Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.2, No.9, 2012

Solutions of Fifth Order Boundary Value Problems

Bhupesh K. Tripathi
Department of mathematics
C.M.P. Degree College, University of Allahabad
Allahabad – 220 011, India
Email: bupeshkt@gmail.com

Abstract
Fifth order boundary value problems have been solved using B-spline functions. In this paper, we propose a simple method for solving fifth order boundary value problems that gives better results than B-spline based method. The effectiveness of the proposed method is tested on several problems.

Keywords – Fifth order boundary value problem, B-spline, Analytic solution.

1. Introduction

The fifth order boundary value problems occur in mathematical modeling of viscoeleastic flows viz-a-viz in various branches of physical sciences and engineering [1-5]. In this paper, we consider the following class of boundary value problems.

$$y^{(5)}(x) = f(x, y) \quad a < x < b \tag{1}$$

subject to the boundary conditions

$$y(a)=A_1, \quad y'(a)=A_2, \quad y''(a)=A_3, \quad y(b)=B_1, \quad y'(b)=B_2$$

f is a real continuous nonlinear function of x and y; and A_i ($i=1,2,3$), B_i($i=1,2$) are real constants. There are various methods available in literature that include iterative method [6], spectral Galerkin and collocation [7, 8], decomposition [9], B-spline functions of different order [1-5]. Details about B-splines can be found in [10, 11]. It is assumed that a unique solution $y(x)$ exists and is analytic in the given interval. The studies related to existence and uniqueness of solutions of such boundary value problems are discussed in [12, 13]. We derive successive relations for obtaining higher derivatives at some point $x=x_0 \in$ [a, b] and then express solution as the linear combination of basic polynomials. Without loss of generality and because of simplicity, we can take x_0 as zero if the desired interval is [0, 1]. Then using Taylor series we may write the solution $y(x)$ as follows.

$$y(x) = y(0) + \frac{1}{1!}y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + ... + \frac{1}{n!}y^{(n)}(0)x^n + ... \tag{3}$$

In (3), some of the derivatives i.e., $y^{(i)}(0)$, $i=5,6,...,$ may be the functions of some of $y(0)$, $y'(0)$, $y''(0)$, $y^{(3)}(0)$, $y^{(4)}(0)$. Using other boundary conditions, we obtain the value of these unknowns. The method is illustrated using some examples from the literature [1].

2. Numerical Results

In this section, we test the effectiveness of our proposed method by applying it on the problems discussed in [1].

2.1 Problem 1: Consider the following equation.

$$\begin{aligned} &y^{(5)}(x) = y(x) - 15e^{x} - 10xe^{x} \quad 0 < x < 1 \\ &\quad y(0)=0, \quad y'(0)=1, \quad y''(0)=0, \quad y(1)=0, \quad y'(1)=-e \end{aligned} \tag{4}$$

Solution: Taking $x \to 0$ in (4), we get $y^{(5)}(0)=-15$.
Differentiating (4) once, we have

$$y^{(6)}(x) = y'(x) - 15e^{x} - 10e^{x} - 10xe^{x}$$

Putting $x=0$ gives

$$y^{(6)}(0) = -24 \tag{5}$$

Differentiating (4) twice, we have

$$y^{(7)}(x) = y^{(2)}(x) - 15e^{x} - 20e^{x} - 10xe^{x}$$

Putting $x=0$ in this equation, we get

$$y^{(7)}(0) = -35 \tag{6}$$

Performing the successive differentiation in (4) and then putting $x=0$, we have the following recursive relation for n[th] derivative

$$y^{(n)}(0) = y^{(n-5)}(0) - 15 - 10(n-5) \quad n \geq 5 \tag{7}$$

We write (7) in the following form ($k=1,2,3,\ldots$)

$$y^{(5k)}(0) = y(0) - 5k(5k-2) \tag{8a}$$

$$y^{(5k+1)}(0) = y'(0) - 5k(5k) \tag{8b}$$

$$y^{(5k+2)}(0) = y''(0) - 5k(5k+2) \tag{8c}$$

$$y^{(5k+3)}(0) = y^{(3)}(0) - 5k(5k+4) \tag{8d}$$

$$y^{(5k+4)}(0) = y^{(4)}(0) - 5k(5k+6) \tag{8e}$$

It may be noted that $y^{(3)}(0)$ and $y^{(4)}(0)$ are unknowns and their values will be found using boundary conditions.
We write the solution in the following form

$$y(x)=y(0)+\frac{1}{1!}y'(0)x+\frac{1}{2!}y''(0)x^2+\frac{1}{3!}y'''(0)x^3+....+\frac{1}{n!}y^{(n)}(0)x^n+... \qquad (9)$$

