Characterization of $\delta$-small submodule

R. S. Wadbude, M.R.Aloney & Shubhanka Tiwari

1 Mahatma Fule Arts, Commerce and Sitaramji Chaudhari Science Mahavidyalaya, Warud.
SGB Amaravati University Amravati  [M.S.]
2 Technocrats Institute of Technology (excellence) Barkttulla University Bhopal. [M.P.]

Abstract
Pseudo Projectivity and M- Pseudo Projectivity is a generalization of Projveativity.
[2], [8] studied M-Pseudo Projective module and small M-Pseudo Projective module. In this paper we consider some generalization of small M-Pseudo Projective module, that is $\delta$-small M-Pseudo Projective module with the help of $\delta$-small and $\delta$- cover.

Key words: Singular module, S.F. small M-Pseudo Projective module, $\delta$-small and projective $\delta$- cover.

Introduction:
Throughout this paper R is an associative ring with unity module and all modules are unitary left R-modules. A sub module K of a module M. K ≤ M. Let M be a module, K ≤ M is said to be small in M if for every L ≤ M, the equality K + L = M implies L = M, ( denoted by $K \ll M$ ). The concept of $\delta$-small sub modules was introduced by Zhon [10]. A sub module K of M is said to be $\delta$-small sub module of M (denoted by $K \ll_\delta M$ ) if whenever M = K+ L, with M/K is singular, then M = L. The sum of all $\delta$-small sub module of M is denoted by $\delta(M)$. $\delta(M)$ is the reject in M of class of all singular simple modules.

Throughout this paper R is an associative ring with unity module and all modules are unitary left R-modules. A sub module K of a module M. K ≤ M. Let M be a module, K ≤ M is said to be small in M if for every L ≤ M, the equality K + L = M implies L = M, ( denoted by $K \ll M$ ). The concept of $\delta$-small sub modules was introduced by Zhon [10]. A sub module K of M is said to be $\delta$-small sub module of M (denoted by $K \ll_\delta M$ ) if whenever M = K+ L, with M/K is singular, then M = L. The sum of all $\delta$-small sub module of M is denoted by $\delta(M)$. $\delta(M)$ is the reject in M of class of all singular simple modules.

1. $\delta$-Small
Definition: 1.1. The sub module $Z(M) = \{ x \in M : r_k(x) is essential in R \}$ is called singular sub module of M. The module M is Called a singular module if $Z(M) = M$. The module M is Called a non- singular module if $Z(M) = 0$.
Definition: 1.2. An R-module N is called small M- pseudo projective module if for every sub module A of M, any epimorphism $f : N \rightarrow \frac{M}{A}$ with $Ker f \ll N$, Can be lifted to a homomorphism $h : N \rightarrow M$

\[ \begin{array}{ccc}
N & \xrightarrow{f} & M/A \\
\downarrow h & & \downarrow 0 \\
M & \xrightarrow{g} & 0
\end{array} \]

i.e. $g \cdot h = f$.
**Definition:** 1.3 An R-module $N$ is called δ-small M- pseudo projective module if for every submodule $A$ of $M$, any small epimorphism $f : N \rightarrow \frac{M}{A}$ with $\text{Ker} f \ll \delta N$, Can be lifted to a homomorphism $h : N \rightarrow M$

\[ h \downarrow f \text{ epic with } \text{Ker} f \ll \delta N \]

\[ M \xrightarrow{g_{small epic}} \frac{M}{A} \rightarrow 0 \]

i.e. $g \cdot h = f$.

**Examples:**

i) Every small sub module of $M$ is δ-small in $M$.
ii) Every non singular semisimple sub module of $M$ is δ-small in $M$.
iii) Every simple module $M$ is hollow.
iv) $Z_\alpha$ as Z-module is not δ-small.

**Example:** 2. Consider the Z-modules $Z_\alpha$ and $Z_\beta$. An epimorphism $f : Z_\alpha \rightarrow Z_\beta$ define by

\[ f(1) = f(\bar{3}) = \bar{1} \text{ and } f(0) = f(\bar{2}) = \bar{0} \]  
Then $f$ is small epimorphism.

Every small sub module of $M$ is δ-small in $M$.

**Example:** 3. Let $R = M = Z_\alpha$. Then two non-trivial sub modules of $M$,

$M_1 = \{0, \bar{3}\}$ and $M_2 = \{0, \bar{2}, \bar{4}\}$ are δ-small in M, but neither $M_1$ and $M_2$ is small in M.

Moreover $M \ll M$

**Proposition:** 1.1 Let $M, L$ and $N$ be R-Modules. If $\alpha : M \rightarrow N$ and $\beta : N \rightarrow L$ are two epimorphisms. Then $\beta \circ \alpha$ is small if and only if both $\alpha, \beta$ are small.

