Endo SS-Coprime Modules

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Abstract

The purpose of this paper is to introduce and investigate the notion of endo strongly S-coprime modules, where an R-module M is called an endo strongly S-coprime module (briefly endo SS-coprime) if, for all \( f, g \in \text{End}(M) \), \( \text{Im}(f \circ g) \) is small in \( M \) implies \( f = 0 \) or \( g = 0 \). We give some properties of endo SS-coprime modules. Several of various relations between such modules and other classes are obtained. Moreover, we give some equivalent statements for endo SS-coprime modules. We also introduce the notion of semi-endo strongly S-coprime modules as generalization of endo strongly S-coprime modules, where an R-module M is called semi-endo strongly S-coprime (briefly semi-endo SS-coprime) if, for each \( f \in \text{End}(M) \), \( \text{Im}(f \circ f) \) is small in \( M \) implies \( f = 0 \). Some results of such modules are given.

Key Words: Endo SS-coprime modules; Semi-endo SS-coprime modules; \( T \)-noncosingular modules; SS-coprime modules; SSS-coprime modules; S-coprime modules.

1. Introduction

Throughout this article, let \( M \) be a left module as a commutative ring with identity. We denote the ring of all endomorphisms of \( M \) by \( \text{End}(M) \) and the Jacobson radical of \( M \) by \( \text{Rad}(M) \). We will denote the left annihilator of \( M \) in \( S = \text{End}(M) \) by \( l_s(M) \), and the direct summand \( N \) of \( M \) by \( N \leq M \). Recall that a submodule \( N \leq M \) is called small and denoted by \( N \ll M \) if, \( N + K \neq M \) for every proper submodule \( K \) of \( M \), [2]. Following [10], an \( R \)-module \( M \) is called \( T \)-noncosingular if, for every nonzero endomorphism \( \varphi \) of \( M \), \( \text{Im} \varphi \) is not small in \( M \). Hadi I.M-A in [7] introduce the notion of strongly S-coprime module (briefly, SS-coprime), where a module \( M \) is called strongly S-coprime (briefly SS-coprime) if, for all \( a, b \in R \), \( abM \ll M \) implies \( aM = 0 \) or \( bM = 0 \). In section 3 of this paper we further investigate the notion of endo strongly S-coprime module (briefly endo SS-coprime), where an \( R \)-module \( M \) is called endo SS-coprime if for each \( f, g \in \text{End}(M) \) with \( \text{Im}(f \circ g) \ll M \) implies \( f = 0 \) or \( g = 0 \). We show that in general the direct sum of endo SS-coprime modules is not endo SS-coprime module. We also prove that endo SS-coprime is inherited by direct summands. We prove some results concerning these types of
modules. It is shown that a divisible \( R \)-module \( M \) over no zero divisor ring \( S = \text{End}(M) \), is faithful endo SS-coprime. For a multiplication module \( M \), we prove that the concepts endo SS-coprime module and a SS-coprime module are coincide. In section 4, the concept of semi-end \( S \)-coprime modules is presented as generalization of endo SS-coprime modules, where an \( R \)-module \( M \) is called semi-end \( S \)-coprime if for any \( f \in \text{End}(M) \) with \( \text{Im}(f \circ f) \ll M \) implies \( f = 0 \). Most of properties of endo SS-coprime modules generalized to semi-end \( S \)-coprime module. Several properties of semi-end \( S \)-coprime modules and some connections between semi-end \( S \)-coprime modules and other related concepts are given. It is proved that, an \( R \)-module \( M \) is semi-end \( S \)-coprime if and only if \( l_s(M) \) is a semiprime ideal of \( S \) and \( M \) is \( T \)-noncosingular, where \( S = \text{End}(M) \).

2. Definitions and Notation

Definition 2.1 An \( R \)-module \( M \) is called \( S \)-coprime if, \( \text{ann}_sM = \frac{M}{N} \) for every small submodule \( N \) of \( M \) [8]. Equivalently, \( M \) is \( S \)-coprime if whenever \( r \in R \), \( rM \ll M \) implies \( rM = 0 \).

Definition 2.2 An \( R \)-module \( M \) is called strongly \( S \)-coprime (briefly \( SS \)-coprime) if, for all \( a, b \in R \), \( abM \ll M \) implies \( aM = 0 \) or \( bM = 0 \) [7].

Definition 2.3 An \( R \)-module \( M \) is called semi-strongly \( S \)-coprime (briefly \( SSS \)-coprime) if, for all \( r \in R \), \( r^2M \ll M \) implies \( rM = 0 \) [7].

Definition 2.4 An \( R \)-module \( M \) is said to be \( T \)-noncosingular if, for each nonzero endomorphism \( \varphi \) of \( M \), \( \text{Im}(\varphi) \) is not small in \( M \) [10].

Remark 2.5 By [7], we have the following implications:

\( \text{SS-coprime} \Rightarrow \text{S-coprime}, \)
\( \text{SS-coprime} \Rightarrow \text{SSS-coprime}, \)
\( T \)-noncosingular \( \Rightarrow \text{S-coprime}. \)

3. Endo strongly \( S \)-coprime modules

In this section, the class of endo strongly \( S \)-coprime modules is defined and investigated. First we obtain some properties of this kind of modules. Also relations between such modules and some other classes of modules will be studied.

Definition 3.1 An \( R \)-module \( M \) is called endo strongly \( S \)-coprime (briefly endo \( SS \)-coprime) if, for each \( f, g \in \text{End}(M) \) with \( \text{Im}(f \circ g) \ll M \) implies \( f = 0 \) or \( g = 0 \).

