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# SubArtex Spaces Of an Artex Space Over a Bi-monoid

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**ABSTRACT**

We define SubArtex Space of an Artex space over a Bi-monoid. We give some examples of SubArtex spaces. We prove the necessary and sufficient condition for a subset of an Artex space over a bi-monoid to be a SubArtex space. We prove another equivalent Proposition for the necessary and sufficient condition for a subset of an Artex space to be a SubArtex space. We prove a nonempty intersection of two SubArtex spaces of an Artex space over a bi-monoid is a SubArtex space. Also we prove a nonempty intersection of a family of SubArtex spaces of an Artex space over a bi-monoid is a SubArtex space. Finally, we prove, in this chapter, by giving an example, that the union of two SubArtex spaces need not be a SubArtex space.

## 1.INTRODUCTION

Most of the People are interested in relations, not only the blood relations, but also the Mathematical concept relations. The word relation suggests some familiar examples of relations such as the relation of father to son, mother to son, brother to sister, etc. Familiar examples in arithmetic are relations such as greater than , less than. We know the relation between the area of a circle and its radius and between the area of a square and its side. These examples suggest relationships between two objects. The relation between parents and child is an example of relation among three objects. This motivated us to think about relation between two different systems and how there are related or how long they can coincide or how much they can be related or how one system can act on another system or how one system can penetrate into another system. As a result, we introduced a new concept in our paper titled "Artex Spaces over Bi-monoids" in the "Research Journal of Pure Algebra". This theory was developed from the lattice theory. George Boole introduced Boolean Algebra in 1854. A more general algebraic system is the lattice. A Boolean Algebra is then introduced as a special lattice. Lattices and Boolean algebra have important applications in the theory and design of computers. There are many other areas such as engineering and science to which Boolean algebra is applied. As the theory of Artex spaces over bi-monoids is developed from lattice theory, we hope, this theory will, in future, play a good role in many fields especially in science and engineering and in computer fields. In Discrete Mathematics this theory will create a new dimension. We hope that the theory of Artex spaces over bi-monoids shall lead to many theories. But a theory can lead only if the theory itself is developed in its own way.

As a development of it, now, we introduce SubArtex spaces of Artex spaces over bi-monoids. From the definition of a SubSartex space that we are going to define, it is clear that not every subset of an Artex space over a bi-monoid is a SubArtex space. The problem that we solve here is to find subsets which qualify to become SubArtex spaces. In our attempt to solve this problem we find some propositions which qualify subsets to become SubArtex spaces. These Propoitions will have important applications in the development of the theory of Artex spaces over bi-monoids.

## 2.PRELIMINARIES

### 2.1.0 Definitions

**2.1.1 Lattice :** A lattice is a partially ordered set (L,≤) in which every pair of elements a,b ϵ L has a greatest lower bound and a least upper bound.

The greatest lower bound of a and b is denoted by aΛb and the least upper bound of a and b is denoted by avb

**2.1.2 Lattice as an Algebraic System :** A lattice is an algebraic system (L,Λ,V) with two binary operations Λ and V on L which are both commutative, associative, and satisfy the absorption laws namely

aΛ(aVb) = a and aV(aΛb) = a

The operations Λ and V are called cap and cup respectively, or sometimes meet and join respectively.

**2.1.3 Bi-monoid :** A system ( M , + , . ) is called a Bi-monoid if

1.( M , + ) is a monoid

2. ( M , . ) is a monoid and

3. a.(b + c) = a.b + a.c and (a + b).c = a.c + b.c , for all a,b,c ϵ M

### 2.2.0 Examples

**2.2.1** Let W={0,1,2,3,…}.

Then (W , + , .), where + and **.** are the usual addition and multiplication respectively, is a bi-monoid.

**2.2.2** Let S be any set. Consider P(S), the power set of S.

Then (P(S) ,∪, ∩) is a bi-monoid.

**2.2.3** Let Q'= $Q^+$ ∪ {0}, where $Q^+$ is the set of all positive rational numbers.

Then (Q', + , . ) is a bi-monoid.

