Common Fixed Point Theorem for \( \psi \)-weakly commuting maps in L-Fuzzy Metric Spaces for integral type

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Abstract
In this paper, we proved a common fixed point theorem \( \psi \)-weakly commuting maps in L-Fuzzy Metric Spaces for integral type inequality.

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1. Introduction
In 1922, Let \((X, d)\) be a complete metric space, \(c \in (0, 1)\) and \(f: X \rightarrow X\) be a mapping such that for each \(x, y \in X\),
\[
d(f(x), f(y)) \leq c \, d(x, y)
\]
Then \(f\) has a unique fixed point \(a \in X\), such that for each \(x \in X\), \(\lim_{n \to \infty} f^nx = a\) by S. Banach [23]. As a generalization of fuzzy sets introduced by L.A.Zadeh [14], K. Atanassov [13] introduced the idea of intuitionistic fuzzy set. Fixed point and common fixed point properties for mappings defined on fuzzy metric spaces by [5], [6], [8], [15], [16]. Intuitionistic fuzzy metric spaces by [7], [21]. A. George and P. Veeramani [2] modified the concept of fuzzy metric space introduced by I. Kramosil and J. Michalek [10] and defined a Hausdorff topology on this fuzzy metric space by [12]. Most of the properties which provide the existence of fixed points and common fixed points are of linear contractive type conditions. L-fuzzy metric spaces have been studied by many authors [11], [24]. H. Adibi et al. [9] introduced the concept of compatible mappings and proved common fixed point theorems for four mappings satisfying some conditions in L-fuzzy metric spaces. In the sequel, we shall adopt the usual terminology, notation and conventions of fuzzy metric spaces introduced by R. Saadati et al. [19] which are a generalization of fuzzy metric spaces and intuitionistic fuzzy metric spaces [20]. R. Saadati, S.Sedghi and H. Zhou [22] by a common fixed point theorem \( \psi \)-weakly commuting maps in L-Fuzzy Metric Spaces

2. Preliminaries
Definition 2.1 [1]: Let \((X, d)\) be a complete metric space, \(c \in (0, 1)\) and \(f: X \rightarrow X\) be a mapping such that for each \(x, y \in X\),
\[
d(f(x), f(y)) \leq c \int_0^1 \varphi(t)dt
\]
where \(\varphi: [0, +\infty) \rightarrow [0, +\infty)\) is a Lebesgue integrable mapping which is summable on each compact subset of \([0, +\infty)\), non negative, and such that for each \(\varepsilon > 0\), \(\int_0^\varepsilon \varphi(t)dt\), then \(f\) has a unique fixed point \(a \in X\) such that for each \(x \in X\), \(\lim_{n \to \infty} f^nx = a\).

B.E.Rhoades [4], extending the result of Branciari by replacing the above condition by the following
\[
d(f(x), f(y)) \leq c \int_0^1 \varphi(t)dt.
\]

Definition 2.2[11] Let \(L = (L, \leq_L)\) be a complete lattice, and \(U\) a nonempty set called a universe.
An L-fuzzy set \(A\) on \(U\) is defined as a mapping \(A: U \rightarrow L\). For each \(u \in U\), \(A(u)\) represents the degree (in \(L\)) to which \(u\) satisfies \(A\).

Lemma 2.1[8]. Consider the set \(\bar{L}^e\) and the operation \(\leq_{\bar{L}^e}\) defined by:
\[L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1 \}, \quad (x_1, x_2) \triangleq (y_1, y_2) \iff x_1 = y_1 \text{ and } x_2 = y_2, \text{ for every} (x_1, x_2), (y_1, y_2) \in L^*.\] Then \((L^*, \leq L^*)\) is a complete lattice.

Classically, a triangular norm \(T\) on \([0, 1]\) is defined as an increasing, commutative, associative mapping \(T: [0, 1]^2 \to [0, 1]\) satisfying \(T(1, x) = x\), for all \(x \in [0, 1]\). These definitions can be straightforwardly extended to any lattice \(L = (L, \leq L)\). Define first \(0_L = \inf L\) and \(1_L = \sup L\).