Remarks and Examples 3.2

1. It is clear that every endo \( SS \)-coprime module is \( T \)-noncosingular, but not conversely, as the following example shows: it is clear that the \( Z \)-module \( Z_a \) is \( T \)-noncosingular. Assume \( f, g \in \text{End}(Z_a) \) defined by \( f(\widetilde{x}) = 2\widetilde{x}, \ g(\widetilde{x}) = 3\widetilde{x} \) for all \( \widetilde{x} \in Z_a \). Now, we have \( \text{Im}(f \circ g) = (0) \ll M \), but neither \( f \) nor \( g \) is zero. This means that the \( Z \)-module \( Z_a \) is not endo \( SS \)-coprime.

2. Let \( M \) be an \( R \)-module, \( S = \text{End}(M) \). Then \( M \) is endo \( SS \)-coprime if and only if \( M \) is \( T \)-noncosingular and \( l_s(M) \) is prime.

Proof. It is obvious. \( \square \)

3. Every endo \( SS \)-coprime module is \( SS \)-coprime.

Proof. Suppose that \( M \) is an endo \( SS \)-coprime module. Let \( a, b \in R \), \( abM \ll M \). Define the endomorphisms \( f, g \) on \( M \) by \( f(m) = am \) and \( g(m) = bm \) for all \( m \in M \). Then \( \text{Im}(f \circ g) = f(bM) = abM \ll M \), but \( M \) is endo \( SS \)-coprime, so either \( f = 0 \) or \( g = 0 \) and this implies \( aM = 0 \) or \( bM = 0 \). \( \square \)
The converse is not true in general, for example: consider the \(Z\)-module \(Z_{\infty} \oplus Z_1\). See [7, Rem.and.Ex. 2.2(3)], \(Z_{\infty} \oplus Z_1\) is SS-coprime \(Z\)-module but not \(T\)-noncosingular, thus by (1), \(Z_{\infty} \oplus Z_1\) is not endo SS-coprime.

By (3) and Remark 2.5, we have the following.

**Corollary 3.3** Every endo SS-coprime module is S-coprime and SSS-coprime.

**Proposition 3.4** Let \(M\) be an R-module, \(\overline{R} = R/\text{ann}M\). Then \(M\) is an endo SS-coprime \(R\)-module if and only if \(M\) is an endo SS-coprime \(\overline{R}\)-module.

**Proof.** It is obvious. \(\square\)

**Proposition 3.5** If \(M_1\) and \(M_2\) are two isomorphic \(R\)-modules. Then \(M_1\) is endo SS-coprime if and only if \(M_2\) is endo SS-coprime.

**Proposition 3.6** Let \(M\) be an \(R\)-module, \(S = \text{End}(M)\). Then \(S\) is an endo SS-coprime \(S\)-module implies \(S\) has no zero divisors.

**Proof.** Suppose \(S\) is an endo SS-coprime \(S\)-module. Let \(f, g \in S\) such that \(f \circ g = 0\), thus \(Sf, Sg \ll S\), so either \(Sf = 0\) or \(Sg = 0\), hence \(f = 0\) or \(g = 0\), and so \(S\) has no zero divisors. \(\square\)

A submodule \(N\) of \(M\) is said to be stable if, \(f(N) \subseteq N\) for each \(R\)-homomorphism \(f : N \to M\). A module \(M\) is called fully stable in case each submodule of \(M\) is stable [1]. An \(R\)-module \(M\) is called multiplication if for every submodule \(N\) of \(M\) there exists an ideal \(I\) of \(R\) such that \(N = IM\) [3].

The following two corollaries are immediately.

**Corollary 3.7** Let \(M\) be a fully stable \(R\)-module, \(S = \text{End}(M)\). Then \(S\) is an endo SS-coprime \(S\)-module implies \(S\) is an integral domain.

**Proof.** Since \(M\) is fully stable, thus by [1, Prop. 1.2.1] \(S = \text{End}(M)\) is a commutative ring. Hence the result is follow by Proposition 3.6. \(\square\)

**Corollary 3.8** Let \(M\) be a multiplication \(R\)-module, \(S = \text{End}(M)\). Then \(S\) is an endo SS-coprime \(S\)-module implies \(S\) is an integral domain.

**Proof.** If \(M\) is multiplication, thus by [13, Prop. 1.1] \(S = \text{End}(M)\) is a commutative ring. Hence the result is follow by Proposition 3.6. \(\square\)

**Proposition 3.9** Let \(M\) be a (multiplication or fully stable) \(R\)-module. If \(M\) is endo SS-coprime then \(S\) is an integral domain, where \(S = \text{End}(M)\).

**Proof.** Let \(f, g \in \text{End}(M)\) such that \(f \circ g = 0\), then \(\text{Im}(f \circ g) \ll M\), but \(M\) is endo SS-coprime, implies \(f = 0\) or \(g = 0\). Thus \(S\) has no zero divisors. Since \(M\) is (multiplication or fully stable), then the result is obtained. \(\square\)

Hadi I.M-A in [7], presented the following result.

**Lemma 3.10** Let \(M\) be a multiplication \(R\)-module. Then \(M\) is an SS-coprime \(R\)-module if and only if \(M\) is an SS-coprime \(S\)-module, where \(S = \text{End}(M)\).

The next result follows directly.

**Proposition 3.11** Let \(M\) be a multiplication \(R\)-module. Then \(M\) is an endo SS-coprime module if and only if \(M\) is a SS-coprime module.