**2.2.4** Let R'= $R^+$ ∪ {0}, where $R^+$ is the set of all positive real numbers.

Then (R', + , . ) is a bi-monoid.

### 2.3.0 Definition

**2.3.1 Artex Space Over a Bi-monoid :** A non-empty set A is said to be an Artex Space Over a bi-monoid (M , + , . ) if 1.(A, Λ ,V) is a lattice and

2.for each m ϵ M , m≠0, and a ϵ A, there exists an element ma ϵ A satisfying the following conditions :

(i) m(a Λ b) = ma Λ mb

(ii) m(a V b) = ma V mb

(iii) ma Λ na ≤ (m+n)a and ma V na ≤ (m+n)a

(iv) (mn)a = m(na) , for all m,n ϵ M,m≠0, n≠0, and a,b ϵ A

(v) 1.a = a , for all a ϵ A

Here, ≤ is the partial order relation corresponding to the lattice (A, Λ , V)

The multiplication ma is called a bi-monoid multiplication with an artex element or simply bi-monoid multiplication in A.

Unless otherwise stated A remains as an Artex space with the partial ordering ≤ need not be "less than or equal to" and M as a bi-monoid with the binary operations + and **.** need not be the usual addition and usual multiplication.

**2.4.0 Examples**

**2.4.1** Let W = {0,1,2,3,…} and Z be the set of all integers.

Then ( W , + , . ) is a bi-monoid , where + and . are the usual addition and multiplication respectively.

(Z, ≤) is a lattice in which Λ and V are defined by a Λ b = mini {a,b} and a V b = maxi {a,b} , for all a,b ϵ Z.

Clearly for each m ϵ W,m≠0, and for each a ϵ Z, there exists ma ϵ Z.

Also, (i) m(a Λ b) = ma Λ mb

(ii) m(a V b) = ma V mb

(iii) ma Λ na ≤ (m +n)a and ma V na ≤ (m + n)a

(iv) (mn)a = m(na) , for all m,n ϵ W, m≠0, n≠0 and a,b ϵ Z

(v) 1.a = a , for all a ϵ Z

Therefore, Z is an Artex Space Over the bi-monoid ( W , + , . )

**2.4.2** As defined in Example 2.4.1, Q, the set of all rational numbers is an Artex space over W

**2.4.3** As defined in Example 2.4.1, R, the set of all real numbers is an Artex space over W

**2.4.4** Let Q'= $Q^+$ ᴜ {0}, where $Q^+$ is the set of all positive rational numbers.

Then (Q', + , . ) is a bi-monoid.

Now as defined in Example 2.4.1, Q, the set of all rational numbers is an Artex space over Q'

**2.4.5** Let R'= $R^+$ ᴜ {0}, where $R^+$ is the set of all positive real numbers.

Then (R', + , . ) is a bi-monoid.

As defined in Example 2.4.1, R, the set of all real numbers is an Artex space over R'

**2.4.6** As defined in Example 2.4.1, R, the set of all real numbers is an Artex space over Q'

**2.4.7** Let A be the set of all sequences $(x_n)$ in Z, where Z is the set of all integers and let W = {0,1,2,3,…}.

Define ≤' ,an order relation, on A by for $(x_n)$, $(y_n)$ in A, $(x_n) \leq' (y_n)$ means $x_n \leq y_n$ , for each n,

where ≤ is the usual relation " less than or equal to "

Clearly ≤' is a partial order relation on A

Now the cap ,cup operations are defined by the following :

$(x_n)$ Λ $(y_n) = (u_n)$, where $u_n$ =mini { $x_n$ , $y_n$ }, for each n.

$(x_n)$ V $(y_n) = (v_n)$, where $v_n$ =maxi { $x_n$ , $y_n$ }, for each n.

Clearly ( A , ≤' ) is a lattice.

The bi-monoid multiplication in A is defined by the following :

For each m∈W,m≠0, and x∈A, where x= $(x_n)$, mx is defined by mx = $m(x_n)=(mx_n)$.

Then clearly A is an Artex space over W.

**2.4.8** If B is the set of all sequences $(x_n)$ in Q, where Q is the set of all rational numbers, then as in Example 2.4.7, B is an Artex space over W.

