ACTION OF THE CYCLIC GROUP C_n ACTING ON THE DIAGONALS OF A REGULAR *n* – GON

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Abstract

The main objective of this paper is to investigate the action of the cyclic group $G = C_n$ on set, X, the diagonals of a regular n-gon. We will first discuss the transitivity and primitivity of this action, after which we will give useful results regarding the suborbits, subdegrees and ranks of this action. It is worth mentioning that most of the results here have been given as Lemmas and Theorems.

1. Introduction

Suborbital graphs of various permutation groups and their actions have been studied by many authors since its introduction by Sims(1967). Kamuti *et al.* (2012)have shown that Γ_∞ (the stabilizer of ∞in Γ) acts transitively and imprimitively on $\mathbb{Z}$.Ndirangu *et al.* (2014) worked on the dihedral group acting on the diagonals of a regular n-gon and computed the rank and subdegrees of this action among other results. In this paper we aim to further research in this area by studying in detail the algebraic properties of the action of the cyclic group C_n on the diagonals of a regular n-gon.

2. Preliminaries

The cyclic group,C_n, is the group of rotational symmetries of a regular n-gon. Throughout this work we will use G to represent the cyclic group which is of order n. The set of the diagonals of a regular n-gon, $n \equiv 0 \ mod \ 2$ will be denoted by X.

Definition 2.1

Let *G* be a group and *X* a non empty set. We say that *G* acts on the set *X* on the left if for each $g \in G$ and each $x \in X$, there is a unique element $gx \in X$ such that the following axioms hold;

a) $1x = x \forall x \in X$, where 1 is the identity element of G.

b) $g(hx) = (gh)x \forall g, h \in G$ and $x \in X$.

That is, the identity element of G is the identity permutation on X and the combined effect of applying h then g is the same as that of applying gh. We can also define the action of G on X from the right in a similar way.

Definition 2.2

Let a group G act on a set X. Then X is partitioned into disjoint equivalence classes called orbits or transitivity classes of the action. For each $x \in X$ the orbit containing x is denoted by $Orb_G(x)$ which is defined as follows;

$$Orb_G(x) = \{y \in X \mid y = gx, g \in G\}.$$

Definition 2.3

Let a group G act on a set X with $x \in X$. The stabilizer of x in G, denoted as $Stab_G(x)$ or G_x, is the set;

$$G_x = \{ g \in G \mid gx = x\}.$$

It is important to note that $Stab_G(x)$ is a subgroup of G.

Theorem 2.1 (Rose, 1978)

Let a group G act on a finite set X with $x \in X$. Then;

$$|Orb_G(x)| = |G : Stab_G(x)|.$$

Definition 2.4

Let G act on a set X. The set of elements of X fixed by $g \in G$ is called the fixed point set of g, denoted by $Fix(g)$. Thus;

$$Fix(g) = \{x \in X \mid gx = x\}.$$

Lemma 2.1 (Rotman, 1973)

Let a group G act on a set X. Then the number of orbits of G on X is given by;

$$\frac{1}{|G|}\sum_{g \in G} |Fix(g)|, \text{ where } Fix(g) = \{x \in X \mid g(x) = x\}.$$

Definition 2.5

A group G acting on a set X is said to be transitive on X if it has only one orbit, and so $Orb_G(x) = X$, $x \in X$. Equivalently G is transitive on X if for every pair of points $x, y \in X$ there exists $g \in G$ such that $gx = y$. A group which is not transitive is called intransitive.

Definition 2.6

If G acts on a set X transitively and $B \subseteq X$ then B is called a block of the action if

$gB \cap B = B$ or $gB \cap B = \emptyset$ for all $g \epsilon G$. We note that the empty set $\emptyset$, the singleton subsets of X and the set X itself are always blocks, referred to as the trivial blocks.

Definition 2.7

Let G act on a set X transitively, then if the action has some non-trivial blocks, then G is said to act imprimitively on X, otherwise G acts primitively on X.

Theorem 2.2 (Wielandt, 1964)

Let X be a set, $|X| > 1$ with $x \in X$. A transitive group G on X is primitive if and only if G_x is a maximal subgroup of G.

Definition 2.8

Let G act transitively on a set X and G_x the stabilizer of $x \in X$, then; $\Delta_0 = \{x\}$, Δ_1, $\Delta_2, \ldots, \Delta_{r-1}$ are suborbits of G. The rank of G in this case is r. The sizes $n_i = |\Delta_i|$ where $i = 0, 1, 2, \ldots, r-1$, often called the lengths of the surborbits Δ_i,

$i = 0, 1, 2, \ldots, r-1$ are known as the subdegrees of G. The values of the ranks and subdegrees are independent of the choices of $x \in X$ due to the transitivity of the action of the group.

