Fixed Point Result with Compact Metric Space

MadhuShrivastava*,K.Qureshi**,A.D.Singh**
1. TIT Group of Institution,Bhopal
2 Ret.Additional Director, Bhopal
3.Govt.M.V.M.College,Bhopal

Abstract
The goal of this research is to study some fixed point results in compact metric space. We have proved some fixed point theorem for self-mapping satisfying a new contractive condition involving rational expressions in compact metric space.

Introduction–
There are several generalizations of classical contraction mapping theorem of Banach [1]. In 1961 Edelstein[3] established the existence of a unique fixed point of a self-map T of a compact metric space satisfying the inequality \( d(Tx, Ty) < d(x, y) \). Which is generalization of Banach.Iseki [4], Fisher [5], have proved some results on compact metric space. Here we are finding some fixed point result involving rational expression in compact metric Space. This paper is the generalization of Bhardwaj et al [2] and Soni [8].

Preliminaries-
Definition 1.1 Fixed point space: Let F be a self-continuous mapping. A space X is called a fixed point space if every continuous mapping F of X into itself, has a fixed point, in the sense that \( f(x_0) = x_0 \).

Definition 1.2 A class \( \{Gi\} \) of open subset of X is said to be an open cover of X, if each point in X belongs to one Gi that is \( U_iGi = X \).

A subclass of an open cover which is at least an open cover is called a sub cover.

A compact space is that space in which every open cover has finite sub cover.

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Main Result-
Theorem- Let \( f \) be a continuous mapping of a compact space \( X \) into itself satisfying the Following condition –
\[
\begin{align*}
d(fx, fy) &< \alpha_1 \left[ \frac{d(x, fx) + d(y, fx) + d(x, fy) + d(x, y)}{1 + d(x, fx) \cdot d(y, fx) \cdot d(y, fy) \cdot d(x, y)} \right] \\
&+ \alpha_2 \left[ \frac{d(x, fx) \cdot d(x, fy) + d(y, fy) \cdot d(y, fx) \cdot d(x, y)}{d(x, y) + d(x, fx) \cdot d(x, fy) \cdot d(y, fy) \cdot d(x, y)} \right] + \\
&\alpha_3 \max\{d(x, fx), d(x, fy), d(y, fx), d(y, fy)\} + \alpha_4 d(x, y)
\end{align*}
\]

For all \( x, y \in X, x \neq y \) and \( 4\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \leq 1 \) where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are non-negative and real, then \( f \) has a unique fixed point.

**Proof**—First we define a function \( T \) on \( X \) as follows \( T(x) = d(x, f(x)) \) for all \( x \in X \).

Since \( d \) and \( f \) are continuous on \( X \), from compactness of \( X \), there exist a point \( p \in X \), such that

\[
T(p) = \inf\{T(x) : x \in X\} \tag{1.2}
\]

If \( T(p) \neq 0 \), it follows that \( p \neq f(p) \), and so

\[
d(f(p), f(f(p))) < \alpha_1 \left[ \frac{d(f(p), f(p)) + d(f(p), f(p)) + d(p, f^2(p)) + d(p, f(p))}{1 + d(f(p), f(p)) \cdot d(f(p), f(p)) \cdot d(f(p), f^2(p)) \cdot d(f(p), f(p))} \right] \\
+ \alpha_2 \left[ \frac{d(f(p), f(p)) \cdot d(f(p), f^2(p)) + d(f(p), f^2(p)) \cdot d(f(p), f(p)) \cdot d(f(p), f(p))}{d(f(p), f^2(p)) + d(p, f^2(p)) \cdot d(p, f^2(p)) \cdot d(p, f^2(p)) \cdot d(p, f^2(p))} \right] \\
+ \alpha_3 \max\{d(f(p), f(p)), d(p, f^2(p)), d(f(p), f^2(p))\} + \alpha_4 d(p, f(p))
\]

\[
d(f^2(p)) < \alpha_1 [2d(p, f(p)) + d(p, f^2(p))] + \alpha_2 d(p, f^2(p)) \\
+ \alpha_3 \max\{d(p, f(p)), d(p, f^2(p)), d(f(p), f^2(p))\} + \alpha_4 d(p, f(p))
\]

\[
< \alpha_1 [2d(p, f(p)) + d(p, f(p)) + d(f(p), f^2(p))] \\
+ \alpha_2 [d(p, f(p)) + d(f(p), f^2(p))] \\
+ \alpha_3 \max\{d(p, f(p)), d(p, f^2(p)), d(f(p), f^2(p))\} \\
+ \alpha_4 d(p, f(p))
\]

