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Boolean Artex Spaces Over Bi-monoids

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ABSTRACT

We define Complimented Artex Space over a Bi-monoid. We define Boolean Artex Space over a Bi-monoid. We give an example of a Boolean Artex space over a bi-monoid. We prove that the homomorphic image of a Complimented Artex Space over a Bi-monoid is a Complimented Artex Space over the Bi-monoid. We also prove that the homomorphic image of a Boolean Artex Space over a bi-monoid M is a Boolean Artex Space over the bi-monoid M. We also prove that the Cartesian product of Complimented Artex Spaces over a Bi-monoid is Complimented Artex Space over the Bi-monoid. Finally we prove the Cartesian product of Boolean Artex Spaces over a bi-monoid M is a Boolean Artex Space over the bi-monoid M.

Keywords : Complimented, Distributive Artex Spaces, Homomorphisms

1 INTRODUCTION

Boolean Algebra was introduced by George Boole in 1854. A more general algebraic system is the lattice. A Boolean Algebra is then introduced as a special lattice. Lattices and Boolean algebra have important applications in the theory and design of computers.. This motivated us to think a lattice in another angle. So, we introduced a new space called an Artex space over a bi-monoid. While we introduce an Artex space over a bi-monoid, our main aim in mind was to introduce Boolean Artex Spaces over Bi-monoids. Here we have. We introduce Boolean Artex Space over a bi-monoid. Like Lattices and Boolean algebra, our Artex Spaces and Boolean Artex Spaces over bi-monoids will have important applications in the theory and design of computers. There are many other areas such as Engineering and Science to which Boolean algebra is applied. Like that our Boolean Artex Spaces over bi-monoids will play a good role in many fields especially in Engineering, Science and Computer fields. Also we hope that this theory of Artex Spaces and Boolean Spaces over bi-monoids will play an important role and may lead the theory and design of computers. In Discrete Mathematics this theory will create a new dimension. We, of course, feel that the theory of Artex spaces and Boolean Spaces over bi-monoids shall lead to many theories. Now the theory of Artex Spaces and Boolean Spaces over bi-monoids has formed a new chapter.

2 Preliminaries

2.1.1 Definition : Bi-monoid : An algebraic system (M , + , .) is called a Bi-monoid if

1.(M , +) is a monoid 2. (M , .) is a monoid

and 3 (i) a.(b+c) = a.b + a.c and (ii) (a+b).c = a.c + b.c , for all a,b,c ϵ M.

2.1.2 Definition : Artex Space Over a Bi-monoid : A non-empty set A is said to be an Artex Space Over a Bi-monoid (M , + , .) if 1.(A, ^ , v) is a lattice and
2.for each mϵM , m$\neq$0, and aϵA, there exists an element ma ϵ A satisfying the following conditions :
(i) m(a ^ b) = ma ^ mb (ii) m(a v b) = ma v mb

(iii) ma ^ na $\leq$ (m +n)a and ma v na $\leq$ (m + n)a

(iv) (mn)a = m(na), for all m,nϵM, m$\neq$0, n$\neq$0, and a,bϵA (v) 1.a = a, for all aϵA

Proposition 2.1.3 : If A and B are any two Artex spaces over a bi-monoid M, then A×B is an Artex Space over M.

Corollary 2.1.4 : If A_1, A_2 ,A_3,….., A_n are Artex spaces over a bi-monoid M, then $A_1\times A_2\times A_3\times \ldots\times A_n$ is also an Artex space over M.

2.1.5 Definition : SubArtex Space : Let (A, Λ ,V) be an Artex space over a bi-monoid (M , + , .) and let S be a nonempty subset of A. Then S is said to be a SubArtex space of A if (S, Λ ,V) itself is an Artex space over M.

Proposition 2.1.6 : Let (A, Λ ,V) be an Artex space over a bi-monoid (M , + , .). Then a nonempty subset S of A is a subartex space of A if and only if for each m,n ϵ M, m$\neq$0, n$\neq$0, and a, b ϵS,ma Λ nb ϵ S and ma V nb ϵ S

2.1.7 Lower Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Lower Bounded Artex Space over M if as a lattice, A has the least element 0.

2.1.8 Upper Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be an Upper Bounded Artex Space over M if as a lattice, A has the greatest element 1.

