On the order of system of entire functions of several complex variables

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Abstract:-

In this paper we study the growth of entire functions represented by Taylor series of several complex variables. The characterization of their order and type has been obtained.

Keywords: entire functions of several complex variables, order and lower order of entire functions**.**

1- Introduction

We denote complex N-space by C^N. Thus , $z \in C^N$ means that $z =(z_1,z_{2,},\ldots,z_N)$, where $z_1,z_{2,},\ldots,z_N$ are complex numbers.

A function f (z), $z \in C^N$ is said to be analytic at a point $\xi \in C^N$ if it can be expanded in some neighborhood of ξ

As an absolutely convergent power series .if we assume ξ=(0,0,….,0) then f(z) has representation :

$$f(z)=\sum_{\|k\|=0}^{\infty} a_{k_{1,k2,\ldots,kN}} \; z_1^{k_1} \, z_2^{k_2}, \ldots\ldots, z_N^{k_N} = \sum_{n=0}^{\infty} a_k \; z^k \quad , \text{ (see[1])} \tag{1.1}$$

Where $k= (k_1, k_{2,},\ldots, k_N) \in N_0^N$ and $n= \|k\| = k_1+k_2 +\ldots+ k_N$. M. M.Dzrbasyan ([2], p.1) has shown that the necessary and sufficient condition for the series (1.1) to represent an entire function of variables $z_1,z_{2,},\ldots,z_N$ is that :

$$\lim_{n\to\infty}(|a_k|)^{\frac{1}{n}}=0, \text{ Or} \tag{1.2}$$

$$\lim_{n\to\infty}\frac{log|a_k|}{n}= - \infty$$

For r>0, the maximum modulus M(r , f) and central index of entire function f(z) is given by (see[3],p.321)

$$M(r)=M(r,f)=\sup\{|z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 = r^2 \tag{1.3}$$

v(r)=v(r,k) ,where k is the index with maximal length n for which maximum term is achieved .

The order of entire function f (z) is defined as [4]:

$$\rho= \lim_{r\to\infty} sup \frac{loglogM(r)}{logr} \;\; , \;\; \rho = \lim_{r\to\infty} sup \frac{log|v(r)|}{logr} \tag{1.4}$$

Also for an entire function f (z), we define the lower order λ of f (z) as:

$$\lambda= \lim_{r\to\infty} inf \frac{loglogM(r)}{logr} \;\; , \lambda= \lim_{r\to\infty} inf \frac{log|v(r)|}{logr} \tag{1.5}$$

A.A.Golderg [5] obtained the order in terms of the coefficients of its Taylor expansion as

$\rho_G = \lim_{\|k\|\to\infty} \frac{\|k\| \, log\|k\|}{-log|a_k|} = \lim_{n\to\infty} \frac{nlogn}{-log|a_k|}$ (1.6)

Susheel Kumar and G.S.Srivastava ([6], p.157) proved the following theorem

Theorem 1.1

The lower order λ of the entire function (1.1) satisfies

$\lambda \geq \lim_{n\to\infty} inf \; \frac{nlogn}{-log\|a_k\|}$ (1.7)

Also if $\frac{|a_k|}{|a_k'|}$, Where $\|k'\|$ = n+1, is non-decreasing function of n, then equality holds in (1.7). If the order of entire function (1.1) is equal to the lower order of (1.1) then the function (1.1) is said to be of regular growth.

Proof

Write

$$\Phi = \lim_{n\to\infty} inf \, \frac{nlogn}{-log\|a_k\|}$$

First we prove that λ≥Φ. The coefficient of an entire Taylors series satisfy Cauchy's inequality, that is

$$\|a_k\| \leq M(r) r^{-n} \tag{1.8}$$

Also from (1.5), for arbitrary ε>0 and a sequence $r=r_s\to\infty$ as s→∞, we have

$$M(r) \leq \exp(r^{\bar{\lambda}}), \; \bar{\lambda} = \lambda+\varepsilon$$

So from (1.8), we get

$$\|a_k\| \leq r^{-n} \exp(r^{\bar{\lambda}})$$

Putting r= $(\frac{n}{\bar{\lambda}})^{1/\bar{\lambda}}$ in the above inequality we get

$$\|a_k\| \leq (\frac{n}{\bar{\lambda}})^{\frac{-n}{\bar{\lambda}}} \exp(n/\bar{\lambda})$$

$$\text{Log } \|a_k\|^{-1} \geq \frac{nlogn}{\bar{\lambda}} [1 - \frac{log\bar{\lambda}}{logn} - \frac{1}{logn}]$$

$$\lim_{n\to\infty} inf \, \frac{nlogn}{-log\|a_k\|} \leq, \bar{\lambda} \, .$$

$$\Phi \leq \bar{\lambda}.$$

Since ε>0 is arbitrarily small so finally we get $\Phi \leq \lambda$.