**Corollary 3.12** Let \(R\) be a ring. Then \(R\) is endo SS-coprime if and only if \(R\) is SS-coprime.

**Proof.** Since \(R\) is a commutative ring with identity, then \(R\) is multiplication. Hence, the result obtained by Proposition 3.11. \(\square\)

Recall that an \(R\)-module \(M\) is called a scalar module if, for all \(\varphi \in \text{End}(M)\), there exists \(r \in R\) such that \(\varphi(m) = rm\) for all \(m \in M\) [14].
We noticed that every endo SS-coprime module is a SS-coprime module but not conversely (see Rem. and Ex. 3.2(3)). In the next result we present condition under which the converse is satisfied.

**Proposition 3.13** Let \( M \) be a scalar \( R \)-module. If \( M \) is a SS-coprime module, then \( M \) is endo SS-coprime.

**Proof.** Let \( f, g \in \text{End}(M) \), \( \text{Im}(f \circ g) \ll M \). Since \( M \) is scalar, so there exist \( a, b \in R \) such that \( f(m) = am \) and \( g(m) = bm \) for all \( m \in M \). Then \( abM = \text{Im}(f \circ g) \) is small in \( M \), but \( M \) is SS-coprime, so either \( aM = 0 \) or \( bM = 0 \) this implies \( f = 0 \) or \( g = 0 \). \( \square \)

The following two results are characterizations of endo SS-coprime modules.

**Proposition 3.14** Let \( M \) be an \( R \)-module, \( S = \text{End}(M) \). Then \( M \) is an endo SS-coprime module if and only if, for each ideals \( A, B \) of \( S \), \( ABM \ll M \) implies that \( AM = 0 \) or \( BM = 0 \).

**Proof.** Assume that \( M \) is an endo SS-coprime module. Let \( A, B \) be ideals of \( S \), \( ABM \ll M \) and \( BM \neq 0 \), so there exists \( g \in B \) such that \( g(M) \neq 0 \). For each \( f \in A \), \( \text{Im}(f \circ g) \leq ABM \ll M \), but \( M \) is endo SS-coprime and \( g \neq 0 \), thus \( f = 0 \) for all \( f \in A \). Hence \( AM = 0 \).

Conversely, let \( f, g \in \text{End}(M) \) with \( \text{Im}(f \circ g) \ll M \). Then \((S_{f} \times S_{g})M \ll M \), so by assumption, \( (S_{f})M = 0 \) or \( (S_{g})M = 0 \), and hence \( f = 0 \) or \( g = 0 \). \( \square \)

**Proposition 3.15** Let \( M \) be an \( R \)-module, \( S = \text{End}(M) \). Then \( M \) is endo SS-coprime if and only if for each \( f, g \in S \), \( \text{Im}(f \circ g) \ll M \) implies \((f(M) \ll M) = l_{s}(M) \) or \((g(M) \ll M) = l_{s}(M) \).

**Proof.** Assume \( M \) is an endo SS-coprime \( R \)-module.

Let \( f, g \in \text{End}(M) \) with \( \text{Im}(f \circ g) \ll M \), so \( f = 0 \) or \( g = 0 \) and hence \((f(M) \ll M) = l_{s}(M) \) or \((g(M) \ll M) = l_{s}(M) \).

Conversely, if \( f, g \in \text{End}(M) \), \( \text{Im}(f \circ g) \ll M \), thus by hypothesis, \((f(M) \ll M) = l_{s}(M) \) or \((g(M) \ll M) = l_{s}(M) \). But \( f \in (f(M) \ll M) \) and \( g \in (g(M) \ll M) \), so either \( f \in l_{s}(M) \) or \( g \in l_{s}(M) \); that is, either \( f = 0 \) or \( g = 0 \). \( \square \)

**Proposition 3.16** Every direct summand of an endo SS-coprime module is also endo SS-coprime.

**Proof.** Let \( M \) be an endo SS-coprime module, and let \( N \subseteq M \), then \( M = N \oplus K \) for some submodule \( K \) of \( M \). Let \( f, g \in \text{End}(N) \), \( \text{Im}(f \circ g) \ll N \). Consider the endomorphisms \( \varphi, \psi \) of \( M \), \( \varphi(n+k) = f(n) \) and \( \psi(n+k) = g(n) \) for all \( n \in N \). Notice that \( \varphi, \psi \) are well-defined. Now, \( \text{Im}(\varphi \circ \psi) = \text{Im}(f \circ g) \ll N \) implies that \( \text{Im}(\varphi \circ \psi) \ll M \), but \( M \) is endo SS-coprime, so either \( \varphi = 0 \) or \( \psi = 0 \) this mean that, \( f = 0 \) or \( g = 0 \).

Hence \( N \) is endo SS-coprime. \( \square \)

**Remarks 3.17**

(1) A homomorphic image of endo SS-coprime module is not necessarily endo SS-coprime, for example: we know that in the \( Z \)-module \( Z \), the zero submodule is the only small, so it is clear that \( Z \) as a \( Z \)-module is endo SS-coprime. Consider the natural epimorphism \( \pi : Z \to Z_{s} \), then \( \text{Im}(\pi) = Z_{s} \) is not endo SS-coprime, to see this: let \( \varphi, \psi \in \text{End}(Z_{s}) \) such that \( \varphi(x) = x \) and \( \psi(x) = 2x \) for all \( x \in Z_{s} \). Then \( \text{Im}(\varphi \circ \psi) = \varphi(\text{Im}(\psi)) = \varphi(\{0, 2\}) = \{0, 2\} \ll Z_{s} \), but neither \( \varphi \) nor \( \psi \) is zero. Also, this example show that, the endo SS-coprime property does not always transfer from a module to each of factor modules.