It can be written as follows

$$y(x)=\sum_{k=0}\left(\frac{y^{(5k)}(0)}{5k!}x^{5k}+\frac{y^{(5k+1)}(0)}{(5k+1)!}x^{5k+1}+\frac{y^{(5k+2)}(0)}{(5k+2)!}x^{5k+2}+\frac{y^{(5k+3)}(0)}{(5k+3)!}x^{5k+3}+\frac{y^{(5k+4)}(0)}{(5k+4)!}x^{5k+4}\right)$$

Using (8a)-(8e) in the above equation and then carrying out some manipulations it gives

$$\begin{aligned}y(x)=&\left(x+y^{(3)}(0)\sum_{k=0}\frac{x^{5k+3}}{(5k+3)!}+y^{(4)}(0)\sum_{k=0}\frac{x^{5k+4}}{(5k+4)!}\right)\\ &-\sum_{k=1}\left(\frac{x^{5k}}{(5k-2)!}+\frac{x^{5k+1}}{(5k-1)!}+\frac{x^{5k+2}}{(5k)!}+\frac{x^{5k+3}}{(5k+1)!}+\frac{x^{5k+4}}{(5k+2)!}\right)\\ &-\sum_{k=1}\left(\frac{x^{5k}}{(5k-1)!}+\frac{x^{5k+1}}{(5k)!}+\frac{x^{5k+2}}{(5k+1)!}+\frac{x^{5k+3}}{(5k+2)!}+\frac{x^{5k+4}}{(5k+3)!}\right)\end{aligned}$$

Or

$$y(x)=\left((y^{(3)}(0)+3)\sum_{k=0}\frac{x^{5k+3}}{(5k+3)!}+(y^{(4)}(0)+8)\sum_{k=0}\frac{x^{5k+4}}{(5k+4)!}\right)-x^2e^x+xe^x$$

Using the boundary conditions it gives
$y^{(3)}(0) = -3$ and $y^{(4)}(0) = -8$.
Thus, the solution is

$$y(x)=-x^2e^x+xe^x \qquad (10)$$

This is the same as the exact solution of the problem.
2.2 Problem 2: Consider the following equation.

$$\begin{aligned}&y^{(5)}(x)=-24e^{-5y(x)}+\frac{48}{(1+x)^5},\quad 0<x<1\\ &y(0)=0,\quad y'(0)=1,\quad y''(0)=-1,\quad y(1)=\ln(2),\quad y'(1)=0.5\end{aligned} \qquad (11)$$

Solution: Performing similar steps as in Problem 1, we get

$$y^{(5)}(0)=24=4! \qquad (12)$$

$$y^{(6)}(0)=-120=-5! \qquad (13)$$

$$y^{(7)}(0)=-5(y^{(6)}(0)-y^{(5)}(0))=-5(120+24)=720=6! \qquad (14)$$

$$y^{(8)}(0)=-5(y^{(7)}(0)-2y^{(6)}(0)+y^{(5)}(0)y'''(0))+5*48(y'''(0)+2y''(0))$$

Or $y^{(8)}(0)=-5(6!+2*(5!)+4!y'''(0)-48y'''(0)+96)$

$$y^{(8)}(0)=-5!(44-y'''(0)) \qquad (15)$$

$$y^{(9)}(0)=-5(y^{(8)}(0)-3y^{(7)}(0)+3y^{(6)}(0)y'''(0)+y^{(5)}(0)y^{(4)}(0))+5*48(y^{(4)}(0)+3y'''(0))$$