Now we prove that $\lambda \leq \Phi$. Let

$$\Psi(n) = \frac{\|a_k\|}{\|a_{k'}\|} \, ,$$

Then ψ (n) →∞ as n→∞, also

Also $\Psi(|k'|) > \psi(n)$.

Now suppose that $\|a_{k^1}\| r^{|k^1|}$ and that $\|a_{k^2}\| r^{|k^2|}$ are two consecutive maximum terms. Then

$$|k^1| \leq |k^2| - 1$$

Let

$$|k^1| \leq n \leq |k^2|$$

Then $|v(r)| = |k^1|$ for $\psi\,(|k^{1^*}|) \leq r < \psi(|k^1|)$. Where $|k^{1^*}|$=$|k^1| - 1$.

Hence from (1.5), for arbitrarily ε>0 and all r>r_0(ε),we have

$$|k^1| = |v(r)| > r^{\lambda'}\ , \lambda' = \lambda - \varepsilon.$$

$$|k^1| = |v(r)| \geq \{\psi(|k^1|) - q\}^{\lambda'} \text{ Where q is constant.}$$

$$\text{Log}\ \psi(|k^1|) \leq o(1) + \frac{log|k^1|}{\lambda'}$$

Further we have $\psi(|k^0|) = \psi(|k^1| + 1) \ldots\ldots \psi(n-1)$.

Now we can write

$$\psi(|k^0|) \ldots \ldots \psi(|k^*|) = \frac{\|a_{k_0}\|}{\|a_k\|} \leq [\psi(|k^*|)]^{n-|k^0|} \text{ where} |k^*| = n - 1\ and\ n > |k^0|.$$

$$\log \|a_k\|^{-1} \leq n log\psi(|k^1|) + o(1)$$

$$\frac{1}{n} \log \|a_k\|^{-1} \leq \frac{logn}{\lambda'}[1+\text{o}(1)]$$

$$\lambda' \leq \frac{nlogn}{-log\|a_k\|}[1 + \text{o}(1)]$$

Now taking limits as n→∞, we get ≤ Φ , Hence the theorem (1.1) is proved .

In this paper we generalized the work of Salimov [1] , where we consider have a system of entire functions represented by Taylors series of several complex variables as follows:-

f_i (z) $=\sum_{n=0}^{\infty} a_k^{(i)}\ z^k$, i=1,2,……., s (1.9)

Then we obtain some relations between the function represented by (1.1) and the system of entire functions (1.9) and study the relations between the coefficients in Taylor expansion of entire functions and their order.

2-Main result

Theorem 2.1

Let f_i (z) $=\sum_{n=0}^{\infty} a_k^{(i)}\ z^k$, i=1, 2,……., s of regular growth then the necessary and sufficient condition that the orders of these functions are same is:

Log $\{|a_k^{(i)}|/|a_k^{(i-1)}|\}$ =o ($\|k\| log\|k\|$), As $\|k\| \to \infty$. (2.1)

Proof

Since the functions $f_i(z)$, i=1,2,……., s are of regular growth , therefore

$\lim_{\|k\|\to\infty} sup \frac{\|k\| \, log\|k\|}{-log\left|a_k^{(i)}\right|} = \rho_i=\lambda_i= \lim_{\|k\|\to\infty} inf \frac{\|k\| \, log\|k\|}{-log\left|a_k^{(i)}\right|}$, (i=1, 2,….,s).

Let the functions $f_i(z)$, i=1, 2,……., s , have the same order that is to say:

$\rho= \rho_i=\lambda_i=\lambda$, i=1,2,……., s , or

$\lim_{\|k\|\to\infty} sup \frac{-log\left|a_k^{(i)}\right|}{\|k\|log\|k\|} = \frac{1}{\rho}$, $\lim_{\|k\|\to\infty} sup \frac{-log\left|a_k^{(i-1)}\right|}{\|k\|log\|k\|} = \frac{1}{\rho}$, Hence

$\lim_{\|k\|\to\infty} \frac{\text{Log}\{\left|a_k^{(i)}\right|/\left|a_k^{(i-1)}\right|\}}{\|k\|log\|k\|} =0$ or

Log $\{\left|a_k^{(i)}\right|/\left|a_k^{(i-1)}\right|\}$ =o ($\|k\|log\|k\|$), As $\|k\| \to \infty$. Let us prove the converse

Let $f_i(z)$, i=1,2,……., s , have the orders ρ_i (i=1,2,…,s) respectively then

$\frac{1}{\rho_i} - \frac{1}{\rho_{i-1}} = \frac{\text{Log}\{\left|a_k^{(i)}\right|/\left|a_k^{(i-1)}\right|\}}{\|k\|log\|k\|} =0$, as a result $\rho_i = \rho_{i-1}$.