(2) The direct sum of endo SS-coprime modules need not be endo SS-coprime module, for example: we know that \( Z_{s} \) as \( Z \)-module is not endo SS-coprime, but we
have \( Z_i \cong Z_i \oplus Z_i \) and each of \( Z_i \) and \( Z_i \) are endo SS-coprime.

**Proposition 3.18** Let \( M \) be an \( R \)-module. If \( M \oplus M \) is an endo SS-coprime \( R \)-module, then \( M \) is so.

**Proof.** Since \( M \cong M \oplus (0) \leq M \oplus M \) and \( M \oplus M \) is endo SS-coprime, so by Proposition 3.16, \( M \oplus (0) \) is an endo SS-coprime module, and hence \( M \) is an endo SS-coprime module. □

The converse is not true in general, as the following shows: we know \( Z \) as \( Z \)-module is endo SS-coprime. Consider the \( Z \)-module \( Z \oplus Z \). Let \( f, g \in \text{End}(Z \oplus Z) \) are defined by \( f(x, y) = (x, 0) \), \( g(x, y) = (0, y) \) for all \( (x, y) \in Z \oplus Z \). Then \( \text{Im}(f \circ g) = f((0) \oplus Z) = (0, 0) \) which is small in \( Z \oplus Z \), but \( f \neq 0 \) and \( g \neq 0 \). Then \( Z \oplus Z \) is not endo SS-coprime as \( Z \)-module.

To prove the following Proposition, we need the following Lemma.

**Lemma 3.19** Let \( M \) be an \( R \)-module, \( S = \text{End}(M) \). Then \( M \) is a \( T \)-noncosingular module if and only if \( (N \cdot_2 M) = l_s(M) \) for any \( N \subseteq M \).

**Proof.** Assume that \( M \) is a \( T \)-noncosingular module. Let \( f \in (N \cdot_2 M) \), then \( f(M) \leq N \leq M \), so \( f(M) \) is small in \( M \), thus \( f = 0 \); that is \( f \in l_s(M) \). Therefore \( (N \cdot_2 M) = l_s(M) \).

Conversely, let \( \varphi \in \text{End}(M) \) with \( \text{Im} \varphi \leq M \), put \( N = \varphi(M) \). Thus \( \varphi \in (N \cdot_2 M) = l_s(M) \), this implies \( \varphi \neq 0 \). □

An \( R \)-module \( M \) is called small prime if, for every nonzero small submodule \( N \) of \( M \), \( \text{ann}_p N = \text{ann}_p M \) [12]. Also, a module \( M \) is called endo-small prime if, \( l_s(N) = l_s(M) \) for all \( N \leq M \), where \( S = \text{End}(M) \) [9]. Equivalently, a module \( M \) is endo-small prime if, for any \( x \in M \), \( \langle x \rangle \leq M \) and \( f(x) = 0 \) implies \( x = 0 \) or \( \text{Im} f = 0 \).

**Remark 3.20** If \( M \) is an endo-small prime module then \( l_s(M) \) is a prime ideal in \( S = \text{End}(M) \).

**Proof.** Let \( f \circ g \in \text{End}(M) \) with \( f \circ g(M) = 0 \). Thus, for any \( x \in M \), \( \langle x \rangle \leq M \) and \( f \circ g(x) = 0 \), implies \( f(g(x)) = 0 \) and \( \langle g(x) \rangle \leq M \), and hence \( g(x) = 0 \) or \( \text{Im} f = 0 \), so \( x = 0 \) or \( \text{Im} g = 0 \) or \( \text{Im} f = 0 \), thus \( \text{Im} g = 0 \) or \( \text{Im} f = 0 \). Thus the result obtained. □

Now, recall the following definition.

**Definition 3.21** An \( R \)-module \( M \) is called \( S \)-relatively divisible if, for all \( f \in \text{End}(M) \), \( f(M) \cap N = f(N) \) for all \( N \leq M \).

**Proposition 3.22** Let \( M \) be a \( S \)-relatively divisible and endo-small prime module then \( M \) is an endo SS-coprime module, provided that \( M \) has a nonzero \( x \in M \) and \( \langle x \rangle \leq M \).

**Proof.** First we shall prove that \( M \) is \( T \)-noncosingular. Assume that there exists \( f \in (N \cdot_2 M) \) and \( f \notin l_s(M) \), \( S = \text{End}(M) \); that is \( f(M) \neq 0 \). As \( M \) is endo-small prime, \( l_s(N) = l_s(M) \) for all \( N \ll M \). Hence \( f(N) \neq 0 \).

But \( f(M) \cap f(N) = f^2(N) \), so \( f(N) = f^2(N) \), this implies that, for any \( n \in N \), \( f(n) = f^2(n) \) for some \( n \in N \). It follows that \( f(n - f(n)) = 0 \). But, we have \( n - f(n) \in N \), so \( \langle n - f(n) \rangle \leq N \ll M \), hence \( \langle n - f(n) \rangle \leq M \). But \( M \) is endo-small prime, we get \( l_s(M) = l_s(M) \) which is a contradiction. Thus, \( (N \cdot_2 M) = l_s(M) \) for all \( N \leq M \). Therefore, \( M \) is \( T \)-noncosingular, by Lemma 3.19. On the other hand, \( M \) is an endo-small prime module, so by Remark 3.20, \( l_s(M) \) is a prime ideal. Thus the result obtained by [Rem. and Ex. 3.2(2)]. □

Recall that an \( R \)-module \( M \) is called \( F \)-regular if, for each submodule \( N \) of \( M \), \( IN = N \cap IM \) for every ideal \( I \) of \( R \) [5]. An \( R \)-module \( M \) is called prime if, for all nonzero submodule \( N \) of \( M \), \( \text{ann}_p N = \text{ann}_p M \) [4].
Corollary 3.23 Let $M$ be an $F$-regular module over $S = \text{End}(M)$. If $M$ is endo-small prime, then $M$ is endo SS-coprime.