**Definition 2.3** [19]. A triangular norm \((t\)-norm\) on \(L\) is a mapping \(T: L^2 \to L\) satisfying the following conditions:

(i) \((\forall x \in L) (T(x, 1_L) = x)\); (boundary condition)

(ii) \((\forall (x, y) \in L^2) (T(x, y) = T(y, x))\); (commutativity)

(iii) \((\forall (x, z) \in L^2) (T(x, T(y, z)) = T(T(x, y), z))\); (associativity)

(iv) \((\forall (x, x', y, y') \in L^4) (x \leq_L x' \land y \leq_L y' \Rightarrow T(x, y) \leq_L T(x', y'))\); (monotonicity)

A \(t\)-norm \(T\) on \(L\) is said to be continuous if for any \(x, y \in X\) and any sequences \([x_n]\) \& \([y_n]\) which converge to \(x\) and \(y\) we have:

\[
\lim_{n \to \infty} T(x_n, y_n) = T(x, y)
\]

**Definition 2.4** [7]. A \(t\)-norm \(T\) on \(L^*\) is called \(t\)-representable if and only if there exist a \(t\)-norm \(T\) and a \(t\)-co norm \(S\) on \([0, 1]\) such that, for all \((x_1, x_2), (y_1, y_2) \in L^*, T(x_1, y_1) = (T(x_1, y_1), S(y_1, y_2))\).

**Definition 2.5** [19]. A negation on \(L\) is any decreasing mapping \(N: L \to L\) satisfying \(N(0_L) = 1_L\) and \(N(1_L) = 0_L\). If \(N(N(x)) = x\), for all \(x \in L\), then \(N\) is called an involutive negation.

**Definition 2.6** [19]. The 3-tuple \((X, M, \mathcal{T})\) is said to be an \(L\)-fuzzy metric space if \(X\) is an arbitrary (non-empty) set, \(\mathcal{T}\) is a continuous \(t\)-norm on \(L\) and \(M\) is an \(L\)-fuzzy set on \(X^2 \times (0, \infty)\) satisfying the following conditions for every \(x, y, z \in X\) and \(t, s \in (0, \infty)\):

(a) \(M(x, y, t) >_L 0_L\);

(b) \(M(x, y, t) = 1_L\) for all \(t > 0\) if and only if \(x = y\);

(c) \(M(x, y, t) = M(y, x, t)\);

(d) \(M(x, y, t), M(y, z, s) \leq_L M(x, z, t + s)\);

(e) \(M(x, y, \cdot): (0, \infty) \to L\) is continuous.

Let \((X, M, \mathcal{T})\) be an \(L\)-fuzzy metric space. For \(t \in (0, \infty)\), we define the open ball \(B(x, r, t)\) with center \(x \in X\) and radius \(r \in L \setminus \{0, 1\}\), as \(B(x, r, t) = \{y \in X : M(x, y, t) >_L N(r)\}\). A subset \(A \subseteq X\) is called open if for each \(x \in A\), there exist \(t > 0\) and \(r \in L \setminus \{0, 1\}\) such that \(B(x, r, t) \subseteq A\). Let \(F^\mathcal{T}_A\) denote the family of all open subsets of \(X\). Then \(F^\mathcal{T}_A\) is called the topology induced by the \(L\)-fuzzy metric \(M\).

**Example 2.1** [21]. Let \((X, d)\) be a metric space. Denote \(\mathcal{T}(a, b) = (a_1 b_1, \min(a_1 + b_1 - 1))\) for all \(a = (a_1, a_2)\) and \(b = (b_1, b_2)\) in \(L^*\) and let \(M\) and \(N\) be fuzzy sets on \(X^2 \times (0, \infty)\) be defined as follows:

\[
M(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{t}{t + d(x, y)}, \frac{d(x, y)}{t + d(x, y)}\right), \text{ in which } m > 1\).
\]

Then \((X, M_{\mathcal{T}, M}, \mathcal{T})\) is an intuitionistic fuzzy metric space.

**Example 2.2** [19]. Let \(X = N\). Define \(\mathcal{T}(a, b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2 b_2)\) for all \(a = (a_1, a_2)\) and \(b = (b_1, b_2)\) in \(L^*\), and let \(M(x, y, t)\) on \(X^2 \times (0, \infty)\) be defined as follows:

\[
M(x, y, t) = \begin{cases} 
\left(\frac{x}{y} - \frac{y}{x}\right) & \text{if } x \leq y \\
\left(\frac{x}{y} - \frac{y}{x}\right) & \text{if } y \leq x
\end{cases}
\]

for all \(x, y \in X\) and \(t > 0\). Then \((X, M, \mathcal{T})\) is an \(L\)-fuzzy metric space.

