

Some Characterized Projective δ-cover

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Abstract

In this paper we characterize some properties of projective δ -cover and find some new results with δ -supplemented module M. Let M be a fixed R-module. A δ -cover in M is an δ -small epimorphism from M onto P.Thes concept introduce by Zhou [14]. A δ -cover is projective δ -cover(M-projective δ -cover) in case M is projective.

Keywords: Singular, non singular, simple, small, δ -small, cover, δ -cover, supplement, δ -supplement.

Introduction:

Throughout this paper R is an associative ring with unity and all modules are unitary R-modules. Let M be a fixed module, a sub module L of module M is denoted by $L \leq M$. submodule L of M is called essential (large) in M, abbreviated $K \leq_e M$, if for every submodule N of M, $L \cap N$ implies N = 0. A sub module N of a module M is called small in M, Denoted by N \ll M, if for every sub module L of M, the equality N + L = M implies L = M. For each X $\subset M$, the right Ann(X) in R is $r_R(X) = \{r \in R : xr = 0 \text{ for all } x \text{ in } x\}$. The sub module $Z(M) = \{x \in M : r_R(x) \text{ is an essential in } R_R\}$ {x} is singleton, is called singular submodule of M. The module M is called simple if $M \neq 0$ and M has only proper submodules. A sub module N of M is called minimal in M if N $\neq 0$ and for every submodules A of M, A \subset N implies A = N.

An epimorphism $f: M \to P$ is called small if ker $f \prec M$. A small epimorphism $f: M \to P$ is called projective cover if M is projective with ker $f \prec M$.[Zhou] introduce the concept of δ -small submodule as generalization of small submodules. Let $K \leq M$, K is called δ -small if whenever M = N + K and M/N is a singular, we have M = N.(denoted by \prec_{δ}). The sum of all δ -small submodules is denoted by $\delta(M)$. A δ -cover in M is an δ -small epimorphism from M onto P. A δ -cover is projective δ -cover(M-projective δ -cover)in case M is projective.

Definition: Let M be a fixed R- module. An R-module U is called (small) M-projective module, if for every (small) epimorphism $f: M \to P$ and homomorphism $g: U \to P$, there exists a homomorphism $v: U \to M$ such that $f \circ v = g$, i.e. following diagram is commute.



$$U$$

$$\downarrow g$$

$$M \xrightarrow{f} P \rightarrow 0 \quad (Kerf \prec M)$$

Example: Every proper sub module of the Z-modules Z_p^{∞} is small in Z_p^{∞} .

Remarks: i) Every M-projective module is a small M-projective cover.

ii) Every self projective module M is self small projective module and converse is true for

M is hollow.

Lemma: [Zhou] Let N be a sub module of M. The following are equivalent:

i) N ≪_δ M
 ii) If M = X + N, then M = X ⊕Y for a projective semisimple sub module Y with Y ⊆N.

Proof: [14]

Lemma: If each $f_i: N_i \to M_i$ are M-projective δ -covers for i = 1,2,3,...n, then

 $\bigoplus_{i=1}^{n} f_i : \bigoplus_{i=1}^{n} N_i \to M_i$ is M-projective δ - cover.

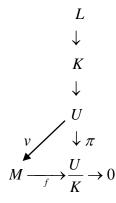
Proof: [12]

Lemma: If N is a direct summand of module M and $A \ll_{\delta} M$, then $A \cap N \ll_{\delta} N$.

Lemma: Let K be a sub module of a M-projective module U. If U/K has a M-projective δ -cover, then it has a M-projective δ -cover of the form $f: \frac{U}{L} \to \frac{U}{K}$ with ker $f = \frac{K}{L}$ where $L \subseteq K$.

Proof: Let K be a sub module of a M-projective module U. Let $f: M \to \frac{U}{K}$ be a M-projective δ -

cover of $\frac{U}{K}$, and $\pi: U \to \frac{U}{K}$ is a canonical epimorphism, U is M-projective module, there exists an homomorphism $v: U \to M$ s.t. $f \circ v = \pi$.