Corollary 3.24 Let $M$ be an $R$-module, $S = \text{End}(M)$. If $S$ a regular ring, then the following statements are equivalent.

(i) $M$ is an endo-small prime $R$-module.
(ii) $M$ is an endo SS-coprime $R$-module.
(iii) $M$ is a prime as $S$-module.

Proof. (i) $\Rightarrow$ (ii) If $S$ is a regular ring, $S/\text{ann}_S(x)$ is also a regular ring, hence $M$ is F-regular as $S$-module, so by previous Corollary, $M$ is endo SS-coprime.

(ii) $\Rightarrow$ (iii) Since $M$ is an endo SS-coprime module, then $l_s(M)$ is a prime ideal of $S$, so $S/l_s(M)$ has no zero divisors. But $S$ is a regular ring, then $S/l_s(M)$ is a regular ring. It follows that $\tilde{S} = S/l_s(M)$ is a division ring. Hence $M$ is a prime $\tilde{S}$-module, and so $M$ is a prime as $S$-module.

(iii) $\Rightarrow$ (i) It is obvious. $\square$

Proposition 3.25 Let $M$ be a divisible module over the ring $S = \text{End}(M)$, where $S$ has no zero divisors, then $M$ is a faithful endo SS-coprime module.

Proof. Assume that $f, g \in \text{End}(M), \text{Im}(f \circ g) \subseteq M$. If $f \circ g \neq 0$ this implies $f \circ g(M) = M$, since $M$ is a divisible $S$-module, so $M \subseteq M$ which is a contradiction. Thus $f \circ g = 0$, but $S$ has no zero divisors, so either $f = 0$ or $g = 0$, and hence $M$ is an endo SS-coprime module. $\square$

Recall that an $R$-module $M$ is called small retractable if, $\text{Hom}(M, N) \neq 0$ for each $N \subseteq M$ [7].

Remark 3.26 Let $M$ be a small retractable and scalar $R$-module. If $M$ is an endo SS-coprime $R$-module, then $\text{Rad}(M) = 0$.

Proof. By Remark 3.2(1), $M$ is an endo SS-coprime module, implies that $M$ is a $T$-noncosingular module, and so by Remark 2.5, $M$ is S-coprime, but $M$ is small retractable and scalar, hence $\text{Rad}(M) = 0$, by [7, Prop. 2.22]. $\square$

Proposition 3.27 Let $M$ be an $R$-module. Then $M$ is a $T$-noncosingular module, wherever $\text{Hom}(M, N) = 0$ for each $N \subseteq M$.

Proof. Assume that $f \in \text{End}(M), \text{Im} f \subseteq M$. Define $g : M \rightarrow \text{Im} f$ by $g(m) = f(m)$ for all $m \in M$. Hence $g \in \text{Hom}(M, \text{Im} f), \text{Im} f \subseteq M$ and so by assumption $g = 0$. Hence $f = 0$. Then $M$ is $T$-noncosingular. $\square$

Corollary 3.28 Let $M$ be an $R$-module with $l_s(M)$ is a prime ideal of $S = \text{End}(M)$. If $\text{Hom}(M, N) = 0$ for all $N \subseteq M$, then $M$ is endo SS-coprime.

Proof. It follows directly by previous Proposition and [Rem.and.Ex.3.2(2)]. $\square$

Proposition 3.29 Let $M$ be an $R$-module, $S = \text{End}(M)$. Then $M$ is an endo SS-coprime module if and only if $\text{Hom}(M, N) = 0$ for every $N \subseteq M$, and $l_s(M)$ is a prime ideal of $S$.

Proof. If $M$ is an endo SS-coprime $R$-module. Let $f \in \text{Hom}(M, N), N \subseteq M$. Thus $\text{Im} f \subseteq N \subseteq M$, but $M$ is an endo SS-coprime module implies that $M$ is a $T$-noncosingular module, and so $f = 0$. Therefore $\text{Hom}(M, N) = 0$. Moreover, since $M$ is an endo SS-coprime $R$-module, then $l_s(M)$ is a prime ideal in $S$, by [Rem.and.Ex.3.2(2)].

Conversely, if $\text{Hom}(M, N) = 0$ for each $N \subseteq M$, then by Proposition 3.27, $M$ is $T$-noncosingular. But $l_s(M)$ is prime, thus by [Rem.and.Ex.3.2(2)], $M$ is an endo SS-coprime module. $\square$

A nonzero module $M$ is called hollow if, every proper submodule is small in $M$ [6].
However, the following result gives a condition under which the concepts of endo SS-coprime module and $T$-noncosingular module are coincide.

**Proposition 3.30** Let $M$ be a hollow $R$-module. Then $M$ is an endo SS-coprime module if and only if $M$ is a $T$-noncosingular module.

**Proof.** Assume that $M$ is a $T$-noncosingular $R$-module. Let $f, g \in \text{End}(M)$, $\text{Im}(f \circ g) \subseteq M$. So either $\text{Im} f$ or $\text{Im} g$ is a proper submodule of $M$. If $\text{Im} f \subseteq M$, then $\text{Im} f \ll M$ and hence $f = 0$. Similarly, $g = 0$. Thus $M$ is an endo SS-coprime module.

