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Some Properties of Li-Yorke Chaos

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Abstract

In this paper we study Li-Yorke chaos in linear operator on Banach space, in addition to establishing some basic properties of Li-Yorke chaos and explanation when the operator be Li-Yorke chaos or not. We also prove the following the theorem, if $\chi_T(\mathcal{D}) \cap \chi_T(\mathbb{C}\backslash\overline{\mathcal{D}}) \neq \emptyset$, where $\mathcal{D}$ is the interior of the unit circle, and $\mathbb{C}\backslash\overline{\mathcal{D}}$ is the exterior of the unit circle then T satisfied Li-Yorke Chaos Criterion.

Key words: Li – Yorke,Choas,Irregular vectors,Li –Yorke chaos criterion.

1. Introduction

The dynamics of linear operators have been widely studied in the last few years. Several notions have been introduced for describing the dynamical behavior of linear operators on infinite-dimensional spaces, such as hypercyclicity, chaos in the sense of Devaney, chaos in the sense of Li-Yorke, mixing and weakly mixing properties, and frequent hypercyclicity, among others. In the paper, we are mainly interested with the notion of Li-Yorke chaos. Let (X, T) be Banach space and T continuous operator from X to itself .The definition of Li-Yorke chaos is based on ideas in [8]. A pair of points $\{x,y\}\subseteq$ X is said to be a Li-Yorke pair if one has simultaneously

$\lim_n \inf\|T^n x - T^n y\| = 0$ and $\lim_n \sup\|T^n x - T^n y\|) > 0$.

A set S $\subseteq$ X is called scrambled if any pair of distinct points $\{x,y\}\subseteq$ S is a Li- Yorke pair. Finally, a system (X, T) is called chaotic in the sense of Li and Yorke if X contains an uncountable scrambled set, and definition a vector $x \in$X is said to be irregular for T if $lim\ \inf_n \| T^n x\| = 0$ and $lim\ \sup_n \| T^n x\| = \infty$ [7]. While [3] give an equivalent definition of irregular vector, that is

A vector x is said to be irregular vector of T if there are two sequences k_n and l_n increasing to ∞ such that $\lim_n T^{k_n}x = 0$ and $\lim_n T^{l_n}x = \infty$.

We will recall some properties of Li-York Chaos that needed later.

1.2. Theorem:[7]

Let T: X → X is an operator. The following assertions are equivalent:

(i) T is Li-Yorke chaotic.
(ii) T admits a Li-Yorke pair.
(iii) T admits an irregular vector.

2. Main Result

Now, we will give our main results

2.1. Proposition:

Li-Yorke Chaos is preserved under conjuacy.

Proof:

let T: X⟶X be conjugate to S:Y⟶Y via φ: X ⟶ Y , then $x\in$X , y∈Y and y=$\varphi(x)$, and suppose S is Li-Yorke chaos, by theorem above , S admits an irregular vector. Let a vector y is an irregular for S, there are two sequences n_k and l_k increasing to ∞ such that $S^{n_k}y \longrightarrow 0$ mean that $S^{n_k}\varphi(x) \longrightarrow 0$ mean that $\varphi T^{n_k}(x) \longrightarrow 0$, we have $T^{n_k}(x) \longrightarrow 0$.

And $\|S^{l_k}y\| \longrightarrow \infty$ mean that $\|S^{l_k}\varphi(x)\| \longrightarrow \infty$ mean that $\|\varphi T^{l_k}(x)\| \longrightarrow \infty$ we have $\|T^{l_k}(x)\| \longrightarrow \infty$.

Then T is Li-Yorke chaos.□

2.2. Proposition:

If at least one of T_1 or T_2 has irregular vectors then $T_1 \otimes T_2$ has irregular vectors.

Proof.

Let x be an irregular vector of T_1. Thus there is a sequence k_n such that $T_1{}^{k_n}x \longrightarrow 0$. This means that $(T_1{}^{k_n}x \otimes T_2{}^{k_n}y) \longrightarrow 0$ which means that $(T_1 \otimes T_2)^{k_n} (x \otimes y) \longrightarrow 0$.

In the same time, there is a sequence l_n such that $\|T_1{}^{l_n}x\| \longrightarrow \infty$ which means that $\|T_1{}^{l_n}x\| \ \|T_2{}^{l_n}y\| \longrightarrow \infty$, then $\|(T_1 \otimes T_2)^{l_n}(x \otimes y)\| \longrightarrow \infty$.□

In following theorem is improved atheorem that found in [1], which proved that if the sum of operators is Li-Yorke chaos then at least one of operator is Li-Yorke while we prove the following:

2.3. Theorem:

T_1 and T_2 are operator on Banach space if T_1 has irregular vector and there is a sequence $k_n$such that $T_1{}^{k_n}x \longrightarrow 0$, where x irregular vector and $T_2{}^{k_n}y \longrightarrow 0$ if and only if $T_1 \oplus T_2$ has irregular vector.