2.1.9 Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Bounded Artex Space over M if A is both a Lower bounded Artex Space over M and an Upper bounded Artex Space over M.

2.1.10 Artex Space Homomorphism : Let A and B be two Artex spaces over a bi-monoid M, where Λ_1 and V_1 are the cap, cup of A and Λ_2 and V_2 are the cap, cup of B. A mapping f : A → B is said to be an Artex Space homomorphism iff for all m ϵ M, m $\neq$ 0 and a,b ϵ A (1) f (a Λ_1b) = f(a) Λ_2f(b) (2) f(a V_1b) = f(a) V_2f(b)
(3) f(ma)) = mf(a).

2.1.11 Artex Space Epimorphism : Let A and B be two Artex spaces over a bi-monoid M. An Artex space homomorphism f : A → B is said to be an Artex Space epimorphism if the mapping f : A → B is onto.

2.1.12 Artex Space Monomorphism : Let A and B be two Artex Spaces over a bi-monoid M. An Artex space homomorphism f : A → B is said to be an Artex Space monomorphism if the mapping f : A → B is one-one.

2.1.13 Artex Space Isomorphism : Let A and B be two Artex spaces over a bi-monoid M. An Artex Space homomorphism f : A → B is said to be an Artex Space Isomorphism if the mapping f : A → B is both one-one and onto, ie, f is bijective.

2.1.14 Isomorphic Artex Spaces : Two Artex spaces A and B over a bi-monoid M are said to be isomorphic if there exists an isomorphism from A onto B or from B onto A.

Proposition 2.1.15 : Let A be a Bounded Artex space over a bi-monoid M and let B be an Artex space over M. Let f : A → B be an epimorphism of A onto B. Then B is a Bounded Artex space over M.

Proposition 2.1.16 : Let A and B be Lower Bounded Artex spaces over a bi-monoid M. If f : A → B is an epimorphism of A onto B, then f(0) = 0',where 0 and 0' are the least elements of A and B respectively.

Proposition 2.1.17 : Let A and B be Upper Bounded Artex spaces over a bi-monoid M.If f : A → B is an epimorphism of A onto B, then f(1) = 1',where 1 and 1' are the greatest elements of A and B respectively.

Proposition 2.1.18 : If B and B' are any two Bounded Artex spaces over a bi-monoid M, then B×B' is also a Bounded Artex Space over M.

Corollary 2.1.19 : If B_1, B_2 ,B_3,……, B_n are Bounded Artex spaces over a bi-monoid M, then $B_1 \times B_2 \times B_3 \times \ldots \times B_n$ is also a Bounded Artex space over M.

2.1.20 Distributive Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Distributive Artex Space over the bi-monoid M if as a lattice, A is a distributive lattice.
In other words, an Artex space A over a bi-monoid M is said to be a Distributive Artex Space over the bi-monoid M if for any a,b,c ϵ A, (i) a ^ (b v c) = (a ^ b) v (a ^ c) (ii) a v (b ^ c) = (a v b) ^ (a v c)

Proposition 2.1.21 : If D and D' are any two Distributive Artex spaces over a bi-monoid M, then D×D' is also a Distributive Artex Space over M.

Corollary 2.1.22 : If D_1, D_2 ,D_3,……, D_n are Distributive Artex spaces over a bi-monoid M, then $D_1 \times D_2 \times D_3 \times \ldots \times D_n$ is also a Distributive Artex space over M.

Proposition 2.1.23 : Let A be a Distributive Artex space over a bi-monoid M and let B be an Artex space over M. Let f : A → B be an epimorphism of A onto B. Then B is a Distributive Artex Space over M.

Proposition 2.1.24 : Let A be an Artex space over a bi-monoid M and let B be an Artex space over M. Let f : A→B be a homomorphism. Let S be a subArtex space of A. Then f(S) is a SubArtex Space of B.

Proposition 2.1.25 : Let A be a Distributive Artex space over a bi-monoid M and let B be an Artex space over M. Let f :A → B be a homomorphism. Let S be a subArtex space of A. Then f(S) is a Distributive SubArtex Space of B.