Conversely, follows by [Rem. and Ex. 3.2(1)].

**Lemma 3.31** Let $M$ be a module and $A$ be a nilideal of $S = \text{End}(M)$. If $M$ is a $T$-noncosingular module, then $AM = 0$.

**Proof.** Let $f \in A$, we claim that $\text{Im} f \ll M$. Assume that $\text{Im} f + N = M$ for some submodule $N$ of $M$. Thus, for all $n \in Z$, $f^n(M) + N = M$. But $f$ is a nilpotent element, so $f^n = 0$ for some $n \in Z$, then $N = M$, and so $\text{Im} f \ll M$. Thus $f = 0$ for any $f \in A$, since $M$ is a $T$-noncosingular. Therefore $AM = 0$.

**Proposition 3.32** Let $M$ be an $R$-module and $A, B$ be two ideals of $S = \text{End}(M)$ such that $AB$ is a nilideal. If $M$ is an endo SS-coprime $R$-module, then $AM = 0$ or $BM = 0$.

**Proof.** Since $M$ is an endo SS-coprime module, then $M$ is $T$-noncosingular module and hence by above Lemma, $ABM = 0$, so $ABM \ll M$. But, $M$ is an endo SS-coprime module, so by Proposition 3.14, $AM = 0$ or $BM = 0$.

Recall that a ring $R$ is semilocal provided that $R/J(R)$ is a semi-simple ring.

**Proposition 3.33** Let $M$ be an $R$-module, $S = \text{End}(M)$ be a semilocal ring and $J(S)$ is a nilideal. Then $M$ is a $T$-noncosingular $R$-module if and only if $M$ is a semisimple $R$-module.

**Proof.** If $M$ is a $T$-noncosingular $R$-module. Since $S$ is a semilocal ring, then $S/J(S)$ is semisimple and hence $M/J(S)M$ is semisimple, by [2, Cor. 15.18]. But $J(S)$ is a nilideal, thus by Lemma 3.31, $J(S)M = 0$ and hence $M$ is semisimple.

Conversely, since $M$ is a semisimple module, then the zero submodule is the only small submodule of $M$, this implies that $M$ is a $T$-noncosingular.

**Proposition 3.34** Let $M$ be a scalar faithful $R$-module. Then $R$ is an endo SS-coprime ring if and only if $S = \text{End}(M)$ is an endo SS-coprime ring.

**Proof.** Since $M$ is a scalar faithful $R$-module, then by [11, Lemma 6.1] $S = \text{End}(M) \cong R$. Hence the result follows by Proposition 3.5.

**Proposition 3.35** Let $M$ be an $R$-module such that $S = \text{End}(M)$ is a regular ring with out zero divisors, then $M$ is endo SS-coprime.

**Proof.** Let $f, g \in \text{End}(M)$, $\text{Im}(f \circ g) \subseteq M$. Since $S$ is a regular ring, so there exists $h \in S$ such that $f \circ g = (f \circ g) \circ h \circ (f \circ g)$, and hence $(f \circ g) \circ h$ is an idempotent element of $S$, so that $\text{Im}((f \circ g) \circ h)$ is a direct summand of $M$. But, $\text{Im}((f \circ g) \circ h) \subseteq \text{Im}(f \circ g) \subseteq M$, thus $\text{Im}((f \circ g) \circ h) \subseteq M$ this implies that $\text{Im}((f \circ g) \circ h) = 0$, and hence either $f \circ g = 0$ or $h = 0$. But, $f \circ g = 0$ implies either $f = 0$ or $g = 0$, since $S$ has no zero divisors. Also, if $h = 0$ then $f \circ g = 0$, and so $f = 0$ or $g = 0$.

**Proposition 3.36** Let $M$ be a multiplication finitely generated faithful module over a PID $R$. Then $M$ is endo SS-coprime if and only if $\text{Rad}(M) = 0$.

**Proof.** If $M$ is an endo SS-coprime $R$-module, then $M$ is $T$-noncosingular, but $M$ is multiplication finitely generated faithful module over a PID $R$, thus by [10, Cor. 2.9] $\text{Rad}(M) = 0$.

Conversely, since $\text{Rad}(M) = 0$, so by [10, Cor. 2.9] $M$ is $T$-noncosingular, means for all $f, g \in \text{End}(M)$,
Im(f ◦ g) ≪ M implies f ◦ g = 0. But M is a finitely multiplication faithful, then M is scalar faithful, thus S ≅ R and so S has no zero divisors. Hence f ◦ g = 0, implies that f = 0 or g = 0. □

4. Semi-Endo SS-coprime modules

In this section, we define and study semi-endo SS-coprime modules which is a generalization of endo SS-coprime modules. We give the relations between such modules and other types of modules.

Definition 4.1 An R-module M is called a semi-endo SS-coprime module (briefly semi-endo SS-coprime) if, for each f ∈ End(M), Im(f ◦ f) ≪ M implies f = 0.

We shall investigate the relation between semi-endo SS-coprime and other classes of modules.

Remarks and Examples 4.2

(1) It is clear that every endo SS-coprime module is semi-endo SS-coprime, but the converse is not true in general, as the following example shows: Z-module Z_6 is semi-endo SS-coprime, but it is not endo SS-coprime. In fact, if f ∈ End(Z_6), f^2(Z_6) ≪ Z_6 this implies that f^2(Z_6) = 0, and since (0) is a semiprime submodule of Z_6, hence f = 0.

