Special Cases for Numerical Radius and Spectral Radius Inequalities

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Abstract

In this paper, the aim of this study is to find special cases of some inequalities for numerical radius and spectral radius of a bounded linear operator on a Hilbert space. Finally, some new results about this subject are obtained.

Keywords: Spectral norm, Numerical radius, Spectral radius.

1. Introduction

Let $B(H)$ denote the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$. For $A \in B(H)$, let $\omega(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of $A$, respectively. It is well known that $\omega(.)$ defines a norm on $B(H)$, and that for every $A \in B(H)$, $w(A) \leq \|A\|$

The concepts of numerical range and numerical radius play an important role in various fields of Con-temporary Mathematics, including Operator Theory, Operator Trigonometry, Numerical Analysis and other see [1], [7],.....

Theorem 1.1[7]

Let $X_i \in B(H)$ $(i = 1, 2, ..., n)$. Then

$$\omega^{\alpha} \left( \sum_{i=1}^{n} X_i \right) \leq \frac{n^{r-1}}{2} \left\| \sum_{i=1}^{n} \left( |X_i|^{2r\alpha} + |X_i|^2r(1-\alpha) \right) \right\| \quad \forall \alpha \in (0,1), r \geq 1 \ldots (1)$$

Theorem 1.2 [1]
Let $A$ and $B$ be self-adjoint operators in $B(H)$, and $r \geq 2$. Then

$$\omega^r (A + B) \leq 2^{r-2} \left\| |A + B|^r + |A - B|^r \right\|. \quad (2)$$

Theorem 1.3[6]

If $A, B \in B(H)$, then

$$r (AB) \leq \frac{1}{4} \left( \|AB\| + \|BA\| \right)$$

$$+ \sqrt{\left( \|AB\| - \|BA\| \right)^2 + 4 \min \left( \|A\| \|BAB\|, \|B\| \|ABA\| \right)} \quad (3)$$

Theorem 1.4[7]

Let $A, B, C, D, S, T \in B(H)$. Then

$$\omega^r (ATB + CSD) \leq 2^{r-2} \left\| \left( A |T^* |A^* \right)^r + (B^* |T |B)^r + (C |S^* |C^* )^r + (D^* |S |D)^r \right\|. \quad (5)$$

In the following results, we find special cases of some inequalities for numerical radius and spectral radius of a bounded linear operator on a Hilbert space.

2. Main results

Theorem 2.1

If $A$ is an operator in $B(H)$, then

$$w (A) \leq \left\| |A|^2 \right\|^{1/2} \quad \ldots \ldots (5)$$

Proof

Let $r=2$ in (2), we get the result.

Theorem 2.2

Let $X_i \in B(H)$ $(i = 1, 2, \ldots, n)$. Then
\[
\omega \left( \sum_{i=1}^{n} X_i \right) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} (|X_i| + |X_i^*|) \right\| \quad (6)
\]

**Proof**

If we put \( r=1 \) in (1), then we obtain the result.

**Corollary 2.1**

If \( X \in B(H) \), then

\[
\omega(X) \leq \frac{1}{2} \left\| |X| + |X^*| \right\| \quad (7)
\]

**Proof**

By taking \( X_i = X \), \( \forall i = 1, 2, \ldots, n \) in (6), so we get the result.

**Theorem 2.3**

If \( A \) is a positive operator in \( B(H) \) and \( n \) is any scalar in \( R \), then

\[
r(A) \leq \left\| A^{n+1} \right\|_{\frac{1}{n+1}} \quad (8)
\]

**Proof**

Take \( B = A^n \) in (3), we obtain

\[
r\left( A^{n+1} \right) \leq \frac{1}{4} \left( \left\| A^{n+1} \right\| + \left\| A^{n+1} \right\| \right)
\]

\[
+ \sqrt{\left( \left\| A^{n+1} \right\| - \left\| A^{n+1} \right\| \right)^2 + 4 \min \left( \left\| A \left\| A^{2n+1} \right\|, \left\| A^n \left\| A^{n+2} \right\| \right\| \right)}
\]

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\[
\frac{1}{4} \left( 2 \left\| A^{n+1} \right\| + \sqrt{4 \min\left( \left\| A \right\| \left\| A^{2n+1} \right\|, \left\| A^n \right\| \left\| A^{n+2} \right\| \right)} \right) \\
\leq \frac{1}{2} \left( \left\| A^{n+1} \right\| + \sqrt{\left\| A^{2n+2} \right\|} \right)
\]

We know that \( r(A^{n+1}) = r(A)^{n+1} \), so we get the result.

**Corollary 2.2**

If \( A \) is a positive operator in \( B(H) \), then

\[
r(A) \leq \left\| A^2 \right\|^\frac{1}{2} (9)
\]

In general for

\[
n \geq 1, r(A) \leq \left\| A^n \right\|^\frac{1}{n}
\]

**Theorem 2.4**

Let \( A, B, C, D, S, T \in B(H) \). Then

\[
\omega \left( ATB + CSD \right) \leq \left( A^* T^* A^* \right)^2 + (B^* T B)^2 + (C^* S C^*)^2 + (D^* S D)^2 \left\|^\frac{1}{2} \right. (10).
\]

**Proof**

Let \( r=2 \) in (5), we get the result.

**Corollary 2.3**
Let $A \in \mathcal{B}(H)$. Then

$$\omega(A^2) \leq \frac{1}{2} \|4(A^* A^*)^{1/2}\| (11).$$

Proof

Let $A = B = C = D = S = T$ in (10), we get the result.

Open Problems

The first open problem is possible to complement the all bounds (5,7,8,9,11) by giving an upper bound estimate for the zeros of

$$p(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1$$

of degree $n \geq 2$, with complex coefficients $a_1, a_2, \ldots, a_n$, where $a_1 \neq 0$.

The second open problem is possible to complement the upper all bounds (5,7,8,9,11) by giving a lower bound estimate for the zeros of $p$. To see this, observe that the zeros of the polynomial

$$q(z) = \frac{z^n}{a_1} p\left(\frac{1}{z}\right)$$

are the reciprocals of those of $p$. Thus, applying the upper bound (9) to the zeros of $q$ yields the desired lower bound estimate for the zeros of $p$. This enables us to present a new annulus containing the zeros of $p$.

References


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