

Some Special Artex Spaces Over Bi-monoids

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ABSTRACT

We introduce a Special Artex Space over a bi-monoid namely Distributive Artex space over a bi-monoid. We give some examples of Distributive Artex space over bi-monoids. We prove the Cartesian product of any two Distributive Artex Spaces over a bi-monoid is a Distributive Artex Space over the bi-monoid. Also we prove the Cartesian product of a finite number of Distributive Artex Spaces over a bi-monoid is a Distributive Artex Space over the bi-monoid. We prove under the Artex space homomorphism $f : A \to B$, the homomorphic image of a SubArtex Space of an Artex space A over a bi-monoid is a SubArtex space of B. We prove the homomorphic image of a Distributive Artex Space over a bi-monoid is a Distributive Artex Space over the bi-monoid. We prove a SubArtex space of a Distributive Artex space over a bi-monoid is a Distributive Artex space over the bi-mpnoid. We solve three problems on Bounded Artex spaces over bi-monoids. 1.A SubArtex space of a Lower Bounded Artex space over a bi-monoid need not be a Lower Bounded Artex space over the bi-monoid. 2. A SubArtex space of an Upper Bounded Artex space over a bi-monoid need not be an Upper Bounded Artex space over the bi-monoid and 3. A SubArtex space of a Bounded Artex space over a bi-monoid need not be a Bounded Artex space over the bi-monoid.

Keywords : Distributive Artex space, Bounded Artex space, Homomorphic image

1.INTRODUCTION

The theory of Groups is one of the richest branches of abstract algebra. Groups of transformations play an important role in geometry. A more general concept than that of a group is that of a semi-group. The aim of considering semi-groups is to provide an introduction to the theory of rings. Two binary operations are defined for a ring. With respect to the first operation written in the order, it is an abelian group and with respect to the second one, it is enough to be a semi-group. We study many examples of rings and we have so many results in rings. However, there are many spaces or sets which are monoids with respect to two or more operations. This motivated us to define a more general concept bi-monoid than that of a ring. So we can have many algebraic systems which are bi-monoids, but not rings. With these concepts in mind we enter into Lattices. In Discrete Mathematics Lattices and Boolean algebra have important applications in the theory and design of computers. There are many other areas such as engineering and science to which Boolean algebra is applied. Boolean Algebra was introduced by George Boole in 1854. A more general algebraic system is the lattice. A Boolean Algebra is introduced as a special lattice. The study of Lattices and Boolean algebra together with our bi-monid motivated us to bring our papers titled "Artex Spaces over Bi-monoids", "Subartex Spaces Of an Artex Space Over a Bi-monoid", and "Bounded Artex Spaces Over Bi-monoids and Artex Space Homomorphisms". As an another development of it now, we introduce a Special Artex Space over a bi-monoid namely Distributive Artex space over a bi-monoid. These Distributive Artex spaces over bi-monoids and the Propositions, we prove here, are important results for further development. The examples of Distributive Artex spaces over bi-monoids are interesting and useful. The main part under consideration deals with Distributive Artex spaces over bi-monoids, SubArtex spaces of Bounded Artx spaces over bi-monoids. The principal concepts that we consider here are those of homomorphism and, epimomorphism. As a result, the theory of Artex spaces over bi-monoids like the theory of Lattices and Boolean Algebra, will, in future, play a good role in many fields especially in science and engineering and in computer fields.

2.Preliminaries

2.1.1 Definition : Distributive Lattice : A lattice (L, ^ , v) is said to be a distributive lattice if

for any a,b,c ϵ L, (i) a ^ (b v c) = (a ^ b) v (a ^ c) (ii) a v (b ^ c) = (a v b) ^ (a v c)

2.1.2 Definition : Bi-monoid : An algebraic system (M , + , .) is called a Bi-monoid if

1.(M , +) is a monoid 2. (M , .) is a monoid

and 3 (i) a.(b+c) = a.b + a.c and (ii) (a+b).c = a.c + b.c , for all a,b,c ϵ M.

Note : The identity elements of (M , + , .) with respect to + and **.** are denoted by 0 and 1 respectively.