3 Boolean Artex Spaces Over Bi-monoids

3.1.1 Definition : Complemented Artex Space over a bi-monoid: A Bounded Artex Space A over a bi-monoid M is said to be a Complemented Artex Space over M if (i) 0.a = 0, for all a ϵ A (ii) m0 = 0, for all m ϵ M and (iii) for every a ϵ A, there exists at least one a' ϵ A such that a v a' =1, and a ^ a'=0.

3.1.2 Note : While the least and the greatest elements of the Complemented Artex Space is denoted by 0 and 1, the identity elements of the bi-monoid (M , + , .) with respect to addition and multiplication are, if no confusion arises, also denoted by 0 and 1 respectively.

3.1.3 Definition : Boolean Artex Space Over a Bi-monoid : A Complemented Distributive Artex Space A over a bi-monoid M is said to be a Boolean Artex Space over the bi-monoid M.

3.1.4 Example : Let V be the standard real inner product space over the field R of real numbers. Let $ be the set of all subspaces of V.

Define the cap ^ and the cup v operations on $ as follows :
For A , B ϵ $, define A ^ B = A ∩ B and A v B = A + B, where ∩ is the intersection of sets and + is the direct sum of subspaces.

Also define the partial order relation ≤ on $ by A ≤ B if and only if A $\subseteq$ B ie A is a subet of B

Claim : ($, ^ ,v)=($,∩, +) is a Boolean Artex Space over the bi-monoid R', where R'= $R^+ \cup \{0\}$

Here R^+ is the set of all positive real numbers.

Subclaim 1: ($, ^ ,v)=($,∩, +) is a Lattice.

(1) Let A, B ϵ $
Clearly A∩B is also a subspace of V and hence A∩B ϵ $
Therefore, ∩ is a binary operation on $
(2) Let A, B ϵ $
Clearly A+B is also a subspace of V and hence A+B ϵ $
Therefore, + is a binary operation on $
(3) Let A, B ϵ $
A∩B = B∩A
Therefore, ∩ is commutative
(4) For any A, B ϵ $, clearly A + B = B+A
Therefore, ∩ is commutative
(5) For any A, B,C ϵ $, clearly A∩(B∩C) = (A∩B)∩C
Therefore, ∩ is associative
(6) For any A, B,C ϵ $, clearly A+(B+C) = (A+B)+C
Therefore, + is associative
(7) Let A, Bϵ $,
Since A is a subset of A+B, A∩(A+B) = A
Since A∩B is a subset A, A+(A∩B) = A
Therefore, the Absorption Laws are satisfied.
Hence, ($,∩, +) is a Lattice.

Subclaim 2: ($, ^ ,v)=($,∩, +) is an Artex Space over the bi-monoid R', where R'= $R^+ \cup \{0\}$

Now, define the bi-monoid multiplication on $ by the following :

For m ϵ R' and A ϵ $, define m.A = { m.a / a ϵ A }, where **.** is the usual multiplication

(1) Since A, for each Aϵ $, is a subspace of V over the field R of real numbers, m.A is nothing but A itself which is in $

Therefore, the bi-monoid multiplication over R' is defined in $.

(2) Let m ϵ R' and A,B ϵ $

For any subspace U of V, m.U, where mϵR', is also a subspace of V which is nothind but U itself.

Since A and B are subspaces V, A∩B is also a subspace of V

Therefore, m(A∩B) is A∩B itself.

Now, m.A = A and m.B = B

Therefore, $m.A \cap m.B = A \cap B$

Therefore, $m(A \cap B) = A \cap B = m.A \cap m.B$

ie $m(A \cap B) = m.A \cap m.B$

(3) Now, for any A, B ϵ \$, A + B is a subspace of V.

m(A+B) is A+B itself.

$m.A = A$ and $m.B = B$

$m.A+m.B = A+B$

Therefore, $m(A+B) = A+B = m.A+m.B$

ie $m(A+B) = m.A+m.B$

(4) Now, for any m, n ϵ R' and for any A ϵ \$,

m+n ϵ R' and therefore (m+n)A is also a subspace and it is A itself.