(2) Every semi-endo SS-coprime module is T-noncosingular.

Proof. Let M be a semi-endo SS-coprime module and f ∈ End(M), Im f ≪ M . But Im(f ◦ f) ≤ Im f , thus Im(f ◦ f) ≪ M , and so f = 0 . □

(3) Let M be an R-module, S = End(M) . Then M is a semi-endo SS-coprime module if and only if M is T-noncosingular and I_s (M) is a semiprime ideal of S.

Proof. It is obvious . □

(4) Let M be an R-module and let S = End(M) be a chained ring. Then M is endo SS-coprime if and only if M is semi-endo SS-coprime.

(5) Every semi-endo SS-coprime module is SSS-coprime.

Proof. Let M be a semi-endo SS-coprime module. Let r ∈ R, r^0 M ≪ M . Consider φ : M → M by φ(m) = rm for all m ∈ M . Thus φ^2(M) = φ(rM) = r M ≪ M , but M is semi-endo SS-coprime, thus rM = Imφ = 0 . Hence M is a SSS-coprime module . □

The converse is not true in general, for example: consider the Z-module Z_6 ⊕ Z_7, then it is SS-coprime and not T-noncosingular see [7, Rem.and.Ex.2.2(3)], this implies Z_6 ⊕ Z_7 is SSS-coprime but not semi-endo SS-coprime.

(6) If M is a semi-endo SS-coprime module, then M is SS-coprime and hence M is S-coprime, whenever ann_s M is a prime ideal.

Proof. It follows by(2) and [ Rem.and.Ex. 3.2 (2),(3)]. □

The next result gives characterizations of semi-endo SS-coprime modules.

Proposition 4.3 Let M be an R-module, S = End(M) . Then the following statements are equivalent.

(i) M is a semi-endo SS-coprime R-module.

(ii) For any ideal A of S, A^2 M ≪ M implies AM = 0 .

(iii) For any ideal A of S, and n ∈ Z_0 . If A^n M ≪ M implies AM = 0 .

Proof. It is easy, so is omitted . □

Proposition 4.4 Let M be an R-module, R/annM = R/annM . Then M is a semi-endo SS-coprime R-module if and only if M is a semi-endo SS-coprime R-module.

Proof. It is obvious . □

Proposition 4.5 If M_1 and M_2 are two isomorphic R-modules. Then M_1 is semi-endo SS-coprime if and only if M_2 is semi-endo SS-coprime.

Proposition 4.6 Let M be a scalar R-module. If M is SSS-coprime, then M is semi-endo SS-coprime.
Proof. Let \( \varphi \in \text{End}(M) \) with \( \varphi^*(M) \ll M \). Since \( M \) is scalar, so there exists \( r \in R \) such that \( \varphi(m) = rm \) for all \( m \in M \), thus \( r^2 M = \varphi^2(M) \) is small in \( M \), but \( M \) is an SSS-coprime module, hence \( rM = 0 \) which implies \( \varphi = 0 \). \( \Box \)

Proposition 4.7 Let \( M \) be an \( R \)-module, \( S = \text{End}(M) \).

Then \( M \) is semi-endo SSS-coprime if and only if for all \( f \in S \), \( \text{Im}(f \circ f) \ll M \) implies \( (f(M) : M) = l_z(M) \).

Proof. Assume that \( M \) is a semi-endo SSS-coprime \( R \)-module. Let \( f \in \text{End}(M) \), \( \text{Im}(f \circ f) \ll M \), so \( f = 0 \) and hence \( (f(M) : M) = (0 : M) = l_z(M) \).

Conversely, let \( \varphi \in \text{End}(M) \) with \( \varphi^*(M) \ll M \), so by hypothesis, \( (\varphi(M) : M) = l_z(M) \). But we have \( \varphi \in (\varphi(M) : M) \), thus \( \varphi(M) = 0 \), hence \( \varphi = 0 \) and \( M \) is a semi-endo SSS-coprime \( R \)-module. \( \Box \)

Proposition 4.8 Let \( M \) be a semi-endo SSS-coprime \( R \)-module and let \( N \) be a direct summand of \( M \). Then \( N \) is semi-endo SSS-coprime.

Proof. Let \( M \) be a semi-endo SSS-coprime \( R \)-module. Assume that \( N \leq^0 M \), then \( M = N \oplus K \) for some submodule \( K \) of \( M \). Let \( \varphi \in \text{End}(N) \), \( \varphi^*(N) \ll N \).

Consider the endomorphism \( \psi : M \to M \) defined by \( \psi(n + k) = \varphi(n) \) for \( n \in N \). Now, \( \varphi^*(N) = \varphi^*(N) \) is small in \( N \), this implies \( \varphi^*(M) \ll M \), but \( M \) is semi-endo SSS-coprime, so \( \varphi = 0 \), and hence \( \varphi = 0 \). \( \Box \)

Remark 4.9 A homomorphic image of semi-endo SSS-coprime module is not necessarily semi-endo SSS-coprime module, for example: it is well known that \( Z \) as \( Z \)-module is endo SSS-coprime, so it is a semi-endo SSS-coprime module. Consider the natural epimorphism \( \pi : Z \to Z \). It is clear that \( \text{Im} \pi = Z \) is not SSS-coprime and hence it is not semi-endo SSS-coprime as \( Z \)-module, by [Rem.and.Ex. 4.2(5)]. In particular case, this example show that, the factor of semi-endo SSS-coprime module need not be semi-endo SSS-coprime module.

Proposition 4.10 Let \( M \) be an \( R \)-module. If \( M \oplus M \) is a semi-endo SSS-coprime \( R \)-module, then \( M \) is so.