Or $y^{(9)}(0)=-5(y^{(8)}(0)-3*6!-3*5!y'''(0)+4!y^{(4)}(0)-48y^{(4)}(0)-3*48y'''(0))$

Or $y^{(9)}(0)=-5(-5!*44-16*4!y'''(0)-3*6!-4!y^{(4)}(0))$

Or $y^{(9)}(0)=5!(342+y^{(4)}(0))$ (16)

Performing successive differentiation in (11) and then putting $x=0$, we have the following recursive relation for n^{th} derivative

$$y^{(n+6)}(0)+5\sum_{k=0}^{n} C_k^n y^{(n+5-k)}(0)y^{(k+1)}(0)=240(y^{(n+1)}+ny^{(n)}) \quad n\geq 1 \qquad (17)$$

We write the solution as the linear combination of fundamental polynomials. The well known such relation is Taylor series. Thus, the solution in the form of Taylor series of about $x=0$, i.e.,

$$y(x)=y(0)+\frac{1}{1!}y'(0)x+\frac{1}{2!}y''(0)x^2+\frac{1}{3!}y'''(0)x^3+....+\frac{1}{n!}y^{(n)}(0)x^n+... \qquad (18)$$

Using the boundary conditions and (12)-(17) in (18), and then after some manipulations it gives
$y^{(3)}(0)=2!$ $\quad y^{(4)}(0)=-3!$
Thus, the solution is

$$y(x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+....=\log(1+x) \qquad (19)$$

This is the same as the exact solution of the problem.

Conclusion

In this paper, we have discussed a very simple method for solving fifth order boundary value problems that provides efficient solutions. In limiting case, the solution matches the exact solution. For numerical solutions, we can take as number of terms as we please in order to reduce the error. We have tested the method on two problems given in literature and their solutions match the exact solutions in limiting case.

References

[1]. H. N. Caglar, S. H. Caglar, E.H. Twizell, The numerical solution of fifth order boundary value problems with sixth-degree B-spline functions, Applied mathematics and Computation, 12 (1999) 25-30.

[2]. A. M. Wazwaz, The numerical solution of fifth order boundary value problems by the decomposition method, Applied mathematics and Computation, 136 (2001) 259-270.

[3]. M.A. Noor, S.T. Mohyud-Din, An efficient algorithm for solving fifth-order boundary problems, Mathematics and Computer Modelling, 45 (2007) 954-964.

[4]. H. Caglar, N. Caglar, K. Elfaituri, B-spline interpolation compared with finite element and finite volume methods which applied to two-point boundary value problems, Appl. Math. Comput. 175(2006) 72-79.

[5]. F.G. Lang, X. P. Xu, A new cubic B-spline method for linear fifth order boundary value problems, J. Appli math Comput. (2010) DOI 10.1007/s12190-010-0390-y.

[6]. M.A. Noor, S. T. Mohyud-Din, A new approach to fifth-order boundary value problems, International Journal of Nonlinear Science 7(2009) 143-148.

[7]. A.R. Davies, A. Karageoghis , T.N. Philips, Spectral Galerkin methods for the primary two-point boundary-value problems in modeling viscoelastic flows. Int . J. Numer. Methods Eng. 26(1988) 647-662

[8]. A. Karageoghis, T.N. Philips , A.R. Davies, Spectral collocation methods for the Primary two point boundary value problems in modeling viscoelastic flows. Int. J. Numer. Meth. Eng. 26 (1998) 805-813

[9]. A.M. Wazwaz, The numerical solution of fifth-order boundary-value problems by Adomian decomposition method. J. Comput. Appl. Math. 136 (2001) 259-270

[10]. J.H. Ahlberg, E.N. Nilson, J.L. Walsh, The theory of splines and their applications, academic press, New York, 1967.

[11]. C. de Boor, A practical guide to Splines, Springer Verlag, New York, 1978.

[12]. R.D. Russell, L.F. Shampine, Numerical methods for singular boundary value problems, SIAM J. Numer. Anal. 12 (1975) 13–36.

[13]. R.P. Agarwal, Boundary Value Problems for High Order Differential Equations, World Scientific, Singapore, (1986).