**Proof:** [8]

**Lemma:** 1.1. Let $N$ be a submodule of $M$. The following are equivalent:

i) $N \ll \delta M$
ii) If $X + N = M$, then $M = X \oplus Y$ for a projective semi simple sub module $Y$ with $Y \subseteq N$.
iii) If $X + N = M$, with $M/X$ Goldie torsion, then $X = M$.

**Proof:** [10].

**Lemma:** 1.2. Let $M$ be a module, then

i) For submodule $N, K, L$ with $K \leq N$, We have
   a) $N \ll \delta M$ if and only if $K \ll \delta M$ and $N/K \ll \delta M/K$
   b) $N + L \ll \delta M$ if and only if $N \ll \delta M$ and $L \ll \delta M$.
ii) If $K \ll \delta M$ and $f : M \rightarrow N$ is an homomorphism, then $f(K) \ll \delta N$.

In particular, if $K \ll \delta M \leq N$ and $K \ll \delta N$.

iii) Let $K_1 \leq M_1$, $K_2 \leq M_2 \leq M$ and $M = M_1 \oplus M_2$,

then $K_1 \oplus K_2 \ll \delta M_1 \oplus M_1$ if and only if $K_1 \ll \delta M_1$ and $K_2 \ll \delta M_2$.

**Proof:** [4].

**Lemma:** 1.3. Let $M$ be a module. Then

i) $\delta(M) = \sum \{L \leq M : L \ll \delta M\} = \cap \{K \leq M : M/K \text{ is singular module}\}$.
ii) If $f : M \rightarrow N$ is an R-homomorphism, then $f(\delta(M)) \leq \delta(N)$. Therefore $\delta(M)$ is fully invariant sub module of $M$. In particular if $K \leq M$, then $\delta(K) \leq \delta(M)$.

iii) If $M = \bigoplus_{\ell \in I} M_\ell$, then $\delta(M) \leq \bigoplus_{\ell \in I} \delta(M_\ell)$.
iv) If every proper sub module of $M$ is contained in a maximal sub module of $m$, then $\delta(M)$ is the unique largest δ-small sub module of $M$. In particular if $M$ is finitely generated, then $\delta(M)$ is δ- small in $M$.

**Proof:** [4]

**Lemma:** 1.4. If $K \leq N \leq M$, $K \ll \delta M$ and $N$ is a direct summand of $M$, Then $K \ll \delta N$.

**Proof:** [10]

**Proposition:** Given a module $M$, each of the following sets is equal $\delta(M)$.

i) $\delta(M) = \sum \{A : A \ll \delta M\}$.
ii) $\delta(M) = \cap \{B : B \leq M \text{ with } M/B \text{ is singular}\}$.
iii) $\delta(M) = \cap \{\text{ker} \Phi : \Phi \in \text{Hom}(M, N) \text{ such that } N \text{ is singular simple}\}$
iv) $\delta(M) = \cap \{\ker \Phi : \Phi \in \text{Hom}(M,N) \text{ such that } N \text{ is singular semi singular}\}$

**Proposition 1.2.** If $f : M \rightarrow N$ is an epimorphism with $\ker f \leq \delta(M)$, then $\delta(N) = f(\delta(M))$.

**Proof:**

**Lemma 1.5.** Let $P$ be a small projective module, then $\delta(M) \ll_{\delta} P$.

**Proof:**

Let $P$ be a small-projective module and $P = \delta(P) + Y$, where $P/Y$ is singular, by hypothesis $P = A \oplus B$ such that $A \leq Y$ and $B \cap Y \leq \delta(P)$, Then $P = A \oplus (B \cap Y)$ and so $P = \delta(P) \oplus A$. Since $A$ is summand of $P$, there exists a sub module $X \leq \delta(P)$ such that $P = X \oplus A$. Since $\delta(X) = X \cap \delta(P) = X$,

$X$ is semi projective and $P/Y$ is epimorphic image of $P/A \cong X =>$ $P/Y$ is projective and singular, we have $P = Y$. Hence $\delta(P) \ll_{\delta} P$.

**Proposition 1.3.** Let $M$ and $N$ be any $R$-modules. Then Following are equivalent:

i) $N \ll_{\delta} M$.

ii) If $X \oplus N = M$, then $M = X \oplus Y$ for projective semi simple sub module $Y$, with $Y \leq N$.

**Proof:**

**Proposition 1.4.** For Hollow module $N$ the following conditions are equivalent:

i) $N$ is $\delta$-small $M$-pseudo projective module.

ii) $N$ is $M$-pseudo projective module.