Theorem 2.2

Let each function of system (1.9) be entire function of order ρ_i , λ_i (i=1,2,….,s) respectively then under the condition

$$|a_k| \sim \prod_{i=1}^{s}\left|a_k^{(i)}\right|^{\alpha_i} , \sum_{i=1}^{s}\alpha_i = 1 , \ i=1,2,\ldots,s \tag{2.2}$$

The function (1.1) is an entire function of order ρ and lower order λ such that

$\frac{1}{\rho} \geq \sum_{i=1}^{s}\frac{\alpha_i}{\rho_i}$, $\frac{1}{\lambda} \leq \sum_{i=1}^{s}\frac{\alpha_i}{\lambda_i}$.

Proof

Since each function of system (1.9) is entire then by (1. 2)

$$\frac{log\left|a_k^{(i)}\right|}{\|k\|} < -M , \ (i=1,2,\ldots,s) \tag{2.3}$$

Where M > 0, from (2.3) we have

$\left|a_k^{(i)}\right|^{\alpha_i} < e^{-M\|k\|\alpha_i}$ Or

$\prod_{i=1}^{s}\left|a_k^{(i)}\right|^{\alpha_i} < e^{-M\|k\|\sum_{i=1}^{s}\alpha_i}$, Where $\sum_{i=1}^{s}\alpha_i = 1$ and $|a_k| \sim \prod_{i=1}^{s}\left|a_k^{(i)}\right|^{\alpha_i}$, i=1 , 2,….,s .

Then we have $\frac{log|a_k|}{\|k\|} < -M$, so we proved that the function (1.1) is entire function.

From (1.6), we have

$\lim_{\|k\|\to\infty} \frac{-log\left|a_k^{(i)}\right|}{\|k\|log\|k\|} = \frac{1}{\rho_i}$, Then

$\lim_{\|k\|\to\infty} \frac{-log\left|a_k^{(i)}\right|^{\alpha_i}}{\|k\|log\|k\|} = \frac{\alpha_i}{\rho_i}$ Or

$\lim_{\|k\|\to\infty} \frac{-log\left|a_k^{(i)}\right|^{\alpha_i}}{nlogn} = \frac{\alpha_i}{\rho_i}$, Then for large number N such that n> N=max ($N_1, N_2, \ldots, N_s$) , ε>0 we have

$\frac{-log\left|a_k^{(i)}\right|^{\alpha_i}}{nlogn} \geq \frac{\alpha_i}{\rho_i} - \frac{\varepsilon}{s}$, Summing the last inequality on i=1,2,…,s , we obtain

$\frac{-log\prod_{i=1}^{s}\left|a_k^{(i)}\right|^{\alpha_i}}{nlogn} \geq \sum_{i=1}^{s}\frac{\alpha_i}{\rho_i} - \varepsilon$

Since ($|a_k| \sim \prod_{i=1}^{s}\left|a_k^{(i)}\right|^{\alpha_i}$) Then we have

$\frac{-log|a_k|}{nlogn} \geq \sum_{i=1}^{s}\frac{\alpha_i}{\rho_i} - \varepsilon$, Or

$\frac{1}{\rho} \geq \sum_{i=1}^{s}\frac{\alpha_i}{\rho_i}$, Further it is easy to prove $\frac{1}{\lambda} \leq \sum_{i=1}^{s}\frac{\alpha_i}{\lambda_i}$.