Definition 2.9

Let Δ be an orbit of G_x on X. Define

$$\Delta^* = \{gx \mid g \epsilon G, x \epsilon g\Delta\}$$

then Δ^* is also an orbit of G_x and is called the G_x – orbit or the G- suborbit paired with Δ.

Clearly $|\Delta| = |\Delta^*|$. If $\Delta^* = \Delta$, then Δ is called a self- paired orbit of G_x.

Theorem 2.3 (Cameron, 1974)

Let G act on the set X, and let $g \epsilon G$, then the number of self-paired suborbits of G is given by;

$$\frac{1}{|G|} \sum_{g \in G} |\, Fix\,(g^2)|$$

where $Fix\,(g^2)$ is the set of elements of X fixed by the permutation g^2.

3. Transitivity And Primitivity Of G Acting On X

For any even n, X has elements of the form;

$$X = \{(1, n/2 + 1), (2, n/2 + 2), (3, n/2 + 3), \ldots, (n/2, n)\}$$

and its order is $^n/_2$, that is $|X| = {^n/_2}$.

Lemma 3.1

The stabilizer of $x \in X$ in *G* is of order 2 $\forall\, n \geq 4$ and $n \equiv 0 \bmod 2$.

Proof

Letting $x \in X$, we determine the stabilizer of the point x. W. L. O. G we let x to be the diagonal joining the vertices 1 and $^n/_2 + 2$, that is $x = (1, {^n/_2} + 1)$. Since $G = \{g^1, g^2, \dots, g^n\}$, we have that the stabilizer of the point x are $1 = g^n$, the identity element of *G*, and $g^{n/2}$ a rotation through 180°. We note that,

$$g^{n/2} = (1\ {^n/_2} + 1)(2\ {^n/_2} + 2)(3\ {^n/_2} + 3) \dots ({^n/_2}\ n).$$

This concludes that $|stab_G(x)| = 2$.

Example 3.1.1

Letting *G* = C_8, then *X* = $\{(1,5),(2,6),(3,7),(4,8)\}$ with $|X| = 4$.

Taking $x = (1,5)$, then $stab_G(x) = \{1,(1\ 5)(2\ 6)(3\ 7)(4\ 8)\}$ and $|stab_G(1,5)| = 2$.

Lemma 3.2

G acts transitively on the set *X*.

Proof

Given an $x \in X$, by Theorem 2.1 we establish that;

$$|Orbit_G(x)| = |G : stab_G(x)| = {^{|G|}/_{|stab_G(x)|}}$$

$$= {^n/_2}.$$

This shows that $|Orbit_G(x)| = |X|$, which implies that the action of G on X has only one orbit. Therefore by definition 2.5, we have that this action of G on X is transitive.

Alternatively, there are only two elements of G which fixes the elements of X. These elements are 1, the identity element, and $g^{n/2}$, the rotation through the angle π. In fact each of them will fix the $n/2$ elements of X. By Lemma 2.1, the number of G – orbits in G is given by;

$$\frac{1}{|G|}\sum_{g\,\in G}|Fix\ (g)| = \frac{1}{n}\left(n/2 + n/2\right) = 1$$

This shows that the action of G on X has got only one orbit hence transitive.

Theorem 3.1

The group G acts on X imprimitively if and only if $n/2$ is not a prime.

Proof

Letting $x = (1, n/2 + 1) \in X$, clearly $stab_G(x) = \{1, g^{n/2}\} = \langle g^{n/2}\rangle$. Assuming G acts on X imprimitively, this implies that $\langle g^{n/2}\rangle$ is not a maximal subgroup of $G = \langle g\rangle$. This implies that there exists an integer k such that $\langle g^{n/2}\rangle < \langle g^k\rangle < G$. Therefore we can find an integer k, $1 < k < n/2$, such that $k/n/2$ concluding that $n/2$ is not a prime.

Conversely, suppose $n/2$ is not a prime, then there exists an integer k such that $k/n/2$. Clearly $\langle g^k\rangle$ is a proper subgroup of $\langle g\rangle = G$ and it is of the form $\langle g^k\rangle = \{g^k, g^{2k}, \ldots, g^{n/2}, \ldots, g^n\}$. This shows that; $stab_G(x) < \langle g^k\rangle < G$.

Thus $stab_G(x)$ is not a maximal subgroup of G establishing the fact that the action of G on X is imprimitive as required completing the proof.

Example 3.1.1

Considering $G = C_{12}$, then $X = \{(1,7),(2,8),(3,9),(4,10),(5,11),(6,12)\}$. The stabilizer of $x = (1,7)$ is constituted as below,

$$Stab_G(1,7) =< (1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12) >=< g^6 >.$$

Since $n/2 = 6$ is not a prime, we let $k = 3$. It's clear that;

$Stab_G(1,7) =< g^6 > \subset < g^3 > \subset G$. Hence $Stab_G(1,7)$ is not a maximal subgroup of G, hence this action of G on X is imprimitive by Theorem 2.2.