**2.4.9** As defined in 2.4.8, B is an Artex space over Q'

**2.4.10** If D is the set of all sequences $(x_n)$ in R, where R is the set of all real numbers, then as in Example 2.4.7, D is an Artex space over W.

**2.4.11** As defined in 2.4.10, D is an Artex space over Q'

**2.4.12** As defined in 2.4.10, D is an Artex space over R'.

**Proposition 2.5.1 :** If A and B are any two Artex spaces over a bi-monoid M and if $\leq_1$ and $\leq_2$ are the partial ordering on A and B respectively, then AXB is also an Artex Space over M, where the partial ordering ≤ on AXB and the bi-monoid multiplication in AXB are defined by the following :

For x,y∈ AXB, where $x=(a_1,b_1)$ and $y=(a_2,b_2)$ , x ≤y means $a_1 \leq_1 a_2$ $b_1 \leq_2 b_2$

For m∈M, m≠0, and x∈ AXB, where x=(a,b),the bi-monoid multiplication in AXB is defined by

mx = m(a,b) = (ma,mb), where ma and mb are the bi-monoid multiplications in A and B respectively.

In other words if $\Lambda_1$ and $V_1$ are the cap, cup of A and $\Lambda_2$ and $V_2$ are the cap, cup of B, then the cap, cup of AXB denoted by Λ and V are defined by x Λ y = $(a_1,b_1)$ Λ $(a_2,b_2)$ = $(a_1 \Lambda_1 a_2 , b_1 \Lambda_2 b_2)$ and xVy = $(a_1,b_1)$ V $(a_2,b_2)$ = $(a_1 V_1 a_2, b_1 V_2 b_2)$.

**Corollary 2.5.2 :** If $A_1$, $A_2$ ,$A_3$,…..., $A_n$ are Artex spaces over a bi-monoid M, then $A_1$ X $A_2$ X $A_3$ X …..X $A_n$ is also an Artex space over M.

## 3 SubArtex Spaces

**3.1 Definition : SubArtex Space :** Let (A, Λ ,V) be an Artex space over a bi-monoid (M , + , . )

Let S be a nonempty subset of A. Then S is said to be a SubArtex space of A

if (S, Λ ,V) itself is an Artex space over M.

**3.1.0 Examples**

**3.1.1** As in Example2.4.1, Z is an Artex space over W = {0,1,2,3,…..} and W is a subset of Z.

W itself is an Artex space over W under the operations defined in Z

Therefore, W is a SubArtex space of Z.

**3.1.2** As in Example 2.4.2,Q is an Artex space over W = {0,1,2,3,…..} and Z is a subset of Q.

Clearly Z itself is an Artex space over W and therefore Z is a SubArtex space of Q.

W is also a SubArtex space of Q.

**3.1.3** As in Example 2.4.3, R is an Artex space over W = {0,1,2,3,…..} and Q is a subset of R

Clearly Q itself is an Artex space over W and therefore Q is a SubArtex space of R over W .

W is also a SubArtex space of R over W.

**3.1.4** As defined in Example 2.4.6, R, the set of all real numbers is an Artex space over Q' and as in Example 2.4.4, Q, the set of all rational numbers is an Artex space over Q' .

Therefore, Q is a SubArtex space of R over Q'.

**3.1.5** In Examples 2.4.7 and 2.4.8, A is a SubArtex space of B over W.

**3.1.6** In Examples 2.4.7, 2.4.8 and 2.4.10, A and B are SubArtex spaces of D over W.

**3.1.7** In Examples 2.4.9 and 2.4.11, B is a SubArtex space of D over Q'.

**Proposition 3.2.1 :** Let (A, Λ ,V) be an Artex space over a bi-monoid (M , + , . )

Then a nonempty subset S of A is a SubArtex space of A if and only if S is closed under the operations Λ,V and the bi-monoid multiplication in A.

**Proof :** Let (A, Λ ,V) be an Artex space over a bi-monoid (M,+, . )

Let S be a nonempty subset of A.

Suppose S is a SubArtex space of A.

Then S itself is an Artex space over M.