Let \((X, M, \mathcal{T})\) be an \(L\)-fuzzy metric space. For \(t \in (0, \infty)\), we define the open ball \(B(x, r, t)\) with center \(x \in X\) and radius \(r \in L \setminus \{0, 1\}\), as \(B(x, r, t) = \{y \in X : M(x, y, t) >_L N(r)\}\). A subset \(A \subseteq X\) is called open if for
each x ∈ A, there exist t > 0 and r ∈ L \ \{0_L, 1_L\} such that B(x, r, t) ⊆ A. Let \( \mathcal{F}_M \) denote the family of all open subsets of X. Then \( \mathcal{F}_M \) is called the topology induced by the L-fuzzy metric M.

**Lemma 2.2 [9].** Let (X, M, \( \mathcal{F} \)) be an L-fuzzy metric space. Then M(x, y, t) is non decreasing with respect to t, for all x, y in X.

**Definition 2.7 [19].** A sequence \( \{x_n\}_{n \in \mathbb{N}} \) in an L-fuzzy metric space (X, M, \( \mathcal{F} \)) is called a Cauchy sequence, if for each \( \varepsilon \in L \setminus \{0_L\} \) and t > 0, there exists \( n_0 \in \mathbb{N} \) such that for all m ≥ n ≥ \( n_0 \), \( n \geq m \geq n_0 \).

M(x_m, x_n, t) ≥ \( \inf_{n \to \infty} \) whenever \( n \to \infty \) for every t > 0. A L-fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

**Definition 2.8 [22].** Let (X, M, \( \mathcal{F} \)) be an L-fuzzy metric space. M is said to be continuous on \( X^2 \times (0, \infty) \) if

\[
\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t)
\]

whenever a sequence \( \{(x_n, y_n, t_n)\} \) in \( X^2 \times (0, \infty) \) converges to a point \((x, y, t) \in X^2 \times (0, \infty) \) i.e.,

\[
\lim_{n \to \infty} M(x_n, y_n, t_n) = \lim_{n \to \infty} M(y_n, x_n, t_n) = 1_L \quad \text{and} \quad \lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t).
\]

**Lemma 2.3 [22].** Let (X, M, \( \mathcal{F} \)) be an L-fuzzy metric space. Then M is continuous function on \( X^2 \times (0, \infty) \).

**Definition 2.9 [22].** Let A and B be maps from an L-fuzzy metric space (X, M, \( \mathcal{F} \)) into itself. The maps are said to be weakly commuting if

\[
M(ABx, BAx, t) \geq_l M(Ax, Bx, t)
\]

for each x in X & t > 0.

**Definition 2.10 [22].** Let A and B be maps from an L-fuzzy metric space (X, M, \( \mathcal{F} \)) into itself. The maps A and B are said to be \( \psi \)-weakly commuting if there exists a positive real function \( \psi: (0, \infty) \to (0, \infty) \) such that

\[
M(ABx, BAx, t) \geq_l M(Ax, Bx, \psi(t))
\]

for each x in X and t > 0.

**Example 2.3 [22].** Let X = R. Let \( \mathcal{F}(a, b) = (a_1, b_1, \min(a_1 + b_2, 1)) \) for all \( a = (a_1, a_2) \) and \( b = (b_1, b_2) \) in \( L^+ \) and let M and N be fuzzy sets on \( X^2 \times (0, \infty) \) be defined as follows:

\[
M_{MN}(x, y, t) = \left( \frac{|x-y|}{t} - 1 \right) \left( \frac{e|

\[
= \left( \frac{2|x-y|^2}{t^2} \right) \left( \frac{e|x-y|^2}{t} - 1 \right) = M_{MN}(Ax, Bx, t/2)
\]

