Special Cases for Numerical Radius and Spectral Radius Inequalities

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Abstract

In this paper, the aim of this study is to find special cases of some inequalities for numerical radius and spectral radius of a bounded linear operator on a Hilbert space, Finally, some new results about this subject are obtained.

Keywords: Spectral norm, Numerical radius, Spectral radius.

1.Introduction

Let B(H) denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle .,. \rangle$. For $A \in B(H)$, let $\omega(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of A ,respectively. It is well known that $\omega(.)$ defines a norm on B(H),and that for every $A \in B(H)$, $w(A) \leq \|A\|$

The concepts of numerical range and numerical radius play an important role in various fields of Con-temporary Mathematics, including Operator Theory, Operator Trigonometry, Numerical Analysis and other see **[1]**, **[7]**,…..

Theorem 1.1[7]

Let $X_i \in B(H)$ $(i = 1, 2, \ldots, n)$. *Then*

$$\omega^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}\left(|X_i|^{2r\alpha} + |X_i^*|^{2r(1-\alpha)}\right)\right\| \forall \alpha \in (0,1), r \geq 1 \text{.....}(1)$$

Theorem 1.2 [1]

Let A and B be self-adjoint operators in B(H), and r ≥ 2. Then

$$\omega^r(A+B) \le 2^{r-2}\left\| |A+B|^r + |A-B|^r \right\|. \quad (2)$$

Theorem 1.3[6]

If A, B ∈ B(H), then

$$r(AB) \le \frac{1}{4}\Big(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\min(\|A\|\|BAB\|, \|B\|\|ABA\|)}\Big) \quad (3)$$

Theorem 1.4[7]

Let A, B,C, D, S, T ∈B(H).Then

$$\omega^r(ATB+CSD) \le 2^{r-2}\left\|(A|T^*|A^*)^r + (B^*|T|B)^r + (C|S^*|C^*)^r + (D^*|S|D)^r\right\|. \quad \textbf{(5)}$$

In the following results , we find special cases of some inequalities for numerical radius and spectral radius of a bounded linear operator on a Hilbert space

2.Main results

Theorem 2.1

If A is an operator in B(H) ,then

$$w(A) \le \left\| |A|^2 \right\|^{\frac{1}{2}} \quad \textbf{......(5)}$$

Proof

Let r=2 in (2) ,we get the result.

Theorem 2.2

Let $X_i \in B(H)\ (i = 1, 2, ..., n)$ *.Then*

$$\omega\left(\sum_{i=1}^{n} X_i\right) \le \frac{1}{2}\left\| \sum_{i=1}^{n}\left(|X_i| + |X_i^*|\right)\right\| \qquad (6)$$

Proof

If we put r=1 in (1) ,then we obtain the result.

Corollary 2.1

If $X \in B(H)$***,then***

$$\omega(X) \le \frac{1}{2}\left\| |X| + |X^*| \right\| \qquad (7)$$

Proof

By taking $X_i = X, \forall i = 1,2,....n$ ***in (6) ,so we get the result.***

Theorem 2.3

If A is positive operator in B(H) and n is any scalar in R , then

$$r(A) \le \left\|A^{n+1}\right\|^{\frac{1}{n+1}} \qquad (8)$$

Proof

Take $B = A^n$ ***in (3),we obtain***

$$r\left(A^{n+1}\right) \le \frac{1}{4}\left(\left\|A^{n+1}\right\| + \left\|A^{n+1}\right\| + \sqrt{\left(\left\|A^{n+1}\right\| - \left\|A^{n+1}\right\|\right)^2 + 4\min\left(\|A\|\left\|A^{2n+1}\right\|, \left\|A^n\right\|\left\|A^{n+2}\right\|\right)}\right)$$

$$\leq \frac{1}{4}\Big(2\left\|A^{n+1}\right\|$$

$$+\sqrt{4\min\left(\|A\|\left\|A^{2n+1}\right\|, \left\|A^{n}\right\|\left\|A^{n+2}\right\|\right)}\Big)$$

$$\leq \frac{1}{2}\Big(\left\|A^{n+1}\right\|$$

$$+\sqrt{\|A\|^{2n+2}}\Big)$$

We know that $r\left(A^{n+1}\right) = r(A)^{n+1}$***,so we get the result.***

Corollary 2.2

If A is positive operator in B(H) , then

$r\left(A\right) \leq \left\|A^{2}\right\|^{\frac{1}{2}}$ *(9) .In general for*

$n \geq 1, r(A) \leq \left\|A^{n}\right\|^{\frac{1}{n}}$

Theorem 2.4

Let A, B,C, D, S, T ∈B(H).Then

$$\omega\left(ATB + CSD\right) \leq \left\|\left(A\,|T^{*}|\,A^{*}\right)^{2} + (\mathrm{B}^{*}\,|\,\mathrm{T}\,|\,\mathrm{B})^{2} + (\mathrm{C}\,|\,\mathrm{S}^{*}\,|\,\mathrm{C}^{*})^{2} + (\mathrm{D}^{*}\,|\,\mathrm{S}\,|\,\mathrm{D})^{2}\right\|^{\frac{1}{2}} (10).$$

Proof

Let r=2 in (5) ,we get the result.

Corollary 2.3

Let A ∈B(H).Then

$$\omega\ (A^3) \le \frac{1}{2}\left\| 4\left(A\,|A^*|\,A^*\right)^2 \right\|^{\frac{1}{2}} \text{(11).}$$

Proof

Let *A= B=C=D= S=T in* **(10) ,**we get the result.

Open Problems

The first open problem is possible to complement the all bounds (5,7,8,9,11) by giving an upper bound estimate for the zeros of

$p(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1$ of degree $n \ge 2$, with complex coefficients $a_1, a_2, \ldots, a_n$, where $a_1 \ne 0$.

The second open problem is possible to complement the upper all bounds (5,7,8,9,11)) by giving a lower bound estimate for the zeros of p .To see this, observe that the zeros of the polynomial $q(z) = \frac{z^n}{a_1} p(\frac{1}{z})$ are the reciprocals of those of p . Thus ,applying the upper bound (9) to the zeros of q yields the desired lower bound estimate for the zeros of p . this enables us to present a new annulus containing the zeros of p,

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