Now $m.A = A$ and $n.A = A$ and $m.A \cap n.A = A \cap A = A$

$m.A \cap n.A = A = (m+n).A$ ie $m.A \cap n.A = (m+n).A$

It can be considered as $m.A \cap n.A \subseteq (m+n).A$ (since $A \le A$ ie $A \subseteq A$)

Now, $m.A + n.A = A+A = A$

Therefore, $m.A+n.A = A = (m+n).A$

ie, $m.A+n.A = (m+n).A$

It can be considered as $m.A+n.A \subseteq (m+n).A$ (since $A \le A$ ie $A \subseteq A$)

(5) Now, for any m, n ϵ R' and for any A ϵ \$,

$(m.n)A = A$ and $n.A = A$ and $m(n.A) = A$

Therefore, $(m.n)A = m(n.A)$

(6) Let 1 be the identity element of the bi-monoid (M, + , **.**) with respect to **.**

Now, for any A ϵ \$,

$1.A = \{1.a / a \in A\}$

$= \{ a / a \in A\}$ (since V is a real inner product space over R, 1.a=a, for all a ϵ V)

$= A$

Hence, (\$,∩, +) is an Artex Space over the bi-monoid R', where $R' = R^{+} \cup \{0\}$

Subclaim 3 : ($,∩, +) is a Bounded Artex Space over R'

(1)Let O = {0}

Then O is clearly a subspace of V and therefore belongs to $

Also for any A ϵ $, O ⊆ A ie O is a subset of any A in $

Hence O is the least element of $

Hence $ is a Lower Bounded Artex Space over R'

(2) Now, V is a subspace of V itself and therefore belongs to $

Also for any A ϵ $, A ⊆ V ie every element A of $ is a subset of V

Therefore, V is the greatest element of $

Hence $ is an Upper Bounded Artex Sapce over R'

Thus $ is a Bounded Artex Sapce over R'

Subclaim 4 : ($,∩, +) is a Complemented Artex Space Over R'

(1) Let m ϵ R'

Then m.O = m.{0}={m.0} = {0}= O, (since V is a real inner product space, m.0 = 0, for all reals and hence for all m ϵ R')

(2) Let A ϵ $

Let 0 be the identity element of the bi-monoid (M , + , **.**) with respect to +

Then 0.A = { 0.a / a ϵ A }

= { 0 / a ϵ A } (since V is a real inner product space, 0.a = 0, for all a ϵ A)

(3) Let A ϵ $

Let A' be the orthogonal complement of A

Clearly A' is a subspace of A and therefore belongs to $

Now, A ^ A' = A ∩ A' = { 0 } = O

and, A v A' = A + A' = V

Therefore, A' is the complement of A in $

($,∩, +) is a Complemented Artex Space Over R'

Subclaim 5 : ($,∩, +) is a Distributive Artex Space Over R'

Let A,B,C ϵ $

Now, to show (i) $A \cap (B+C) = (A \cap B)+(A \cap C)$ and (ii) $A+(B \cap C) = (A+B) \cap (A+C)$

To show (i) $A \cap (B+C) = (A \cap B)+(A \cap C)$

Let $x \in (A \cap B)+(A \cap C)$

Then $x = u + v$, for some $u \in A \cap B$ and $v \in A \cap C$

- ⇨ $u \in A$ and $u \in B$ and $v \in A$ and $v \in C$
- ⇨ $u \in A$ and $v \in A$ and $u \in B$ $v \in C$
- ⇨ $u+v \in A$ and $u+v \in B+C$
- ⇨ $u+v \in A \cap (B+C)$

Therefore, $(A \cap B)+(A \cap C) \subseteq A \cap (B+C)$ ----------(*)

Let $x \in A \cap (B+C)$

Then $x \in A$ and $x \in B+C$ => $x = a$, for some $a \in A$ and $x = b+c$, for some $b \in B$ and $c \in C$.