Define \( A(x) = 2x - 1, B(x) = x^2 \), then

\[
M_{MN}(x, y, t) = \left( \frac{|x-y|}{t} - 1 \right) \left( \frac{e|x-y|^2}{t} - 1 \right) = M_{MN}(Ax, Bx, t/2)
\]

for all \( t > 0 \). Then (X, M_{MN}, \( \mathcal{F} \)) is an intuitionistic fuzzy metric space.
\[ \psi(t) = \frac{t}{2}, \quad A \text{ and } B \text{ are } \psi \text{ weakly commuting. But } A \text{ and } B \text{ are not weakly commuting since the exponential function is strictly increasing.} \]

### 3. Main Results

**Theorem 3.1.** Let \((X, M, \tau)\) be a left L-fuzzy metric space and let \(A\) and \(B\) be \(\psi\) weakly commuting self-mappings of \(X\) satisfying the following conditions:

- (3.1.1) \(A(X) \subseteq B(X)\);
- (3.1.2) either \(A\) or \(B\) is continuous;
- (3.1.3) \(\int_0^1 M(Ax, Ay, t) \, \xi(t) \, dt \geq \int_0^1 \left\{ C(M(Bx, By, t), M(Bx, Ax, t), M(Ax, By, t)) \right\} \, \xi(t) \, dt\)

where \(C: L \to L\) is a continuous function such that \(C(\alpha) > \alpha\) for each \(\alpha \in L \setminus \{0_L, 1_L\}\), for every \(x, y\) in \(X\). Then \(A\) and \(B\) have a unique common fixed point in \(X\).

**Proof.** Let \(x_0 \in X\) be an arbitrary point in \(X\). By (3.1.1), there exists \(x_1 \in X\) such that \(Ax_0 = Bx_1\). In general choose \(x_{n+1}\) such that \(Ax_n = Bx_{n+1}\). Then for \(t > 0\),

\[ \int_0^1 M(Ax_n, Ax_{n+1}, t) \, \xi(t) \, dt \geq \int_0^1 C(M(Bx_n, Bx_{n+1}, t), M(Bx_n, Ax_n, t), M(Ax_n, By, t)) \, \xi(t) \, dt \]

\[ \geq \int_0^1 C(M(Ax_{n-1}, Ax_n, t), M(Ax_{n-1}, Ax_n, t), M(Ax_n, Ax_n, t)) \, \xi(t) \, dt \]

\[ = \int_0^1 M(Ax_{n-1}, Ax_n, t) \, \xi(t) \, dt \]

Thus, \(\{M(Ax_n, Ax_{n+1}, t) ; n \geq 0\}\) is an increasing sequence in \(L\) and therefore, tends to a limit \(\alpha \leq 1_L\). We claim that \(\alpha = 1_L\) if \(\alpha < 1_L\), when \(n \to \infty\) in the above inequality we get \(\alpha \geq 1_L\) contradiction. Hence \(\alpha = 1_L\), i.e.

\[ \lim_{n \to \infty} M(Ax_n, Ax_{n+1}, t) = 1_L. \]

If we define (2.9) \(c_n(t) = M(Ax_n, Ax_{n+1}, t)\) then \(\lim_{n \to \infty} c_n(t) = 1_L\). Now, we prove that \(\{Ax_n\}\) is a Cauchy sequence in \(A(X)\). Suppose that \(\{Ax_n\}\) is not a Cauchy sequence in \(A(X)\). For convenience, let \(y_n = Ax_n\) for \(n = 1, 2, 3, \ldots\). Then there is an \(\epsilon \in L \setminus \{0_L, 1_L\}\) such that for each integer \(k\), there exists integers \(m(k)\) and \(n(k)\) with \(m(k) > n(k) \geq k\) such that

\[ d_k(t) = M(y_{n(k)}, y_{m(k)}, t) \leq N(\epsilon) \text{ for } k = 1, 2, 3, \ldots \]

We may assume that

**Example 2.3**

by choosing \(m(k)\) to be the smallest number exceeding \(n(k)\) for which (2.10) holds. Using (2.9), we have

\[ N(\epsilon) \geq d_k(t) \geq \frac{T(M(y_{n(k)}, y_{m(k)-1}, t/2), M(y_{m(k)}, y_{m(k)-1}, t/2))}{T(c_k(t/2), N(\epsilon))} \]

\[ \geq T \left( c_k(t/2), N(\epsilon) \right) \]
Hence, \( d_k(i) \rightarrow N(\varepsilon) \) for every \( t > 0 \) as \( k \rightarrow \infty \).