Proof:

Let x be an irregular vector of T_1. thus are sequence k_n such that $T_1{}^{k_n}x \to 0$ and $T_2{}^{k_n}y \to 0$.We have $T_1{}^{k_n}x \oplus T_2{}^{k_n}y \longrightarrow 0$, that mean $(T_1 \oplus T_2)^{k_n}(x \oplus y) \longrightarrow 0$.

In the same time, there is a sequence$\|T_1{}^{\ell_n}x\| \longrightarrow \infty$, we have $\|T_1{}^{\ell_n}x\|^2 + \|T_2{}^{\ell_n}y\|^2 \longrightarrow \infty$, means that $\|T_1{}^{\ell_n}x \oplus T_2{}^{\ell_n}y\|^2 \longrightarrow \infty$, thus $\|(T_1 \oplus T_2)^{\ell_n}(x \oplus y)\|^2 \longrightarrow \infty$. Then $T_1 \oplus T_2$ has irregular vector. The conversely is similarity.

2.4. Theorem:

If $T \in$ B(H), and T is Li-Yorke, then T^* has no eigenvectors.

Proof:

Suppose that T is Li-Yorke chaos and $T^*v = \lambda v$ when $v \neq 0$. If A vector $x \in$H is irregular for T then $\limsup \| < T^n x, v > \| = \limsup \| < x, T^{n^*} v > \| = \limsup \| < x, \lambda^n v > \| = \limsup \| \overline{\lambda^n} < x, v > \| = \infty$ and

$$\liminf \| < T^n x, v > \| = \liminf \| < x, T^{n^*} v > \| = \liminf \| < x, \lambda^n v > \| = \liminf \| \overline{\lambda^n} < x, v > \| = 0$$

If $|\lambda| < 1$ or $< x, v> = 0$ then the set is bounded and if $|\lambda| \geq 1$ and $< x, v> \neq 0$ then the last set is bounded below, then T is not Li-Yorke chaos.□

2.5. Corollary:

If X is finite dimensional, then T has not Li-Yorke chaos in X.

Proof:

Suppose T is Li-Yorke Chaos in X. since X is finite dimensional, hense T^* has eigenvalues a contradiction.

The following theorem described the relation between irregular vectors commuting operators.

2.6. Theorem:

Let S and T be commuting operators on X. then the set of all irregular vectors for S is T-invariant.

Proof:

Let M be the set of all irregular vectors for S, $x \in$M to prove $Tx \in$M, by definition of irregular vector , there exist two sequence $k_n$and l_n increasing to ∞ such that $S^{k_n}x \longrightarrow 0$ and $\|S^{l_n}x\| \longrightarrow \infty$. By induction we can show that $S^{k_n}Tx = TS^{k_n}x$. Then $S^{k_n}Tx = TS^{k_n}x \longrightarrow 0$ and$\|S^{l_n}Tx\| = \|TS^{l_n}x\| = \|T\|\|S^{l_n}x\| \longrightarrow \infty$.□

2.7. Theorem:

Let $T \in B(X)$, if there exist sequence $k_n$increasing to ∞ such that $\lim_{n\to\infty}\|T^{k_n}\| = 0$ and $\|T\| > 1$ Then T is Li-Yorke chaos.

Proof:

Let R=$\|T\| > 1$. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ is a sequence of positive numbers decreasing to zero. First of all, fix $N_1 \in \mathbb{N}$ (for example, set $N_1 = 2$). Then there is x_1 such that $\|x_1\| = 1$ and

$$\lim_{n\to\infty}\|T^{k_n}x_1\| = 0 \text{ and } \sup\left\|T^{i}x_1\right\| = \infty \text{ , i=1,…,} N_1.$$

So we can choose a positive integer M_1 such that $\|T^{k_n}x\| < \varepsilon_1$ for any n≥ M_1. For convenience. Then $\|T^{N_1}x_1\| \geq 1$.

Now we will construct a sequence of points $\{x_k\}_{k=1}^{\infty}$ associated with two sequences of integers $\{N_k\}_{k=1}^{\infty}$and $\{M_k\}_{k=1}^{\infty}$ such that for every k≥2,

1) $\|x_k\| = R^{-M_{k-1}}.2^{-k}\varepsilon_{k-1}$;
2) $\left\|T^{i}x_k\right\| \geq 1$, i=1,…,N_k;
3) $\sum_{j=1}^{k}\left\|T^{k_n}x_j\right\| < \varepsilon_k$, for any $k_n \geq M_k$.