2.1.3 Definition : Artex Space Over a Bi-monoid : A non-empty set A is said to be an Artex Space Over a Bi-monoid (M , + , .) if 1.(A, ^ , v) is a lattice and

2.for each mϵM , m$\neq$0, and aϵA, there exists an element ma ϵ A satisfying the following conditions :

(i) m(a ^ b) = ma ^ mb

(ii) m(a v b) = ma v mb

(iii) ma ^ na $\leq$ (m +n)a and ma v na $\leq$ (m + n)a

(iv) (mn)a = m(na), for all m,nϵM, m$\neq$0, n$\neq$0, and a,bϵA

(v) 1.a = a , for all aϵA

Here, $\leq$ is the partial order relation corresponding to the lattice (A, ^ , v)

The multiplication ma is called a bi-monoid multiplication with an artex element or simply bi-monoid multiplication in A.

Unless otherwise stated A remains as an Artex space with the partial ordering $\leq$ which need not be "less than or equal to" and M as a bi-monoid with the binary operations + and . need not be the usual addition and usual multiplication.

Proposition 2.1.4 : If A and B are any two Artex spaces over a bi-monoid M and if $\leq_1$ and $\leq_2$ are the partial ordering on A and B respectively, then AXB is also an Artex Space over M, where the partial ordering $\leq$ on AXB and the bi-monoid multiplication in AXB are defined by the following :

For x,y ϵ AXB, where x=(a_1,b_1) and y=(a_2,b_2) , x $\leq$ y means $a_1 \leq_1 a_2$ and $b_1 \leq_2 b_2$

For $m \in M$, $m \neq 0$, and $x \in AXB$, where $x=(a,b)$, the bi-monoid multiplication in AXB is defined by

$$mx = m(a,b) = (ma,mb),$$ where ma and mb are the bi-monoid multiplications in A and B respectively.

In other words if $\wedge_1$ and v_1 are the cap, cup of A and $\wedge_2$ and v_2 are the cap, cup of B, then the cap, cup of AXB denoted by Λ and V are defined by $x \Lambda y = (a_1,b_1) \Lambda (a_2,b_2) = (a_1 \wedge_1 a_2 , b_1 \wedge_2 b_2)$

$$x \, V \, y = (a_1,b_1) \, V \, (a_2,b_2) = (a_1 \, v_1 \, a_2 , b_1 \, v_2 \, b_2).$$

Corollary 2.1.5 : If $A_1, A_2 ,A_3,\ldots, A_n$ are Artex spaces over a bi-monoid M, then $A_1 X A_2 X A_3 X \ldots X A_n$ is also an Artex space over M.

2.1.6 Definition : Complete Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Complete Artex Space if as a lattice, A is a complete lattice, that is each nonempty subset of A has a least upper bound and a greatest lower bound.

2.1.7 Remark : Every Complete Artex space must have a least element and a greatest element. The least and the greatest elements, if they exist, are called the bounds or units of the Artex space and are denoted by 0 and 1 respectively.

Note : The identity elements of (M , + , .) with respect to + and **.** are also denoted by 0 and 1 respectively.

2.1.8 Definition : Lower Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Lower Bounded Artex Space over M if as a lattice, A has the least element 0.

2.1.9 Definition : Upper Bounded Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be an Upper Bounded Artex Space over M if as a lattice, A has the greatest element 1.

2.1.10 Definition : Artex Space Homomorphism : Let A and B be two Artex spaces over a bi-monoid M, where $\wedge_1$ and v_1 are the cap, cup of A and $\wedge_2$ and v_2 are the cap, cup of B. A mapping $f : A \rightarrow B$ is said to be an Artex space homomorphism if

1. $f(a \wedge_1 b) = f(a) \wedge_2 f(b)$
2. $f(a \, v_1 b) = f(a) \, v_2 f(b)$
3. $f(ma)) = mf(a)$, for all $m \in M$, $m \neq 0$ and $a,b \in A$.

2.1.11 Definition : Artex Space Epimorphism : Let A and B be two Artex spaces over a bi-monoid M. An Artex space homomorphism $f : A \rightarrow B$ is said to be an Artex space epimorphism if the mapping $f : A \rightarrow B$ is onto.

2.1.12 Definition : Artex Space Monomorphism : Let A and B be two Artex Spaces over a bi-monoid M. An Artex space homomorphism $f : A \rightarrow B$ is said to be an Artex Space monomorphism if the mapping $f : A \rightarrow B$ is one-one.