4 Suborbits, Subdegrees And Ranks Of The Action Of *G* On *X*

Theorem 4.1

The action of G on X has a rank of $n/2$ with subdegrees $\underbrace{1, 1, 1, \dots, 1}_{n/2\ times}$, that is $n/2$ of them.

Proof

Let G act on X, taking $x = (1, n/2 + 1) \in X$, the stabilizer of x consist of two elements, that is;

$$Stab_G(x) = \{1, (1\ \ n/2 + 1)(2\ \ n/2 + 2)(3\ \ n/2 + 3) \dots (n/2\ \ n)\}.$$

Recall, $X = \{(1, n/2 + 1), (2, n/2 + 2), (3, n/2 + 3), \dots, (n/2, n)\}$, thus the suborbits of G on X are as follows;

$$\Delta_0 = Orb_{G_x}(1, n/2 + 1) = \{1, n/2 + 1\}$$

$$\Delta_1 = Orb_{G_x}(2, n/2 + 2) = \{2, n/2 + 2\}$$

$$\vdots\vdots\vdots$$

$$\Delta_{n/2 - 1} = Orb_{G_x}(n/2, \quad n) = \{n/2, n\}$$

We now compute the rank, r, of this action as below,

$r - 1 = n/2 - 1 \Rightarrow r = n/2$.

Therefore the rank is $n/2$ as required.

Letting $n_i = |\Delta_i|$ be the subdegrees, we have that $n_0 = n_1 = n_2 = \cdots = n_{n/2-1} = 1$. This establishes that the subdegrees of the action of *G* on *X* are; $\underbrace{1, 1, 1, \dots, 1}_{n/2 \ times}$.

Example 4.1.1

Let $G = C_{14} =< g >$ and $X = \{(1,8), (2,9), (3,10), (4,11), (5,12), (6,13), (7,14)\}$, the stabilizer of the first diagonal is given by;

$$Stab_{G_x}(1,8) = \{1, g^7\}.$$

The suborbits of *G* on *X* are as below;

$$\Delta_0 = Orb_{G_x}(1,8) = \{1,8\}$$

$$\Delta_1 = Orb_{G_x}(2,9) = \{2,9\}$$

$$\Delta_2 = Orb_{G_x}(3,10) = \{3,10\}$$

$$\Delta_3 = Orb_{G_x}(4,11) = \{4,11\}$$

$$\Delta_4 = Orb_{G_x}(5,12) = \{5,12\}$$

$$\Delta_5 = Orb_{G_x}(6,13) = \{6,13\}$$

$$\Delta_6 = Orb_{G_x}(7,14) = \{7,14\}$$

To compute the rank, r, of this action we solve $r - 1 = 6, \Rightarrow r = 7$, that is $r = n/2$.

The subdegrees are;

$$n_0 = n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = 1,$$

where; $n_i = |\Delta_i|$, $i = 1, 2, \dots, 6$.

Corollary 4.1.1

In general the ${}^{n}/_{2}$ suborbits of G on X are given by;

$$\Delta_i = \{(i+1, {}^{n}/_{2} + (i+1))\},\ i = 0,1,2,3,\dots, {}^{n}/_{2} - 1$$

Theorem 4.2

The action of G on X has two self – paired suborbits when $n \equiv 0 \ mod\ 4$ and only one self-paired suborbit when $n \equiv 2\ mod\ 4$

Proof

Considering the case $G = C_n = \langle g \rangle$and $n \equiv 0\ mod\ 4$, G has two elements of order 4 and one element of order 2. Letting $g \in G$ be any of these elements, either of order two or four, we have that g^2 fixes all the elements of X. In addition to these elements the identity element in G also fixes all the elements of X when squared. Thus by Theorem 2.3,we can count all the self-paired suborbits of G, this is given by;

$$\frac{1}{|G|}\sum_{g\in G}|Fixg^2| = \frac{1}{n}\left\{\frac{n}{2}+\frac{n}{2}+\frac{n}{2}+\frac{n}{2}\right\}$$

$$= \frac{1}{n} \times 2n$$

$$= 2$$

For the case when $n \equiv 2\ mod\ 4$, we only have two elements of G that will fix the elements of X when squared. These elements are; the identity element and the element which is of order 2.

In fact they will fix all the elements of X when squared. Therefore the number of self-paired suborbits of G on X is in this case given by;

$$\frac{1}{|G|}\sum_{g\in G}|Fixg^2| = \frac{1}{n}\left\{\frac{n}{2}+\frac{n}{2}\right\}$$

$$= \frac{1}{n} \times n$$

$$= 1$$

Example 4.2.1

Considering the case when $n \equiv 0 \; mod \; 4$, in particular we let $n = 12$.