Select there is x_2 such that $\|x_2\| = R^{-M_1}.2^{-2}\varepsilon_1$ and

$$\lim_{n\to\infty}\|T^{k_n}x_2\| = 0 \text{ and } \sup\|T^{N_2}x_2\| \geq 1$$

So we can choose M_2such that $\|T^{k_n}x_1\| + \|T^{k_n}x_2\| < \varepsilon_2$ for any n≥ M_2.

Continue in this manner. If we have obtained $\{x_k\}_{k=1}^{\infty}$, $\{N_k\}_{k=1}^{\infty}$and $\{M_k\}_{k=1}^{\infty}$ such that for each k=2,…, m.

1) $\|x_k\| = R^{-M_{k-1}}.2^{-k}\varepsilon_{k-1}$;
2) $sup_n\left\|T^{i}x_k\right\| \geq 1$, i=1,…,$N_k$;
3) $\sum_{j=1}^{k}\left\|T^{k_n}x_j\right\| < \varepsilon_k$, for any n≥ M_k.

Select there is x_{m+1} such that $\|x_{m+1}\| = R^{-M_m}.2^{-(m+1)}\varepsilon_m$ and

$$\lim_{n\to\infty}\|T^{k_n}x_{m+1}\| = 0 \text{ and } \sup\|T^{N_{m+1}}x_{m+1}\| \geq 1 .$$

So we can choose M_{m+1}such that $\sum_{j=1}^{m+1}\left\|T^{k_n}x_j\right\| < \varepsilon_{m+1}$ for any n≥ M_{m+1}.

If we have obtained $\{x_k\}_{k=1}^{\infty}$, $\{N_k\}_{k=1}^{\infty}$ and $\{M_k\}_{k=1}^{\infty}$ such that for each k=2,…,m,

1) $\sum_{k=1}^{\infty}\|x_k\|$ is finite.

2) For each p, $\|T^i x_k\| < 2^{-k}\varepsilon_{k-1}$, for any $k > p$ and any 1 ≤ i ≤ M_p. Hence, $\sum_{k=p+1}^{\infty}\|T^i x_k\| < \sum_{k=p+1}^{\infty} 2^{-k}\varepsilon_{k-1} < \varepsilon_p$ for any 1 ≤ i ≤ M_p
3) For each k, $\|T^i x_k\| \geq 1, i = N_k', \dots, N_k$.
4) $M_k > N_k > M_{k-1}$ for each k.
5) $\sum_{j=1}^{k-1}\|T^{k_n}x_j\| < \varepsilon_{k-1}$, for n=1, …, N_k.
6) For each p, $\sum_{k=p+1}^{\infty}\|T^{k_n}x_k\| < \varepsilon_p$, for n=1, …, N_p.

Let $\sum_2 = \{0,1\}^{\mathbb{N}}$ be a symbolic space with two symbols. According to condition (1), we can define a map *f*: $\sum_2 \to X$ as

$$f(\xi) = \sum_{k=1}^{\infty} \xi_k x_k$$

For every element $\xi = (\xi_1, \xi_2, \dots) \in \sum_2$.

Obviously one can get an uncountable subset D∈ $\sum_2$ such that for any two distinct points $\xi, \xi' \in D$, ξ and ξ' have infinite coordinates that are different and infinite coordinates that are equivalent. Then

$$\|f(\xi) - f(\xi')\| = \|\sum_{k=1}^{\infty}(\xi_k - \xi_k')x_k\|.$$

Set $\theta = (\theta_1, \theta_2, \dots) = (\xi_1 - \xi_1', \xi_2 - \xi_2', \dots)$.Then$\|f(\xi) - f(\xi')\| = \|\sum_{k=1}^{\infty}(\theta_k)x_k\|$. Note that the possible values of $\xi_k - \xi_k'$are only 0,-1, or 1, and θ has infinite coordinates being zero and infinite coordinates bring nonzero.

Now we will prove that $\{ f(\xi), f(\xi')\}$ is a Li-Yorke chaotic pair .

Let $z = \sum_{k=1}^{\infty}\theta_k x_k$. Suppose $\{k_q\}_{q=1}^{\infty}$ is the infinite subsequence such that the k_qth coordinate of θ is nonzero (1 or -1) and $\{k_r\}_{r=1}^{\infty}$ is the infinite subsequence such that the k_rth coordinate of θ is zero.

