space, then it is each of (a) hereditarily-reflexive, (b) dually hereditarily-reflexive, (c) strongly hereditarily-reflexive, and (d) dually strongly hereditarily-reflexive.

Proof: We know that:

(i) If $E$ is a Banach space, then closed subspace $F$ and the quotient $E/F$ are also Banach spaces ([4], §14, 3(2), and §14, 4(3), resp.).
(ii) If $E[\tau]$ is reflexive Banach space and $F$ is its closed subspace, then $F$ as well as $E/F$ are reflexive [4],§23,5(11).
(iii) The strong dual of a reflexive Banach space is also a reflexive Banach space ([4], §23, 5(11)).

Now assume that $E[\tau]$ is a reflexive Banach space. Let $F$ (resp. $E/F$) be any closed subspace (separated quotient) of $E$ and let $H$ (resp. $E'/H$) be any closed subspace (resp. separated quotient) of the strong dual $E'[\tau_b(E)]$.

(a) $F$ and $E/F$ are reflexive by (ii). Hence $E[\tau]$ is hereditarily-reflexive.
(b) $F$ is reflexive by (ii) and $H$ are reflexive by (iii) and (ii). Hence $E[\tau]$ is dually hereditarily-reflexive.
(c) By (i) and (ii), $F$ and $E/F$ are reflexive Banach spaces, so, by (a), they are hereditarily-reflexive and so $E[\tau]$ is strongly hereditarily-reflexive.
(d) $E[\tau]$ is strongly hereditarily-reflexive (by (c)) and $E'[\tau_b(E)]$ is a reflexive Banach space (by(iii)) and so it is strongly hereditarily-reflexive, by(c). Hence $E[\tau]$ is dually strongly hereditarily-reflexive.

There are separated quotients of a reflexive (F)-space which are not reflexive. So reflexive (F)-space is not hereditarily-reflexive, in general. However, we have:

3.4 Corollary: A reflexive (F)-space $E[\tau]$ is each of (a) hereditarily-reflexive, (b) dually hereditarily-reflexive, (c) strongly hereditarily-reflexive, and (d) dually strongly hereditarily-reflexive if any one of the following conditions holds:

(i) $E[\tau]$ has a fundamental sequence of bounded sets
(ii) $E[\tau]$ contains a bounded absorbent set.
Proof: If (i) holds, then $E[\tau]$ is an (F)-space and it has a fundamental sequence of bounded sets, so $E[\tau]$ is a Banach space, by [4], §29.1(3). On the other hand, if (ii) holds, then $E[\tau]$ is an (F)-space and it contains a bounded absorbent set, so $E[\tau]$ is a Banach space, by [4], §29.1(9). Hence theorem 3.3 completes the proof.

Following [3], a locally convex space $E[\tau]$ is called pair-wise-reflexive, if the dual pair $(E', E)$ is reflexive in the sense of Köthe i.e. $(E'[\tau_0(E)])' = E$ and $(E[\tau_0(E')])' = E'$.

Following result is due to [4]:

3.5 Theorem: A locally convex space $E[\tau]$, where $\tau$ is the Mackey topology is reflexive if and only if it is pair-wise-reflexive.

We also have:

3.6 Corollary: A reflexive locally convex space is $E[\tau]$ is pair-wise-reflexive.

Proof: $E[\tau]$ is reflexive, so it is barreled and so $\tau = \tau_0(E')$.

References