2.1.13 Definition : Artex Space Isomorphism : Let A and B be two Artex spaces over a bi-monoid M. An Artex Space homomorphism $f : A \rightarrow B$ is said to be an Artex Space Isomorphism if the mapping $f : A \rightarrow B$ is both one-one and onto, ie, f is bijective.

2.1.14 Definition : Isomorphic Artex Spaces : Two Artex spaces A and B over a bi-monoid M are said to be isomorphic if there exists an isomorphism from A onto B or from B onto A.

2.1.15 Definition : SubArtex Space : Let (A, ^ , v) be an Artex space over a bi-monoid (M , + , .)

Let S be a nonempty subset of A. Then S is said to be a Subartex space of A if (S, ^ , v) itself is an Artex space over M.

Proposition 2.1.16 : Let (A, ^ , v) be an Artex space over a bi-monoid (M , + , .)

Then a nonempty subset S of A is a SubArtex space of A if and only if for each m,n ϵ M, m≠0, n≠0, and

a, b ϵ S,ma ^ nb ϵ S and ma v nb ϵ S

3 Homomorphic image of a SubArtex Space of an Artex space over a bi-monoid

Proposition 3.1.1 : Let A and B be Artex spaces over a bi-monoid M. Let f : A → B be an Artex space homomorphism. Let S be a SubArtex space of A. Then f(S) is a SubArtex Space of B. In other words, under the Artex space homomorphism f : A → B, the homomorphic image of a SubArtex Space of the Artex space A over M is a SubArtex space of B over M.

Proof : Let A and B be Artex spaces over a bi-monoid M.

Let S be a SubArtex space of A.

Let $\leq_1$ and $\leq_2$ be the partial orderings of A and B respectively.

Let $^_1$ and v_1 be the cap and cup of A and let $^_2$ and v_2 be the cap and cup of B.

Suppose f : A → B is an Artex space homomorphism.

To show f(S) is a SubArtex Space of B.

Let a', b' ϵ f(S) and m , n ϵ M, where m ≠ 0, and n ≠ 0.

Then there exists a and b in A such that f(a) = a' and f(b) = b'

ma' $^_2$ nb' = mf(a) $^_2$ n f(b) = f(ma) $^_2$ f(nb) = f(ma $^_1$ nb) (since f is a homomorphism)

Since S is a subArtex space of A, ma $^_1$ nb ϵ A

Therefore, f(ma $^_1$ nb) ϵ f(S)

ma' $^_2$ nb' ϵ f(S)

ma' v_2 nb' = mf(a) v_2 n f(b) = f(ma) v_2 f(nb) = f(ma v_1 nb).

Since S is a subArtex space of A, ma v_1 nb ϵ A

Therefore, f(ma v_1 nb) ϵ f(S)

ma' v_2 nb' ϵ f(S)

Hence by the Proposition 2.1.16, f(S) is a SubArtex space of B.

4 Distributive Artex Space over a bi-monoid : An Artex space A over a bi-monoid M is said to be a Distributive Artex Space over the bi-monoid M if as a lattice, A is a distributive lattice.

In other words, an Artex space A over a bi-monoid M is said to be a Distributive Artex Space over the bi-monoid M if for any a,b,c ϵ A, (i) a ^ (b v c) = (a ^ b) v (a ^ c) (ii) a v (b ^ c) = (a v b) ^ (a v c)

4.1.1 Example : Let A be the set of all sequences (x_n) in Z, where Z is the set of all integers and let W = {0,1,2,3,…}.

Define ≤' ,an order relation, on A by for (x_n), (y_n) in A, $(x_n) \leq' (y_n)$ means $x_n \leq y_n$, for each n,

where ≤ is the usual relation " less than or equal to "

Let $x \in A$, where $x = (x_n)$

Clearly $x_n \leq y_n$, for each n

So, $(x_n) \leq' (x_n)$

Therefore, ≤' is reflexive.

Let $x,y \in A$, where $x = (x_n)$ and $y = (y_n)$ be such that $x \leq' y$ and $y \leq' x$, that is, $(x_n) \leq' (y_n)$ and $(y_n) \leq' (x_n)$.

Then $(x_n) \leq' (y_n)$ implies $x_n \leq y_n$, for each n

$(y_n) \leq' (x_n)$ implies $y_n \leq x_n$, for each n

Now, $x_n \leq y_n$, for each n , and $y_n \leq x_n$, for each n ,implies $x_n = y_n$, for each n.