By (5),(6) and (2), for n=1, …, N_{k_q},

$$\|T^{k_n}z\| \geq \left\|T^{k_n}(\theta_{k_q}x_{k_q})\right\| - \sum_{j=1}^{k_q-1}\left\|T^{k_n}x_j\right\| - \sum_{j=k_q+1}^{\infty}\left\|T^{k_n}x_j\right\| > 1 - \varepsilon_{k_{q-1}} - \varepsilon_{k_q}.$$

Since $\{\varepsilon_k\}_{k=1}^{\infty}$ decrease to zero, then

$$\limsup_{n\to\infty}\|T^{k_n}(f(\xi)) - T^{k_n}(f(\xi'))\|$$

$$= \limsup_{n\to\infty}\|T^{k_n}(z)\|$$

$$\geq \limsup_{q\to\infty}\left\|T^{N_{k_q}}(z)\right\|$$

$$\geq 1$$

On the other hand,

$$\|T^{k_n}z\| \leq \left\|T^{k_n}(\theta_{k_r}x_{k_r})\right\| - \sum_{j=1}^{k_r-1}\left\|T^{k_n}x_j\right\| - \sum_{j=k_r+1}^{\infty}\left\|T^{k_n}x_j\right\| < \varepsilon_{k_r-1} - \varepsilon_{k_r}.$$

$$\liminf_{n\to\infty}\|T^{k_n}(f(\xi)) - T^{k_n}(f(\xi'))\|$$

$$= \liminf_{n\to\infty}\|T^{k_n}(z)\|$$

$$\leq \liminf_{q\to\infty}\left\|T^{N_{k_q}}(z)\right\|$$

$$\leq 0$$

Therefore, $\{ f(\xi), f(\xi')\}$ is a Li-Yorke chaotic pair for any distinct $\{ \xi, \xi'\}$∈D , then T is Li-Yorke chaos.

3. The criterion for Li-Yorke chaos

The following criterion for Li-Yorke was introduced in [7]. Some definitions and theorems on Li-Yorke Chaos Criterion.

3.1. Definition[7]

An operator T: $X \to X$ satisfies the Li-Yorke Chaos Criterion (LYCC) if there exist an increasing sequence of integers$(\mathrm{n_k})_\mathrm{k}$ and a subset $X_0 \subset X$ such that

(a) $\lim_{k\to\infty} T^{n_k}x = 0\,, x \in X_0$,

(b) sup $\|T^{\mathrm{n}}|_{\mathrm{Y}}\| = \infty$, where Y:=$\overline{\mathrm{span}(X_0)}$ and $T^{\mathrm{n}}|_{\mathrm{Y}}$ denotes the restriction operator of $T^n$to Y.

3.2. Definition[5]

Given an arbitrary operator $T \in B(X)$ on a complex Banach space X and any X and any closed set F$\subseteq\mathbb{C}$, the glocal spectral subset $\chi_T(F)$ consists of all $x \in X$ for which there exists an analytic function $f: \mathbb{C}\backslash\mathrm{F} \to \mathrm{X}$ with the property that $(T - \lambda I)f(\lambda) = x$ for all $\lambda \in \mathbb{C}\backslash\mathrm{F}$.

3.3. Theorem[5, P.225]

Let $T \in B(X)$ be an operator on a Banach space X and let $f: U \to \mathbb{C}$ be an analytic function on neighbourhood U of $\sigma(T)$. Then $X_{f(T)}(F) = \chi_{\varphi(T)}(F) = \chi_T(\varphi^{-1}F)$ For every closed subset F of $\mathbb{C}$.

3.4. Theorem:

Suppose that the operator T∈B(X) on the Banach space X. if $\chi_T(\mathcal{D}) \cap \chi_T(\mathbb{C}\backslash\mathcal{D}) \neq \emptyset$, and then T satisfied Li-Yorke Chaos Criterion.

Proof:

To show that T satisfies Li-Yorke chaos, we will prove that every vector $x \in \chi_T(\mathcal{D}) \cap \chi_T(\mathbb{C}\backslash\overline{\mathcal{D}})$ is irregular vector for T.

If $x \in \chi_T(k)$ for some compact k$\subset$ $\mathcal{D}$, then there exists resolvent function f: $\mathbb{C}\backslash k \to \chi$ such that $(T - \lambda I)f(\lambda) = x$ for all $\lambda \in \mathbb{C}\backslash k$. Choose 0<p<1 so that k⊂B(0,p).