The set $X = \{(1,7)(2,8)(3,9)(4,10)(5,11)(6,12)\}$. Taking $x = (1,7) \in X$, then the stabilizer of x is;

$$G_x = \{1, (1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12)\}.$$

The orbits of G_x are as listed below;

$$\Delta_0 = \{(1,7)\},\ \Delta_1 = \{(2,8)\}, \Delta_2 = \{(3,8)\},\ \Delta_3 = \{(4,9)\},\ \Delta_4 = \{(5,10)\},\ \Delta_5 = \{(6,12)\}.$$

We notice that;

$$\Delta_0{}^* = \Delta_0,\ \Delta_1{}^* = \{(6,12)\} = \Delta_5,\ \Delta_2{}^* = \{(5,11)\} = \Delta_4,\ \Delta_3{}^* = \{(4,10)\} = \Delta_3.$$

This shows that we have two self-paired suborbits of this action which are; $\Delta_0$and Δ_3.

Example 4.2.2

The case when $n \equiv 2 \; mod \; 4$, we take $G = C_{10}$. So $X = \{(1,6)(2,7)(3,8)(4,9)(5,10)\}$.

Taking $x = (1,6) \in X$, The stabilizer of x is given by;

$$G_x = \{1, (1\ 6)(2\ 7)(3\ 8)(4\ 9)(5\ 10)\}.$$

The orbits of G_x are as follows;

$$\Delta_0 = \{(1,6)\},\ \Delta_1 = \{(2,7)\}, \Delta_2 = \{(3,8)\},\ \Delta_3 = \{(4,9)\},\ \Delta_4 = \{(5,10)\}.$$

Now, we notice that;

$$\Delta_0{}^* = \Delta_0,\ \Delta_1{}^* = \{(5,10)\} = \Delta_4,\ \Delta_2{}^* = \{(4,9)\} = \Delta_3.$$

This establishes that it is only the trivial suborbit of G that is self –paired.

Corollary 4.2.1

Let G act on X, then the suborbit Δ_i of G is paired with the suborbit $\Delta_{n/_2 - 1}$, $i = 1,2,3, \dots, {}^n/_2 - 1$.

Proof

Let $G = C_n$ and $\Delta_i = \left\{\left(i+1, {}^n/_2 + (i+1)\right)\right\}$. To find the suborbit paired with Δ_i we find a $g^k \in G$,

$1 \leq k \leq n$ such that, $g^k\left(\left(i+1, {}^n/_2 + i + 1\right)\right) = \left(1, {}^n/_2 + 1\right)$. To obtain the value of k we solve

$(k+i+1) mod\ n \equiv 1$, that is $k+i+1 = n+1$. This gives $k = n-i$ implying that

$g^{n-i}(i+1, {}^{n}/_{2}+i+1) = (1, {}^{n}/_{2}+1)$. To find the suborbit paired with Δ_i we evaluate

$$g^{n-i}(\Delta_0) = g^{n-i}(1, {}^{n}/_{2}+1)$$

$$= (n-i+1, {}^{n}/_{2}+1+n-i)$$

$$= (n-i+1, {}^{n}/_{2}+1-i)$$

$$= ({}^{n}/_{2}-i+1, n-i+1)$$

$$= \Delta_{n/_{2}-1}.$$

Hence Δ_i is paired with $\Delta_{n/_{2}-1}$, that is $\Delta_0{}^{*} = \Delta_{n/_{2}-1}$.

Example 4.2.1

Considering the case when $n \equiv 2\ mod\ 4$, we let $G = C_6 = \langle 1\ 2\ 3\ 4\ 5\ 6 \rangle$ and so we have $X = \{(1,4), (2,5), (3,6)\}$. The stabilizer of $x = (1,4) \in X$ is given by $G_x = \{1, (1\ 4)(2\ 5)(3\ 6)\}$. The suborbits of this action are;

$$\Delta_0 = \{(1,4)\},\ \Delta_1 = \{(2,5)\}, \Delta_2 = \{(3,6)\}.$$

For self-pairing we have that;

$$\Delta_0{}^{*} = \Delta_0,\ \Delta_1{}^{*} = \{(3,6)\} = \Delta_2$$

This confirms that; $\Delta_1{}^{*} = \Delta_{6/_{2}-1} = \Delta_2$ as required.

Corollary 4.2.2.

If $n \equiv 0\ mod 4$, the action of G on X has two self-paired suborbits. These are the trivial suborbit, Δ_0, and the suborbit $\Delta_{n/_{4}}$. If $n \equiv 2\ mod 4$, this action has only one self-paired suborbit, which is the trivial suborbit

Δ_0.

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