Proof. By Proposition 4.8, \( M \oplus (0) \) is a semi-endo SSS-coprime module of \( M \oplus M \). But \( M \oplus (0) \cong M \), so \( M \) is semi-endo SSS-coprime, by Proposition 4.5. \( \Box \)

Proposition 4.11 Let \( M \) be a module has a projective cover \( \varphi : p \to M \). If \( P \) is semi-endo SSS-coprime, then so is \( M \).

Proof. Since \( M \) is has a projective cover \( \varphi : p \to M \), then \( \varphi \) is an epimorphism and \( \text{Ker} \varphi \ll P \), thus we have \( P/\text{Ker} \varphi \cong M \). It is enough to show that \( P/\text{Ker} \varphi \) is semi-endo SSS-coprime. Assume \( \psi \in \text{End}(P/\text{Ker} \varphi) \), \( \psi^*(P/\text{Ker} \varphi) \ll P/\text{Ker} \varphi \). Consider \( \pi : p \to P/\text{Ker} \varphi \) the natural epimorphism. Since \( P \) is projective, so there exists a homomorphism \( \lambda : P \to P \) such that \( \psi \circ \pi = \pi \circ \lambda \).

\[
\begin{array}{c}
P \xrightarrow{\pi} P/\text{Ker} \varphi \\
\text{Im} \psi \downarrow \\
\psi \downarrow \\
P \xrightarrow{\pi} P/\text{Ker} \varphi
\end{array}
\]

So \( \psi \circ \pi = \pi \circ \lambda \). Hence, \( \pi \circ \lambda(P) = \psi^*(P/\text{Ker} \varphi) \) is small in \( P/\text{Ker} \varphi \), and hence \( \lambda(P) + \text{Ker} \varphi \ll \frac{P}{\text{Ker} \varphi} \), and since \( \text{Ker} \varphi \ll P \), thus \( \lambda(P) \ll P \). But \( P \) is semi-endo SSS-coprime, hence \( \lambda = 0 \), and so \( \psi \circ \pi = 0 \). Thus \( \psi = 0 \). \( \Box \)

Corollary 4.12 Let \( R \) be a ring. Then the following statements are equivalent.

(i) Every projective \( R \)-module is semi-endo SSS-coprime.

(ii) Every \( R \)-module has a projective cover is semi-endo SSS-coprime.

Proof. (i) \( \Rightarrow \) (ii) It follows by previous Proposition.
(ii) \(\Rightarrow\) (i) Let \(M\) be a projective \(R\)-module. Consider the identity mapping \(i : M \rightarrow M\), \(\ker i = 0 \ll M\), thus \(M\) has a projective cover. Hence by (ii), \(M\) is semi-endomorphic module.

**Theorem 4.13** Let \(M\) be a multiplication \(R\)-module. Then \(M\) is a SSS-coprime \(R\)-module if and only if \(M\) is a semi-endomorphic SS-coprime \(R\)-module.

**Proof.** Since \(M\) is a SSS-coprime \(R\)-module, then by [7, Th. 3.10] \(M\) is a SSS-coprime as \(S\)-module, where \(S = \text{End}(M)\). This implies that \(M\) is a semi-endomorphic SS-coprime as \(R\)-module.

**Proposition 4.14** Let \(M\) be a scalar faithful \(R\)-module. Then \(R\) is a semi-endomorphic SS-coprime ring if and only if \(S = \text{End}(M)\) is a semi-endomorphic SS-coprime ring.

**Proof.** Since \(M\) is a scalar faithful \(R\)-module, so \(S \cong R\). Hence the result is obtained.

**Remark 4.15** Let \(M\) be an \(R\)-module, \(S = \text{End}(M)\) be a semilocal ring with \(J(R)\) is a nilideal. If \(M\) is a semi-endomorphic SS-coprime \(S\)-module, then \(M\) is \(T\)-noncosingular and hence \(M\) is semisimple, by Proposition 3.33.

For every module \(M\), let \(S(M) = \{\varphi \in \text{End}(M) : \text{Im} \varphi^2 \ll M\}\). It is easy to see that \(S(M)\) is an ideal of \(\text{End}(M)\). By the semi-endomorphic SS-coprime submodule of \(M\) we mean \(\overline{Z}_S(M) = \bigcap_{\varphi \in S(M)} \ker \varphi\).

**Proposition 4.16** Let \(M\) be an \(R\)-module. Then \(M\) is semi-endomorphic SS-coprime if and only if \(\overline{Z}_S(M) = M\).

**Proof.** Suppose that \(M\) is a semi-endomorphic SS-coprime module. Then, for each \(\varphi \in S(M)\), \(\varphi = 0\) and hence \(M = \ker\varphi = \bigcap_{\varphi \in S(M)} \ker\varphi = \overline{Z}_S(M)\).

Conversely, assume \(\overline{Z}_S(M) = M\). Let \(\varphi \in \text{End}(M)\) and \(\text{Im}\varphi^2 \ll M\), hence \(\varphi \in S(M)\). By hypothesis, we have \(M = \overline{Z}_S(M) = \bigcap_{\varphi \in S(M)} \ker\varphi\). Thus \(M \subseteq \ker\varphi\); that is \(\varphi(M) = 0\). Hence \(M\) is semi-endomorphic SS-coprime.

**Acknowledgement**

This paper was written during a visit by the second author to the Department of Mathematics, College of Education Ibn Al-Haitham, Baghdad University, and he thanks the first author for giving him many useful suggestions which help to modify the presentation of this article.

**References**