**Case (i)**—If

\[
\max\{d(p, f(p)), d(p, f^2(p)), d(f(p), f^2(p))\} = d(f(p), f^2(p)) < \\
\alpha_1 [3d(p, f(p)) + d(f(p), f^2(p))] + \alpha_2 [d(p, f(p)) + d(f(p), f^2(p))] \\
+ \alpha_3 d(f(p), f^2(p)) + \alpha_4 d(p, f(p))
\]

\[
(1 - \alpha_1 - \alpha_2 - \alpha_3)d(f(p), f^2(p)) < (3\alpha_1 + \alpha_2 + \alpha_4) d(p, f(p))
\]

\[
d(f^2(p)) = \frac{(3\alpha_1 + \alpha_2 + \alpha_4)}{(1 - \alpha_1 - \alpha_2 - \alpha_3)} d(p, f(p))
\]

\[
d(f^2(p)) = \lambda_1 d(p, f(p))
\]

Where \( \lambda_1 = \frac{3\alpha_1 + \alpha_2 + \alpha_4}{1 - \alpha_1 - \alpha_2 - \alpha_3} \)

But \( 4\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \leq 1 \), which is a contradiction.

**Case (ii)**—If \( \max\{d(p, f(p)), d(p, f^2(p)), d(f(p), f^2(p))\} = d(p, f^2(p)) \)
Then,
\[ d(f(p), f^2(p)) < \alpha_1 [3d(p, f(p)) + d(f(p), f^2(p))] + \alpha_2 [d(p, f(p)) + d(f(p), f^2(p))] + \alpha_3 d(p, f^2(p)) + \alpha_4 d(p, f(p)) \]
\[ < \alpha_1 [3d(p, f(p)) + d(f(p), f^2(p))] + \alpha_2 [d(p, f(p)) + d(f(p), f^2(p))] + \alpha_3 d(p, f(p)) + \alpha_4 (1 - \alpha_1 - \alpha_2 - \alpha_3) d(f(p), f^2(p)) < (3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) d(p, f(p)) \]
\[ d(f(p), f^2(p)) = \frac{(3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}{(1 - \alpha_1 - \alpha_2 - \alpha_3)} d(p, f(p)) \]

Where \( \lambda_2 = \frac{(3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}{(1 - \alpha_1 - \alpha_2 - \alpha_3)} \)

But \( 4\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \leq 1 \), which is a contradiction.

**Case-(iii)-If** \( \max\{d(p, f(p)), d(p, f^2(p)), d(f(p), f^2(p))\} = d(p, f(p)) \)

Then,
\[ d(f(p), f^2(p)) < \alpha_1 [3d(p, f(p)) + d(f(p), f^2(p))] + \alpha_2 [d(p, f(p)) + d(f(p), f^2(p))] + \alpha_3 d(p, f(p)) + \alpha_4 d(p, f(p)) \]
\[ (1 - \alpha_1 - \alpha_2) d(f(p), f^2(p)) < (3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) d(p, f(p)) \]
\[ d(f(p), f^2(p)) = \frac{(3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}{(1 - \alpha_1 - \alpha_2)} d(p, f(p)) \]

Where \( \lambda_2 = \frac{(3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}{(1 - \alpha_1 - \alpha_2)} \)

But \( 4\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \leq 1 \), which is a contradiction

Here \( \lambda = \max\{\lambda_1, \lambda_2, \lambda_3\} \)

\[ \lambda = \max\left\{ \frac{3\alpha_1 + \alpha_2 + \alpha_4}{1 - \alpha_1 - \alpha_2 - \alpha_3}, \frac{3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_1 - \alpha_2 - \alpha_3}, \frac{3\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_1 - \alpha_2} \right\} \]

Hence \( p = f(p) \) and \( p \) is a fixed point of \( f \).

**Unique**—Now next we prove the uniqueness of \( p \). Let If possible \( q \neq p \) be another fixed point \( f \).

Then,
\[ d(p, q) = d(f(p), f(q)) \]
\[
< \alpha_1 \frac{[d(p, f(p)) + d(q, f(p)) + d(p, f(q)) + d(q, q)]}{[1 + d(p, f(p)).d(q, f(p)).d(q, f(q)).d(p, q)]}
+ \alpha_2 \frac{[d(p, f(p)).d(q, f(q)) + d(q, f(q)).d(q, f(p))]}{[d(p, q) + d(p, f(p)).d(p, f(q)).d(q, f(p)).d(p, q)]} + \\
\alpha_3 \max\{d(p, f(p)), d(p, f(q)), d(q, f(p)), d(q, f(q))\} + \alpha_4 d(p, q)
\]
\[
d(p, q) < (3 \alpha_1 + \alpha_3 + \alpha_4) d(p, q)
\]
This is contradiction, because \(3 \alpha_1 + \alpha_3 + \alpha_4 < 1\)

Hence \(p\) is unique.

**Remark-** If we put \(\alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = 1\), then we get the result of Edelstein [3].

**References**