Therefore, $(x_n) = (y_n)$, that is x = y

Therefore, ≤' is anti-symmetric.

Let $x,y,z \in A$, where $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$ be such that $x \leq' y$ and $y \leq' z$,

That is, $(x_n) \leq' (y_n)$

and $(y_n) \leq' (z_n)$.

Then $(x_n) \leq' (y_n)$ implies $x_n \leq y_n$, for each n

$(y_n) \leq' (z_n)$ implies $y_n \leq z_n$, for each n

Now, $x_n \leq y_n$, for each n , and $y_n \leq z_n$, for each n, implies $x_n \leq z_n$ for each n.

Therefore, $(x_n) \leq' (z_n)$

Therefore, ≤' is transitive.

Hence, ≤' is a partial order relation on A

Now the cap ,cup operations are defined by the following :

$(x_n) \wedge (y_n) = (u_n)$, where $u_n = \text{mini}\ \{ x_n , y_n \}$, for each n.

$(x_n) \vee (y_n) = (v_n)$, where $v_n = \text{maxi}\ \{ x_n , y_n \}$, for each n.

Clearly (A , ≤') is a lattice.

The bi-monoid multiplication in A is defined by the following :

For each $m \in W, m \neq 0$, and $x \in A$, where $x = (x_n)$, mx is defined by $mx = m(x_n) = (mx_n)$.

Clearly $(mx_n) \in A$

Let x,y ϵ A, where x = (x_n) , y = (y_n) and let m ϵ W,m≠0

Then, it is clear that (i) m(x ^ y) = mx ^ my

(ii) m(x v y) = mx v my

(iii) mx ^ nx ≤ (m +n)x and mx v nx ≤ (m + n)x

(iv) (mn)x = m(nx)

(v) 1.x = x , for all m,nϵW, m≠0, n≠0, and x,y ϵ A

Therefore, A is an Artex space over W.

Clearly, (x_n) ^ [(y_n) v (z_n)] = [(x_n) ^ (y_n)] v [(x_n) ^ (z_n)]

and (x_n) v [(y_n) ^ (z_n)] = [(x_n) v (y_n)] ^ [(x_n) v (z_n)].

Hence, A is a Distributive Artex Space over the bi-monoid W.

4.1.2 Example : If B is the set of all sequences(x_n) in Q, where Q is the set of all rational numbers, then as in Example 4.1.1, B is a Distributive Artex Space over the bi-monoid W.

4.1.3 Example : If D is the set of all sequences(x_n) in R, where R is the set of all real numbers, then as in Example 4.1.1, D is a Distributive Artex Space over the bi-monoid W.

Proposition 4.2.1: A SubArtex space S of a Distributive Artex space A over a bi-monoid M is a Distributive Artex space over M.

Proof : Let A be a Distributive Artex space over a bi-monoid M.

Let S be SubArtex space of A.

As a SubArtex space of A, by the definition of a SubArtex space, S ≠ φ and S itself is an Artex space over M.

Let a,b,c ϵ S

Since S is a subset of A, a,b,c ϵ A

Since A is a Distributive Artex space over M,(i) a ^ (b v c) = (a ^ b) v (a ^ c)

and (ii) a v (b ^ c) = (a v b) ^ (a v c)

Hence, S is a Distributive Artex space over M.

Proposition 4.2.2 : If D and D’ are any two Distributive Artex spaces over a bi-monoid M, then DXD’ is also a Distributive Artex Space over M, If $\leq_1$ and $\leq_2$ are the partial orderings on D and D’ respectively, then the partial ordering ≤ on DXD’ and the bi-monoid multiplication in DXD’ are defined by the following :

For x,yϵ DXD’, where x=(a_1,b_1) and y=(a_2,b_2) , x ≤ y means $a_1 \leq_1 a_2$ and $b_1 \leq_2 b_2$

For mϵM, m ≠ 0, and x ϵ DXD’, where x=(a,b),the bi-monoid multiplication in DXD’ is defined by mx = m(a,b) = (ma,mb), where ma and mb are the bi-monoid multiplications in D and D’ respectively.