Theorem 2.3

Suppose that each function f_i (z) of the system (1.9) of order ρ_i and lower order λ_i (i=1.2,….,s) respectively then under the condition

$$s\,(log|a_k|^{-1})^{-1} \sim \sum_{i=1}^{s}(log\left|a_k^{(i)}\right|^{-1})^{-1} \qquad (2.4)$$

The function (1.1) is entire function of order ρ and lower order λ, such that

$$\sum_{i=1}^{s-1}\lambda_i \leq (s\lambda-\lambda_s\ ,\ s\rho-\rho_s) \leq \sum_{i=1}^{s-1}\rho_i \qquad (2.5)$$

Proof

Integrity of function (1.1) is proved similarly to theorem (2.2) using conditions (2.4). According to (1.6) we have

$$\frac{\|k\|log\|k\|}{log\left|a_k^{(i)}\right|^{-1}} \leq \rho_i+\varepsilon \quad , (i=1,2,\ldots,s) \quad \text{for } \varepsilon>0,\ k>N \text{ , where N is large number .} \qquad (2.6)$$

Summing in (2.6) for i=1,2,….,s we obtain

$\sum_{i=1}^{s}\frac{\|k\|log\|k\|}{log\left|a_k^{(i)}\right|^{-1}} < \sum_{i=1}^{s}\rho_i + \varepsilon\, s$, Or

$\|k\|log\|k\|\sum_{i=1}^{s}(log\left|a_k^{(i)}\right|^{-1})^{-1} < \sum_{i=1}^{s}\rho_i + \varepsilon\, s$

Taking into account the condition (2.4) in the last inequality we obtain

$$s\rho - \rho_s \leq \sum_{i=1}^{s-1} \rho_i \qquad (2.7)$$

And also easy to prove

$$s\lambda - \lambda_s \leq \sum_{i=1}^{s-1} \rho_i \,, \sum_{i=1}^{s-1} \lambda_i \leq s\lambda - \lambda_s \,,$$

$$s\,\rho - \rho_s \geq \sum_{i=1}^{s-1} \lambda_s \qquad (2.8)$$

From (2.7) and (2.8) we obtain the proof of theorem 2.3.

Theorem 2.4

Let each function f_i (z) of the system (1.9) of order ρ_i and lower order λ_i (i=1.2,….,s) respectively and if

$$\mathrm{Log}|a_k|^{-1} \sim \prod_{i=1}^{s} (log\left|a_k^{(i)}\right|^{-1})^{\alpha_i} \qquad (2.9)$$

Where $0< \alpha_i<1$, $\sum_{i=1}^{s} \alpha_i=1$, then the function (1.1) is entire function of order ρ and of lower order λ, such that

$$\prod_{i=1}^{s-1} \lambda_i^{\alpha_i} \leq \left(\frac{\lambda}{\lambda_s^{\alpha_s}}, \frac{\rho}{\rho_s^{\alpha_s}} \right) \leq \prod_{i=1}^{s-1} \rho_i^{\alpha_i} \qquad (2.10)$$

Theorem 2.5

Let each function fi (z) of the system (1.9) of order ρ_i and lower order λ_i (i=1.2,….,s) respectively and if

$$\mathrm{Log}|a_k| \sim \log \prod_{i=1}^{s} \left|a_k^{(i)}\right|^{\alpha_i} \qquad (2.11)$$

Where α_i = constant, for i=1,2,…,s , then the function (1.1) is entire of order ρ and of lower order λ , such that

$$\sum_{i=1}^{s-1} \frac{\alpha_i}{\rho_i} \leq \left\{ \frac{1}{\rho} - \frac{\alpha_s}{\rho_s}, \frac{1}{\lambda} - \frac{\alpha_s}{\lambda_s} \right\} \leq \sum_{i=1}^{s-1} \frac{\alpha_i}{\lambda_i} \quad .$$

Theorem (2.4) and theorem (2.5) is proved as theorem (2.3).

3-Acknowledgemnt
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4-Reference

[1] Salimov F.Q., "On growth of entire functions represented by multiple Dirichlet series", Caspian journal of Applied Mathematics, Ecology and Economics, v.2, No.2, 2014, ISSN 1560-4055.

[2] M. M.Dzrbasyan,"On the theory of some class of integral functions ", Izv.Akademi .Nauk Armyanskoi. S.S.R. VIII, No.4, 1955, pp.1-23.

[3] A. Janik, "On approximation of entire functions and generalized order", Univ.lagel.Acta Math. 24(1984), 321-326.

[4] T.Winiarski, "Application of approximation and interpolation method of entire functions of n complex variables', Ann.pol. Math 28(1973), pp. 97-121.

[5] A.A.Goldberge, "Elementary remarks on the formulas for defining order and type of functions of several variables", Akad, Nauk Armjan. SSR. Dokl. , 29 (1959), pp.145-152.

[6] Susheel Kumar and G.S.Srivastava, "Maximum term and lower order of entire functions of several complex variables", v.3 issue (2011), pp. 156-164.