Since B+C is the direct sum and $x = a$ and $x = b+c$ are two expressions for x, $a+0 = b+c$ implies $a = b$ and $c = 0$

=> $x \in A$ and $x = a$, $a = b$ implies $a \in B$ and $c = 0$, $0 \in A$ and $0 \in C$

=> $x = a + 0$, $a \in A \cap B$ and $0 \in A \cap C$

Therefore, $x = a + 0 \in (A \cap B)+(A \cap C)$

Therefore, $A \cap (B+C) \subseteq (A \cap B)+(A \cap C)$ ----------(**)

From (*) and (**) it is clear that $A \cap (B+C) = (A \cap B)+(A \cap C)$

To show (ii) $A+(B \cap C) = (A+B) \cap (A+C)$

Let $x \in A+(B \cap C)$

Then $x = u+v$, for some $u \in A$ and $v \in B \cap C$

- ⇨ $u \in A$ and $v \in B$ and $v \in C$
- ⇨ $u \in A$ and $v \in B$ and $u \in A$ and $v \in C$
- ⇨ $u+v \in A+B$ and $u+v \in A+C$
- ⇨ $u+v \in (A+B) \cap (A+C)$

Therefore, $A+(B \cap C) \subseteq (A+B) \cap (A+C)$ ---------- (***)

Let $x \in (A+B) \cap (A+C)$

- ⇨ $x \in A+B$ and $x \in A+C$
- ⇨ $x = a+b$ for some $a \in A$ and $b \in B$ and $x = a'+c$, for some $a' \in A$ and $c \in C$
- ⇨ Since the sum is a direct sum $x=a+b$ and $x=a'+c$ implies $a=a'$ and $b=c$
- ⇨ $a \in A$ and $b=c \in C$
- ⇨ $a \in A$ and $b \in B \cap C$
- ⇨ $x = a+b$, where $a \in A$ and $b \in B \cap C$
- ⇨ $x \in A+(B \cap C)$

Therefore, $(A+B) \cap (A+C) \subseteq A+(B \cap C)$ ---------- (****)

From (***) and (****) it is clear that $A+(B\cap C) = (A+B)\cap(A+C)$

Hence ($,∩, +) is a Distributive Artex Space Over R'

Thus, ($\$$, ^ ,v) = ($\$$, ∩ , +) is a Boolean Artex Space over the bi-monoid R', where $R' = R^{+} \cup \{0\}$.

Proposition 3.2.1 Let A be a Complemented Artex space over a bi-monoid M and let B be an Artex Space over the bi-monoid M. Let f : A → B be an Artex Space epimorphism. Then B is a Complemented Artex Space over M. In other words, the homomorphic image of a Complemented Artex Space over a bi-monoid is a Complemented Artex space over the bi-monoid.

Proof : Let A be a Complemented Artex space over a bi-monoid M and B be an Artex Space over the bi-monoid M.

Let f : A → B be an Artex Space epimorphism of A onto B .

To show that f(A) = B is a Complemented Artex Space over the bi-monoid M.

Being a Complemented Artex Space, A is a Bounded Artex Space over M.

Let 0 and 1 be the least and the greatest elements of A

By the Propositions 2.1.16 and 2.1.17, f(0) and f(1) are the least and the greatest elements of f(A) = B

Let f(0) = 0' and f(1) = 1'

(1) Let m ϵ M

Now, m0' = mf(0)

= f(m0) (since f is an Artex Space homomorphism)

= f(0) (since A is a Complemented Artex Space over M, m0 = 0 for all m ϵ M)

= 0'

(2) Let b ϵ B

Since f : A → B is onto, there exists an element a ϵ A such that f(a) = b

Now, 0.b = 0f(a)

= f(0.a), (since f is an Artex Space homomorphism)

= f(0), (since A is a Complemented Artex Space over M, 0.a = 0 for all a ϵ A)

= 0'

(3) Now, it is enough to show that for each b ϵ B there exists an element b' in B such that

b v b' =f(1) = 1' and b ^ b'=f(0) = 0'.

Let b ϵ B

Since f : A → B is onto, there exists an element a ϵ A such that f(a) = b

Since A is a Complemented Artex Space over the bi-monoid M, there exists an element a' ϵ A such that a v a' =1 and a ^ a' = 0.

Therefore, f(a v a') = f(1) and f(a ^ a') = f(0)

f(a) v f(a') = f(1) and f(a) ^ f(a') = f(0) (since f is an Artex space homomorphism)

Let f(a') = b'

Then b' ϵ f(A) = B and b v b' = f(1) and b ^ b' = f(0)

ie b v b' = f(1) = 1' and b ^ b' = f(0) = 0'

ie b v b' = 1' and b ^ b' = 0'

Therefore, b' is a complement of b in B.