Then $M = \ker g \oplus \operatorname{Im} v$. By lemma [Zhou] $M = N \oplus \operatorname{Im} v$ for semi simple sub module N, with N \subseteq Kerf since $\ker(f \operatorname{I}_{\operatorname{Im} v}) \prec_{\delta} \operatorname{Im} v$. So $f \operatorname{I}_{\operatorname{Im} v}$ is also M-projective δ -cover of $\frac{U}{K}$. But $\frac{U}{\ker v} \cong \operatorname{Im} v$ by isomorphism theorem. Since $f \circ v = \pi$ and $\ker v \subseteq K$. If we consider the isomorphism $v': \frac{U}{\ker v} \to \operatorname{Im} v$ defined by $v'(\ker v + u) = u \quad \forall u \in U, \operatorname{Im} v \leq^{\oplus} U$. Then we obtain $\ker(f_{\operatorname{Im} v}, v') \prec_{\delta} \frac{U}{\ker v}.$

Lemma: A pair (M, f) is a M-projective δ -cover of finitely generated module U, The there exists a finitely generated direct summand M' of M such that f I_{M'} is a M-projective δ -cover of U.

Theorem: An R module M has a M-projective δ -cover, then for every epimorphism $f: M \to P$, the following are equivalent:

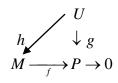
- i) $f: M \to P$ is a M-projective cover.
- ii) M is projective, for every epimorphism $f': M' \to P$, with $M' \leq^{\oplus} M$, there exists a necessarily split epimorphism $h: M' \to M$ such that $f \circ h = f'$.
- iii) For every small epimorphism $g: M \to N$, there exists an epimorphism $h: P \to M$ such that $f \circ h = g$

Corollary: Let $f: M \to P$ and $f': M' \to P$, $M' \leq^{\oplus} M$, be a M-projective cover. Then there is an isomorphism $h: M \to M'$ such that $f' \circ h = f$. In fact if $h: M \to M'$ is a homomorphism with $f' \circ h = f$, then h is an isomorphism.

Proposition: Let $f: M \to P$ be a M-projective δ -cover. If U is M-projective and $g: U \to P$ is an homomorphism, then there exists decomposition $M = A \oplus B$ and $U = X \oplus Y$ such that

- i) $A \cong X$
- ii) $fI_A: A \to P$ is a M-projective δ -cover.
- iii) $hI_X: X \to P$ is a M-projective δ -cover.
- iv) B is a Projective semi simple with $B \subseteq \ker f$ and $Y \subseteq \ker h$

Proof: Since U is M-projective,



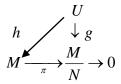
Then there exists $h: U \to M$ such that $f \circ h = g$. Thus we have $M = \operatorname{Im} h + \ker f$ and $\ker f \prec_{\delta} M$, we have $M = \operatorname{Im} h + B$ for a semi-simple module B with $B \subseteq \ker f$, by lemma 9. $fI_A: A \to P$ is a M-projective δ -cover. Since direct summand of projective module is projective, so A is projective and homomorphism $h: U \to A$ splits, then there exists $t: A \to U$ such that $h \circ t = I_A$. Thus $U = X \oplus Y = \operatorname{Im} t + \ker h$ this implies $A \cong t(A) = X$. Since $\ker(hI_A) \prec_{\delta} A$ $(M = A \oplus B)$, we have $\ker(hI_X) = \ker(hI_A) \prec_{\delta} t(A) = X$. $g(X) = (f \circ h)(X + Y) = (f \circ h)(U) = P$. Thus $hI_X: X \to P$ is a M-projective δ -cover.//

Lemma: Let U be a M-projective module and $N \leq^{\oplus} M$, then the following are equivalent;

- i) $\frac{M}{N}$ has a M-projective δ -cover.
- ii) $M = M_1 \oplus M_2$ for some M_1 and M_2 , with $M_1 \subset N$ and $M_2 \cap N \prec_{\delta} M$.

Proof: i)=> ii) Assume that $\frac{M}{N}$ has a M-projective δ -cover. Let $g: U \to \frac{M}{N}$ be a M-projective δ -

cover and $\pi: M \to \frac{M}{N}$ is canonical epimorphism, then there exists an homomorphism $h: U \to M$ such that the diagram



is commute. Therefore $M = \operatorname{Im} h + \ker \pi = \operatorname{Im} h + N$. By lemma [Zhou 3.1] there exists a decomposition $M = M_1 \oplus M_2$ such that $\pi I_X : M_2 \to M$ is a M-projective δ -cover and $M_1 \subseteq \ker \pi = N$. Thus $M_2 \cap N = \ker(\pi I_X) \prec_{\delta} X$. Since $M_2 \leq^{\oplus} M$ then $M_2 \cap N \prec_{\delta} M$. ii)=> i) it is clear.