In other words if $^_1$ and v_1 are the cap, cup of D and $^_2$ and v_2 are the cap, cup of D’, then the cap, cup of DXD’ denoted by ^ and v are defined by x ^ y = (a_1,b_1) ^ (a_2,b_2) = $(a_1 \wedge_1 a_2 , b_1 \wedge_2 b_2)$

and $x \vee y = (a_1,b_1) \vee (a_2,b_2) = (a_1 \vee_1 a_2 , b_1 \vee_2 b_2)$.

Proof : Let A = DXD'

We know that if (D, $\leq_1$) and (D', $\leq_2$) are any two Artex spaces over a bi-monoid M, then by the Proposition 2.1.4 DXD' is an Artex space over the bi-monoid M . Therefore, it is enough to prove that A = DXD' is Distributive.

Let x,y,z ϵ A = DXD' , where x = (b,b'), y = (c,c'), z = (d,d')

Now, $x \wedge (y \vee z) = (b,b') \wedge ((c,c') \vee (d,d'))$

$= (b,b') \wedge (c \vee_1 d, c' \vee_2 d')$

$= (b \wedge_1 (c \vee_1 d), b' \wedge_2 (c' \vee_2 d'))$

$= ((b \wedge_1 c) \vee_1 (b \wedge_1 d) , (b' \wedge_2 c') \vee_2 (b' \wedge_2 d'))$

$= (b \wedge_1 c , b' \wedge_2 c') \vee (b \wedge_1 d, b' \wedge_2 d')$

$= ((b,b') \wedge (c,c')) \vee ((b,b') \wedge (d,d'))$

$= (x \wedge y) \vee (x \wedge z)$

To show $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Now, $x \vee (y \wedge z) = (b,b') \vee ((c,c') \wedge (d,d'))$

$= (b,b') \vee (c \wedge_1 d, c' \wedge_2 d')$

$= (b \vee_1 (c \wedge_1 d) , b' \vee_2 (c' \wedge_2 d'))$

$= ((b \vee_1 c) \wedge_1 (b \vee_1 d),(b' \vee_2 c') \wedge_2 (b' \vee_2 d'))$

$= (b \vee_1 c , b' \vee_2 c') \wedge (b \vee_1 d , b' \vee_2 d')$

$= ((b,b') \vee (c,c')) \wedge ((b,b') \vee (d,d'))$

$= (x \vee y) \wedge (x \vee z)$

Hence, A = DXD' is a Distributive Artex space over M.

Corollary 4.2.3 : If $D_1, D_2 ,D_3,\ldots, D_n$ are Distributive Artex spaces over a bi-monoid M, then

$D_1 \text{ X } D_2 \text{ X } D_3 \text{ X } \ldots \text{X } D_n$ is also a Distributive Artex space over M.

Proof : The proof is by induction on n

When n=2, by the Proposition $B_1 \text{ X } B_2$ is a Distributive Artex space over M

Assume that $B_1 \text{ X } B_2 \text{ X } B_3 \text{ X } \ldots \text{X } B_{n-1}$ is a Distributive Artex space over M

Consider $B_1 \text{ X } B_2 \text{ X } B_3 \text{ X } \ldots \text{X } B_n$

Let $B= B_1 \text{ X } B_2 \text{ X } B_3 \text{ X } \ldots \text{X } B_{n-1}$

Then $B_1 X B_2 X B_3 X \ldots X B_n = (B_1 X B_2 X B_3 X \ldots X B_{n-1}) X B_n$

$= B X B_n$

By assumption B is a Distributive Artex space over M.

Again by the Proposition $B X B_n$ is a Distributive Artex space over M

Hence, $B_1 X B_2 X B_3 X \ldots X B_n$ is a Distributive Artex space over M

Proposition 4.2.4 : Let A be a Distributive Artex space over a bi-monoid M and let B be an Artex space over M. Let $f : A \rightarrow B$ be an Artex space epimorphism of A onto B. Then B is a Distributive Artex Space over M. In other words, the homomorphic image of a Distributive Artex Space over a bi-monoid is a Distributive Artex space over the bimonoid.

Proof : Let A be a Distributive Artex Space over a bi-monoid M and let B be an Artex space over M.

Let $\leq_1$ and $\leq_2$ be the partial orderings of A and B respectively.

Let $\wedge_1$ and v_1 be the cap and cup of A and let $\wedge_2$ and v_2 be the cap and cup of B.

Suppose $f : A \rightarrow B$ is an Artex space epimorphism of A onto B.

To show that B = f(A) is a Distributive Artex Space over the bi-monoid M.