Therefore, B is a Complemented Artex Space over M.

Hence, the homomorphic image of a Complemented Artex Space over a bi-monoid is a Complemented Artex space over the bi-monoid.

Corollary 3.2.2 : Let A be a Complemented Artex Space over a bi-monoid M and let B be an Artex Space over the bi-monoid M. Let f :A → B be an Artex Space homomorphism.Then f(A) is a Complemented SubArtex Space of B.

Proof : Let A be a Complemented Artex Space over a bi-monoid M.

Let f : A → B be an Artex Space homorphism of A onto B.

Since f : A → B is an Artex Space homorphism, by the Proposition 2.1.24, f(A) is a SubArtex Space of B and hence f(A) is an Artex Space over M.

Let g : A → f(A) be defined by g(a) = f(a)

Then g: A → f(A) is clearly an epimorphism of A onto f(A)

Then by the Proposition, f(A) is a Complemented Artex Space over M and hence a Complemented SubArtex Space of B.

Proposition 3.2.3 : Let A be a Boolean Artex space over a bi-monoid M and let B be an Artex Space over the bi-monoid M. Let f : A → B be an Artex Space epimorphism. Then B is a Boolean Artex Space over M. In other words, the homomorphic image of a Boolean Artex Space over a bi-monoid is a Boolean Artex space over the bi-monoid.

Proof : The proof is immediate from the Propositions 2.1.23 and 3.2.1

Corollary 3.2.4 : Let A be a Boolean Artex Space over a bi-monoid M and let B be an Artex Space over the bi-monoid M. Let f : A → B be an Artex Space homomorphism. Then f(A) is a Boolean SubArtex space of B.

Proof : The proof is immediate from the Propositions 3.2.3 and 2.1.24 or from 3.2.2 and 2.1.25.

Proposition 3.2.5 : If A and B are any two Complemented Artex Spaces over a bi-monoid M, then A×B is also a Complemented Artex Space over M.
If $\leq_1$ and $\leq_2$ are the partial orderings on A and B respectively, then the partial ordering $\leq$ on A×B and the

bi-monoid multiplication in $A\times B$ are defined by the following : For $x,y \in A\times B$, where $x=(a_1,b_1)$ and $y=(a_2,b_2)$, $x \le y$ means $a_1 \le_1 a_2$ and $b_1 \le_2 b_2$ .For $m \in M$ and $x \in A\times B$, where $x=(a,b)$,the bi-monoid multiplication in $A\times B$ is defined by $mx = m(a,b) = (ma,mb)$, where ma and mb are the bi-monoid multiplications in A and B respectively. In other words if $\wedge_1$ and v_1 are the cap, cup of A and $\wedge_2$ and v_2 are the cap, cup of B, then the cap, cup of $A\times B$ denoted by $\wedge$ and v are defined by $x \wedge y = (a_1,b_1) \wedge (a_2,b_2) = (a_1 \wedge_1 a_2 , b_1 \wedge_2 b_2)$ and $y = (a_1,b_1)\ v\ (a_2,b_2) = (a_1\ v_1 a_2 , b_1\ v_2\, b_2)$.

Proof : Let $A^* = A\times B$

We know that if ($A, \le_1$) and ($B, \le_2$) are any two Distributive Artex spaces over a bi-monoid M, then $A\times B$ is also a Distributive Artex Space over the bi-monoid M.

If 0_1 and 0_2 are the least elements of A and B respectively, then $0 =(0_1,0_2)$ will be the least element of $A\times B$

If 1_1 and 1_2 are the greatest elements of A and B respectively, then $1= (1_1,1_2)$ will be the greatest element of $A\times B$

Let $m \in M$ and $x \in A^* = A\times B$, where $x = (a,b)$

Now, $m0 = m(0_1,0_2) = (m0_1,m0_2)$

$= (0_1,0_2)$, (since A and B are Complemented Artex Spaces, $m0_1= 0_1$ and $m0_2= 0_2$)

$= 0$

Let $x \in A^* = A\times B$, where $x = (a,b)$ and let 0 be the identity element of the bi-monoid (M , + , **.**) with respect to +