Let γ and Γ respectively denote the positively oriented circles $\{\lambda: |\lambda| = p\}$ and $\{\lambda: |\lambda| = \|T\| + 1\}$ respectively, then By [6,P.140]

$$T^n x = \frac{1}{2\pi i}\int_\gamma \lambda^n g(\lambda)d\lambda \qquad \text{for every n≥0 and hence by [2,P. 205]}$$

$$T^n x = \frac{-1}{2\pi i}\int_\Gamma \lambda^n g(\lambda)d\lambda = \frac{-1}{2\pi i}\int_\gamma \lambda^n g(\lambda)d\lambda \qquad \text{for every n≥0}$$

In particular, for every $x \in \chi_T(\mathcal{D})$, it follows that $T^n x \to 0$ as $n \to \infty$. Thus the first condition, in the irregular vector for T is satisfied.

Now for the second condition, if $x \in \chi_T(k)$ for some compact k$\subset$ $\mathbb{C}\backslash\overline{\mathcal{D}}$, then there exists resolvent function $g: \mathbb{C}\backslash k \to \chi$ such that $(T - \lambda I)g(\lambda) = x$ for all $\lambda \in \mathbb{C}\backslash k$.

Choose 1<p_1<p_2, so that k is contained in the annulus {λ:$p_1 < \lambda$<p_2}. let $\gamma_1$and $\gamma_2$be the inner and outer boundaries of the annulus respectively, each with counterclockwise orientation, and let $\gamma = \gamma_1 - \gamma_2$.

If $a \in k$, then a is the inside of $\gamma$and hence n$(\gamma, a) = 1$. Thus n$(\gamma, k) = 1$.

If $a \in \overline{\mathcal{D}}$, then a is the outside of $\gamma$and hence n$(\gamma, a) = 0$. Thus n$(\gamma, \overline{\mathcal{D}}) = 0$.

Define

$$T^n x = \frac{1}{2\pi i}\int_\gamma \lambda^n g(\lambda)d\lambda \qquad \text{for every n≥0}$$

In particular, for every $x \in \chi_T(\mathbb{C}\backslash\overline{\mathcal{D}})$, it follows that $T^n x \to \infty$ as $n \to \infty$.

Thus the second condition, in the irregular vector for T is satisfied.□

3.5. Corollary:

Suppose that $T \in B(X)$ and φ analytic in a neighborhood of $\sigma(T)$. If there exists open sets U, V⊂$\mathbb{C}$ so that each of the subspace $\chi_T(U) \cap \chi_T(V) \neq \varphi$, then $\varphi(T)$ is Li-Yorke Chaos Criterion if φ separates U and V is the sense that $\varphi(U)\subseteq\mathcal{D}$ and $\varphi(V)\subseteq\mathbb{C}\backslash\overline{\mathcal{D}}$.

Proof:

Since φ is analytic in a neighborhood of $\sigma(T)$ then theorem (3.3). $\chi_{\varphi(T)}(F) = \chi_T(\varphi^{-1}F)$ For every closed subset F of $\mathbb{C}$. If H is an open subset of $\mathbb{C}$, let

$$\chi_{\varphi(T)}(H) = \cup \{\chi_{\varphi(T)}(F): F \text{ is a closed subset of } H\}$$
$$= \cup \{\chi_T(\varphi^{-1}(F)): F \text{ is a closed subset of } H\}$$
$$= \chi_T(\varphi^{-1}(H))$$

For every open subset H of $\mathbb{C}$. Since $\varphi(U) \subseteq \mathcal{D}$ and $\varphi(V) \subseteq \mathbb{C} \backslash \overline{\mathcal{D}}$ it follows that $U \subseteq \varphi^{-1}(\mathcal{D})$ and $V \subseteq \varphi^{-1}(\mathbb{C} \backslash \overline{\mathcal{D}})$. Hence

$$\chi_T(U) \subseteq \chi_T\left(\varphi^{-1}(\mathcal{D})\right) = \chi_{\varphi(T)}(\mathcal{D})$$
$$\chi_T(V) \subseteq \chi_T\left(\varphi^{-1}(\mathbb{C} \backslash \overline{\mathcal{D}})\right) = \chi_{\varphi(T)}(\mathbb{C} \backslash \overline{\mathcal{D}})$$

Thus

$$\chi_T(U) \subseteq \chi_{\varphi(T)}(\mathcal{D}) \text{ and } \chi_T(V) \subseteq \chi_{\varphi(T)}(\mathbb{C} \backslash \overline{\mathcal{D}})$$

Since $\chi_T(U) \cap \chi_T(V) \neq \varphi$ then$\chi_T(\mathcal{D}) \cap \chi_T(\mathbb{C} \backslash \overline{\mathcal{D}}) \neq \varphi$. Thus $\varphi(T)$ is Li-Yorke Chaos Criterion.□

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