Let x', y', z' ϵ B

To show (i) $x' \wedge_2 (y' v_2 z') = (x' \wedge_2 y') v_2 (x' \wedge_2 z')$ and (ii) $x' v_2 (y' \wedge_2 z') = (x' v_2 y') \wedge_2 (x' v_2 z')$.

Since $f : A \rightarrow B$ is an Artex space epimorphism of A onto B, there exist elements x, y, z ϵ A such that

f(x) = x', f(y) = y', and f(z) = z'.

To show (i) $x' \wedge_2 (y' v_2 z') = (x' \wedge_2 y') v_2 (x' \wedge_2 z')$

Now, $x' \wedge_2 (y' v_2 z') = f(x) \wedge_2 (f(y) v_2 f(z))$

$= f(x) \wedge_2 f(y\, v_1\, z)$ (since f is an Artex space homomorphism)

$= f(x \wedge_1 (y\, v_1\, z))$ (since f is an Artex space homomorphism)

$= f((x \wedge_1 y)\, v_1 (x \wedge_1 z))$ (since A is a Distributive Artex Space)

$= f(x \wedge_1 y)\, v_2\, f(x \wedge_1 z)$ (since f is an Artex space homomor.)

$= (f(x) \wedge_2 f(y))\, v_2\, (f(x) \wedge_2 f(z))$ (since f is an Artex space homomor.)

$= (x' \wedge_2 y')\, v_2\, (x' \wedge_2 z')$.

To show (ii) $x' v_2 (y' \wedge_2 z') = (x' v_2 y') \wedge_2 (x' v_2 z')$

Now, $x' v_2 (y' \wedge_2 z') = f(x)\, v_2\, (f(y) \wedge_2 f(z))$

$= f(x)\, v_2\, f(y \wedge_1 z)$ (since f is an Artex space homomorphism)

$= f(x \vee_1 (y \wedge_1 z))$ (since f is an Artex space homomorphism)

$= f((x \vee_1 y) \wedge_1 (x \vee_1 z))$ (since A is a Distributive Artex Space)

$= f(x \vee_1 y) \Lambda_2 f(x \vee_1 z)$ (since f is an Artex space homomorphism)

$= (f(x) \vee_2 f(y)) \wedge_2 (f(x) \vee_2 f(z))$ (since f is an Artex space homomor.)

$= (x' \vee_2 y') \wedge_2 (x' \vee_2 z')$.

Hence, B is a Distributive Artex Space over M.

SubArtex Spaces Of Bounded Artex Spaces Over Bi-monoids

Problem 1 : A SubArtex space of a Lower Bounded Artex space over a bi-monoid need not be a Lower Bounded Artex space over the bi-monoid.

Proof : Let A be the set of all constant sequences (x_n) in $[0,\infty)$ and let W = {0,1,2,3,…}.

Define $\leq'$, an order relation, on A by for (x_n), (y_n) in A, $(x_n) \leq' (y_n)$ means $x_n \leq y_n$, for each n

Where $\leq$ is the usual relation " less than or equal to "

Since the sequences in A are all constant sequences, $x_n \leq y_n$, for some n implies $x_n \leq y_n$, for each n

Therefore, $x_n \leq y_n$, for each n and $x_n \leq y_n$, for some n in this problem are the same.

Let $x \in A$, where $x = (x_n)$

Clearly $x_n \leq x_n$, for each n

So, $(x_n) \leq' (x_n)$

Therefore, $\leq'$ is resflexive.

Let $x,y \in A$, where $x = (x_n)$ and $y = (y_n)$ be such that $x \leq' y$ and $y \leq' x$, that is, $(x_n) \leq' (y_n)$ and $(y_n) \leq' (x_n)$.

Then $(x_n) \leq' (y_n)$ implies $x_n \leq y_n$, for each n

and $(y_n) \leq' (x_n)$ implies $y_n \leq x_n$, for each n

Now, $x_n \leq y_n$, for each n , and $y_n \leq x_n$, for each n ,implies $x_n = y_n$, for each n.

Therefore, $(x_n) = (y_n)$, that is $x = y$

Therefore, $\leq'$ is anti-symmetric.

Let $x,y,z \in A$, where $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$ be such that $x \leq' y$ and $y \leq' z$, that is, $(x_n) \leq' (y_n)$ and $(y_n) \leq' (z_n)$.