Now, $0.x = 0(a,b) = (0.a,0.b)$

$= (0_1,0_2)$ (since A and B are Complemented Artex Spaces, $0.a =0_1$ and $0.b_2= 0_2$)

$= 0$

Therefore, it is enough to prove that for each $x \in A^* = A\times B$, there exists an element $x' \in A^*$ such that $x \wedge x' = 0$ and $x\ v\ x' = 1$

Let $x \in A\times B$, where $x=(a,b)$, $a \in A$ and $b \in B$

Since A is a Complemented Artex Space over M and $a \in A$, there exists an element $a' \in A$ such that $a \wedge a' = 0_1$ and $a\ v\ a' = 1_1$

Since B is a Complemented Artex Space over M and $b \in B$, there exists an element $b' \in B$ such that $b \wedge b' = 0_2$ and $b\ v\ b' = 1_2$

Let $x' = (a'\ b')$. Then $a' \in A$ and $b' \in B$ implies $x' = (a',b') \in A\times B$

Now, $x \wedge x' = (a,b) \wedge (a',b') = (a \wedge a', b \wedge b') = (0_1,0_2) = 0$

$x\ v\ x' = (a,b)\ v\ (a',b') = (a\ v\ a', b\ v\ b') = (1_1,1_2) = 1$

Therefore, x' is a complement of x in A^*

Hence, $A^* = A\times B$ is a Complemented Artex Space Over M.

Corollary 3.2.6 : If $A_1, A_2, A_3, \ldots, A_n$ are Complemented Artex spaces over a bi-monoid M, then $A_1 \times A_2 \times A_3 \times \ldots \times A_n$ is also a Complemented Artex space over M.

Proof : The proof is by induction on n

When n=2, by the theorem $A_1 \times A_2$ is a Complemented Artex space over M

Assume that $A_1 \times A_2 \times A_3 \times \ldots \times A_{n-1}$ is a Complemented Artex space over M

Consider $A_1 \times A_2 \times A_3 \times \ldots \times A_n$

Let $A= A_1 \times A_2 \times A_3 \times \ldots \times A_{n-1}$

Then $A_1 \times A_2 \times A_3 \times \ldots \times A_n = (A_1 \times A_2 \times A_3 \times \ldots \times A_{n-1}) \times A_n = A \times A_n$

By assumption A is a Complemented Artex space over M.

Again by the theorem $A \times A_n$ is a Complemented Artex space over M

Hence $A_1 \times A_2 \times A_3 \times \ldots \times A_n$ is a Complemented Artex space over M.

Proposition 3.2.7 : If A and B are any two Boolean Artex Spaces over a bi-monoid M, then A×B is also a Boolean Artex Space over M. If $\leq_1$ and $\leq_2$ are the partial orderings on A and B respectively, then the partial ordering $\leq$ on A×B and the bi-monoid multiplication in A×B are defined by the following : For $x,y \in A \times B$, where $x=(a_1,b_1)$ and $y=(a_2,b_2)$, $x \leq y$ means $a_1 \leq_1 a_2$ and $b_1 \leq_2 b_2$ .For $m \in M$ and $x \in A \times B$, where $x=(a,b)$,the bi-monoid multiplication in A×B is defined by $mx = m(a,b) = (ma,mb)$, where ma and mb are the bi-monoid multiplications in A and B respectively. In other words if $\wedge_1$ and v_1 are the cap, cup of A and $\wedge_2$ and v_2 are the cap, cup of B, then the cap, cup of A×B denoted by ^ and v are defined by $x \wedge y = (a_1,b_1) \wedge (a_2,b_2) = (a_1 \wedge_1 a_2 , b_1 \wedge_2 b_2)$ and $x \vee y = (a_1,b_1) \vee (a_2,b_2) = (a_1 \vee_1 a_2 , b_1 \vee_2 b_2)$.

Proof : The Proof is immediate from the Propositions 2.1.21 and 3.2.5.

Corollary 3.2.8 : If $A_1, A_2, A_3, \ldots, A_n$ are Boolean Artex spaces over a bi-monoid M,

then $A_1 \times A_2 \times A_3 \times \ldots \times A_n$ is also a Boolean Artex space over M.

Proof : We can prove this corollary by induction on n, but the proof is immediate from Corollaries 2.1.22 and 3.2.6.

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