**Proof:**

(i) $\Rightarrow$ (ii) obvious.

(ii) $\Rightarrow$ (i) Let R-module $N$ be a $M$- pseudo projective module. $A \leq M$, any small epimorphism $f : N \rightarrow \frac{M}{A}$ and natural epimorphism $\pi : M \rightarrow \frac{M}{A}$. For a sub module $A$ of $M$, $A \ll_{\delta} M$, then $A$ is direct summand of $M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N + F$ and $B \cap N \cap F \ll_{\delta} M$, there exists a homomorphism $h : N \rightarrow M$ such that the diagram

\[
\begin{array}{c}
N \\
\downarrow f \\
M \xrightarrow{\pi} \frac{M}{A} \\
\downarrow \\
0
\end{array}
\]

i.e. $gh = f$. Hence $N$ is $M$-pseudo projective module.//

**Proposition 1.5.** Let $M$ be a $\delta$-small $M$-Pseudo projective module, then following conditions are equivalent:

i) $M$ is S.F. If $M/A$ is isomorphic to direct summand of $M$, $A \leq M$.

ii) $M$ is direct sum of sub modules of $A, B$ with $A \leq A \cap F$ and $B \cap N \cap F \ll_{\delta} M$.

**Proof:**

(i) $\Rightarrow$ (ii) by S1

(ii) $\Rightarrow$ (i) Since $M = A \oplus B$, i.e. $M = A + B$ and $A \cap B = 0$. Now $M = A + B$, for some $B \leq M$ and $\frac{M}{B}$ is singular, then $S = A + (S \cap B)$. Suppose that $A \cap B \neq A$, then $\frac{\delta(M)}{A} = \frac{S}{A}$ is f.g. this is contradiction. Thus $A = A \cap B \leq B$. We have $M = A + B = B$. So $A \ll_{\delta} M$../

2. $\delta$-Cover

**Definition 2.1.** Let $P$ and $M$ be modules. $\delta$-small f: $P \rightarrow M$ is called a $\delta$-cover of $M$ in case kerf $\ll_{\delta} P$.

**Definition 2.2.** A $\delta$-cover f: $P \rightarrow M$ is called a projective $\delta$-cover in case P is a projective module.
Some module may not have projective \( \delta \)-cover and some module have projective \( \delta \)-cover but not projective cover.

**Definition:** 2.3. A module \( M \) is called a semi perfect module if any homomorphic image of \( M \) has a projective \( \delta \)-cover.

**Lemma:** 2.1. If \( f : P \to M \) and \( g : M \to N \) are \( \delta \)-covers then \( g \circ f : P \to N \) is a \( \delta \)-cover.

**Lemma:** 2.2. If \( f_1 : P_1 \to M_1 \) is a \( \delta \)-cover for \( i = 1, 2, 3, \ldots, n \), then \( \bigoplus_{i=1}^{n} f_i : \bigoplus_{i=1}^{n} P_i \to M_i \) is a \( \delta \)-cover.

**Lemma:** 2.3. If \( N \) is a direct summand of \( M \) and \( A \ll_{\delta} M \), then \( A \cap N \ll_{\delta} N \).

**Lemma:** 2.4. Let \( K \) be a sub module of a projective module \( M \). if \( M/K \) has a \( \delta \)-cover, then, it has a \( \delta \)-cover of the form \( f : M \to M/K \), with \( \ker f = K/L \), where \( L \leq K \).

**Proof:** Let small epimorphism \( f : P \to M \) be a \( \delta \)-cover of \( M/K \) and \( \pi : M \to M/K \) is a natural epimorphism. Since \( M \) is projective module, there exists \( h : M \to P \) such that the following diagram commute.

\[
\begin{array}{ccc}
M & \xrightarrow{\pi} & M/K \\
\downarrow h & & \\
P & \xrightarrow{f} & M & \to 0
\end{array}
\]

i.e. \( f \cdot h = \pi \). Then \( P = \ker f + \text{Im} h \) by lemma 1. \( P = Y + \text{Im} h \) for a semi simple \( Y \) with \( Y \subseteq \ker f \) also by lemma 2. \( \ker(f \mid_\text{Im}h) \ll_{\delta} \text{Im} h \). So \( f \mid_\text{Im}h \) is also \( \delta \)-cover of \( M/K \). But \( \frac{M}{\ker h} \cong \text{Im} h \) (by Isomorphic the.) and \( f \cdot h = \pi, \; \ker h \subseteq K \). If we consider the isomorphism \( h' : \frac{M}{\ker h} \to \text{Im} h \), then we obtain \( \ker(f \mid_\text{Im}h) \ll_{\delta} M/\text{Im} h \) by lemma 2. //

**References**


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