Then $(x_n) \leq' (y_n)$ implies $x_n \leq y_n$, for each n

$(y_n) \leq' (z_n)$ implies $y_n \leq z_n$, for each n

Now, $x_n \leq y_n$, for each n , and $y_n \leq z_n$, for each n, implies $x_n \leq z_n$ for each n.

Therefore, $(x_n) \leq' (z_n)$

Therefore, $\leq'$ is transitive.

Hence, $\leq'$ is a partial order relation on A

Now the cap ,cup operations are defined by the following :

$(x_n) \wedge (y_n) = (u_n)$, where $u_n = \text{mini}\{x_n, y_n\}$, for each n.

$(x_n) \vee (y_n) = (v_n)$, where $v_n = \text{maxi}\{x_n, y_n\}$, for each n.

Clearly (A , $\leq'$) is a lattice.

The bi-monoid multiplication in A is defined by the following :

For each $m \in W$, $m \neq 0$, and $x \in A$, where $x = (x_n)$, mx is defined by $mx = m(x_n) = (mx_n)$.

Since (x_n) is a constant sequence belonging to A , (mx_n) is also a constant sequence belonging to A.

Therefore $(mx_n) \in A$

Let $x,y \in A$, where $x = (x_n)$, $y = (y_n)$ and let $m \in W, m \neq 0$

Then, it is clear that (i) $m(x \wedge y) = mx \wedge my$

(ii) $m(x \vee y) = mx \vee my$

(iii) $mx \wedge nx \leq (m+n)x$ and $mx \vee nx \leq (m+n)x$

(iv) $(mn)x = m(nx)$, for all $m,n \in W$, $m \neq 0$, $n \neq 0$, and $x,y \in A$

(v) $1.x = x$, for all $x \in A$

Therefore, A is an Artex space over W.

The sequence (0_n), where 0_n is 0 for all n, is a constant sequence belonging to A

Also $(0_n) \leq' (x_n)$, for all the sequences (x_n) belonging to A

Therefore, (0_n) is the least element of A.

That is, the sequence 0,0,0,…… is the least element of A

Hence A is a Lower Bounded Artex space over W.

Now let S be the set of all constant sequences (x_n) in $(0,\infty)$.

Clearly S is a SubArtex Space A.

But S has no least element.

Therefore, S is not a Lower Bounded Artex Space over W.

Hence, a SubArtex space of a Lower Bounded Artex space over a bi-monoid need not be a Lower Bounded Artex space over the bi-monoid.

Problem 2 : A SubArtex space of an Upper Bounded Artex space over a bi-monoid need not be an Upper Bounded Artex space over the bi-monoid.

Proof : Let A be the set of all constant sequences (x_n) in $(-\infty,0]$ and let W = {0,1,2,3,…}.

Define $\leq$' ,an order relation, on A by for (x_n), (y_n) in A, $(x_n) \leq' (y_n)$ means $x_n \leq y_n$, for n = 1,2,3,…

where $\leq$ is the usual relation " less than or equal to "

Then a in Example 3.3.1, A is an Artex space over W.

Now, the sequence (1_n), where 1_n is 0, for all n, is a constant sequence belonging to A

Also $(x_n) \leq' (1_n)$, for all the sequences (x_n) in A

Therefore, (1_n) is the greatest element of A.

That is, the sequence 0,0,0,… is the greatest element of A

Hence A is an Upper Bounded Artex Space over W.

Now let S be the set of all constant sequences (x_n) in $(-\infty,0)$.

Clearly S is a SubArtex Space A.

But S has no greatest element.

Therefore, S is not an Upper Bounded Artex Space over W.

Hence, a SubArtex space of an Upper Bounded Artex space over a bi-monoid need not be an

Upper Bounded Artex space over the bi-monoid.

Problem 3 : A SubArtex space of a Bounded Artex space over a bi-monoid need not be a Bounded Artex space over the bi-monoid.

Proof : The proof is clear from the Problems 1 and 2

In other way, if A is taken as the set of all constant sequences in the extended real line and S is taken as the set of all sequences in the real line, then as in the Problems 1 and 2 the proof follows. Here by convention the symbols $-\infty$ and ∞ are considered to be the least and the greatest elements of the extended real line, but in the real line there exist no such elements. Therefore S is not a Bounded Artex space over W.

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