Lemma: If $f: U \to M$ and $g: M \to N$ are δ -covers, then $g \circ f$ is a δ -cover.

Proof: [12]

Lemma: Let M, N, P be R-modules, for some homomorphisms $f: M \to P$, $g: M \to N$ and $h: N \to P$ such that $h \circ g = f$ then,

- i) F is a small epimorphism if and only if $N = \ker h + \operatorname{Im} g$.
- ii) A pair (M, f) is a projective δ -cover if and only if g(M) is a δ -supplement of kerh in N and ker $g \prec_{\delta} M$

Proof: i) it is clear by lemma R

(ii) =>Suppose a pair (M, f) is a δ -cover, by (i) we have $N = \ker h + \operatorname{Im} g$ i.e. f is small

epimorphism, we get $g(\ker f) = \ker h + \operatorname{Im} g$ and $\ker f \prec_{\delta} M$. By lemma [1,1 K. Al-

Thakman] $g(\ker f) \prec_{\delta} \operatorname{Im} g$, hence Img is δ -supplement of kerh in N.

(Assume that the <math>g(M) is a δ -supplement of ker h in N, then $N = \operatorname{Im} g + \ker h$ and $\operatorname{Im} g \cap \ker h \prec_{\delta} \operatorname{Im} g$. Since f is epimorphism, consider ker f + S = M and $\frac{M}{S}$ is singular. So $g(\ker f) + g(S) = g(M)$ but $g(\ker f) = \ker h \cap \operatorname{Im} g$, Hence $g(M) = g(\ker f) \cap \operatorname{Im} g + g(S)$,

since $\frac{g(M)}{g(S)}$ is singular, being a homomorphic image of singular module and $\operatorname{Im} g \cap \ker h \prec_{\delta} \operatorname{Im} g$. We have g(M) = g(S) and so $M = S + \ker g$, by assumption $\ker g \prec_{\delta} M$ and $\frac{M}{S}$ is singular, so M = S. Hence $\ker f \prec_{\delta} M$.//

Theorem: If $M = M_1 + M_2$ then the following are equivalent:

- i) M_2 is a small- M_1 -projective.
- ii) For any sub module N of M such that M_1 is a δ -supplement of N in M. There exists a sub N_1 of N such that $M = M_1 \oplus N_1$.

Proof: [14].

Proposition: If U is a sub module of R-module M, then following are equivalent:

- i) $\frac{M}{M_1}$ has a M-projective δ -cover,
- ii) If $M_2 \leq M$ and $M = M_1 + M_2$, M_2 has a δ -supplemented $M'_1 \subseteq M_2$ such that M'_1 has a M-projective δ -cover.
- iii) M_2 has a δ -supplemented M'_1 , which has a M-projective δ -cover.

Proof: (i) \Rightarrow (ii) Assume that $\frac{M}{M_1}$ has a M-projective δ -cover. Therefore $f: U \to \frac{M}{M_1}$ be a M-

projective δ -cover. Since $M = M_1 + M_2, g: M_2 \to \frac{M}{M_1}$ is an epimorphism .Given that U is M-

projective module, then there exists an homomorphism $h: U \to M_2$ such that $f = g \circ h$. By lemmaQ] $M = M_1 + \operatorname{Im} h = M_1 + h(U)$, where $h(U) \prec_{\prec_{\delta}} M_2$. Since ker $f \prec_{\prec_{\delta}} U$, we have $M_1 \bigcap h(U) = h(\ker f) \prec_{\prec_{\delta}} h(U)$ and h(U) is δ -supplement of M_1 in M. Since ker $h \leq \ker f \prec_{\prec_{\delta}} U$, $h: U \to h(U)$ is M-projective δ -cover.

(ii) \Rightarrow (iii) it is clear.

(iii) \Rightarrow (i) Let $f: U \to M_1$ 'be a M-projective δ -cover. Since M_1 ' is a δ -supplement of M, the natural epimorphism $g: M_1' \to \frac{M_1'}{M_1 \cap M_1'} \cong \frac{M_1 + M_1'}{M_1} = \frac{M}{M_1}$ is M-projective δ -cover. Hence

 $f: U \to \frac{M}{M_1}$ is a M-projective δ -cover, by lemma [A], where $h: \frac{M_1'}{M_1 \cap M_1'} \to \frac{M_1 + M_1'}{M_1}$ is an isomorphism. //

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