We know that
\[
d_k(t) = M(y_{n(k)}, y_{m(k)}, t)
\geq T^2 \left[ M(y_{n(k)}, y_{m(k)+1}, t/3), M(y_{n(k)+1}, y_{m(k)+1}, t/3) \right]
\geq T^2 \left[ \left( c_k(t/3), C(M(y_{n(k)}, y_{m(k)+1}, t/3)) \right), c_k(t/3) \right]
= T^2 \left[ \left( c_k(t/3), C(d_k(t/3)) \right), c_k(t/3) \right]
\]

Thus, as \( k \rightarrow \infty \) in the above inequality we have \( N(\varepsilon) \geq C(N(\varepsilon)) > N(\varepsilon) \) which is a contradiction.

Thus, \( \{Ax_n\}_n \) is a Cauchy and by the completeness of \( X \), \( \{Ax_n\}_n \) converges to \( z \) in \( X \). Also \( \{Bx_n\}_n \) converges to \( z \) in \( X \). Let us suppose that the mapping \( A \) is continuous. Then \( \lim_{n \to \infty} AAx_n = Az \) and

\[
\lim_{n \to \infty} ABx_n = Az.
\]

Further we have since \( A \) and \( B \) be \( \psi \)-weakly commuting

\[
M(ABx, BAx, t) \geq_L M(Ax, Bx, \psi(t))
\]

On letting \( n \to \infty \) in the above inequality we get \( \lim_{n \to \infty} Bx_n = z \), by lemma (2.3). We now prove that

\[
z = Az. Suppose \ z \neq Az \ then \ M(z, Az, t) \leq 1. By (3.1.3)
\]

\[
\int_0^T \xi(t) dt \geq L \int_0^T \xi(t) dt
\]

Letting \( n \to \infty \) in the above inequality we get

\[
\int_0^T \xi(t) dt \geq L \int_0^T \xi(t) dt > L \int_0^T \xi(t) dt
\]

a contradiction. Therefore, \( z = Az \). Since \( A(X) \subseteq B(X) \) we can find \( z_1 \) in \( X \) such that \( z = Az = Bz_1 \).

Now,

\[
\int_0^T \xi(t) dt \geq L \int_0^T \xi(t) dt
\]

Letting \( n \to \infty \) in the above inequality we get

\[
\int_0^T \xi(t) dt \geq L \int_0^T \xi(t) dt \geq L \int_0^T \xi(t) dt
\]

Since \( C(1_L) = 1_L \), this implies that \( Az = Az_1 \), i.e. \( z = Az = Az_1 = Bz_1 \). Also for any \( t > 0 \),

\[
M(Az, Bx, t) = M(ABz_1, BAx_1, t) \geq_L M(Az_1, Bz_1, \psi(t)) = 1_L
\]

which again implies that \( Az = Bz \). Thus \( z \) is a common fixed point of \( A \) and \( B \). Now, to prove uniqueness suppose \( z \neq \hat{z} \) is another common fixed point of \( A \) and \( B \). Then there exists \( t > 0 \) such that \( M(z, \hat{z}, t) \leq 1 \) and

\[
\int_0^T \xi(t) dt = \int_0^T \xi(t) dt
\]

Which is contradiction. Therefore, \( z = \hat{z} \) i.e. \( z \) is a unique common fixed point \( A \) and \( B \).
Example 2.4[22]. Consider Example 2.1 in which $X = [0,1]$.

Define $A(x) = 1$ and $B(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ on $X$. It is evident that $A(X) \subseteq B(X)$, $A$ is continuous

and $B$ is discontinuous. Define $C : L^p \to L^p$ by $C(a) = \left(\frac{a_1}{a_2}, \frac{a_2}{a_1}\right)$, then $C(a) = (\sqrt[3]{a_1}, a_2^2)$ for $0 < a_1 < 1, i = 1,2$ and $M(Ax, Ay, t) \geq L^p C(M(Bx, By, t))$ for all $x, y \in X$, $A$ and $B$ be $\psi$ weakly commuting. Thus all the conditions of last theorem are satisfied and 1 is a common fixed point of $A$ and $B$.

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References:
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