Therefore, S is closed under the operations Λ ,V and the bi-monoid multiplication in A.

Conversely, suppose S is closed under the operations Λ ,V and the bi-monoid multiplication in A.

As an Artex space, (A, Λ ,V) is a lattice.

We know that a non-empty subset S of a lattice (A, Λ ,V) is a sublattice iff S is closed under the operations Λ and V.

By assumption S is closed under the operations Λ and V.

Therefore, (S, Λ ,V) is a sublattice and hence a lattice.

Now, let m ϵ M, m≠0, and a ϵ S

By assumption the bi-monoid multiplication in A is also defined in S.

Therefore, for each m ϵ M, m≠0, and a ϵ S, ma ϵ S.

Now, let m,n ϵ M, m≠0, n≠0, and a, b ϵ S

Since a,b ϵ S,and S ⊆ A, a,b ϵ A.

Since A is an Artex space over M, m,n ϵ M, m≠0, n≠0, and a, b ϵ A,

(i) m(a Λ b) = ma Λ mb

(ii) m(a V b) = ma V mb

(iii) ma Λ na ≤ (m+n)a and ma V na ≤ (m+n)a

(iv) (mn)a = m(na)

(v) 1.a = a

Therefore, (S, Λ ,V) itself is an Artex space over M.

Hence (S, Λ ,V) is a SubArtex space of A.

**Proposition 3.2.2 :** Let (A, Λ ,V) be an Artex space over a bi-monoid (M , + , . )

Then a nonempty subset S of A is a SubArtex space of A if and only if for each m,n ϵ M, m≠0, n≠0, and a, b ϵS,ma Λ nb ϵ S and ma V nb ϵ S

**Proof :** Let (A, Λ ,V) be an Artex space over a bi-monoid (M, + , . )

Let S be a nonempty subset of A.

Suppose S is a SubArtex space of A.

Then S itself is an Artex space over M.

Therefore, for each m,n ϵ M, m≠0, n≠0, and a, b ϵ S, ma ϵ S and nb ϵ S

and hence maΛnb ϵ S and maVnb ϵ S .

Conversely, suppose for each m,nϵM, m≠0, n≠0, and a, b ϵ S, ma Λ nb ϵ S and ma V nb ϵ S.

Since ( M , + , . ) is a bi-monoid, M contains the multiplicative identity, that is, the identity corresponding to the second operation **.** in M .

Let 1 be the identity corresponding to the second operation **.** in M.

Take m=1 and n=1

ma Λ nb ϵ S implies 1.a Λ 1.b ϵ S

Since a,b ϵS,and S ⊆ A, a,bϵA.

Since A is an Artex space over M, 1.a=a and 1.b=b

Therefore,1.a Λ 1.b ϵ S implies a Λ b ϵ S.

Similarly, ma V nb ϵ S implies 1.a V 1.b ϵ S and hence a V b ϵ S.

Therefore, S is closed under the operations Λ ,V

Now, let mϵM, m≠0, aϵS

By assumption, for each m,nϵM, m≠0, n≠0, and a, b ϵS, ma Λ nb ϵ S and ma V nb ϵ S.

Take n=m and b=a

Then ma Λ nb ϵ S implies ma Λ ma ϵ S

Since (A, Λ ,V) is a lattice, every element is idempotent in A, that is a Λ a=a, for all a ϵ A

and hence in S

Therefore, ma Λ ma ϵ S implies maϵ S

That is, for each m ϵ M, m≠0,and a ϵ S, ma ϵ S

Thus S is closed under the operations Λ ,V and the bi-monoid multiplication in A.

By Proposition 3.2.1, S is a SubArtex space of A.

**Proposition 3.2.3 :** A nonempty intersection of two SubArtex spaces of an Artex space A over a bi-monoid M is a SubArtex space of A.

**Proof :** Let (A, Λ ,V) be an Artex space over a bi-monoid (M,+, . )

Let S and T be two SubArtex spaces of A such that S ∩ T is nonempty.

Let B = S ∩ T.

Let m,n ϵ M, m≠0, n≠0, and a, b ϵ B.

a, b ϵ B implies a,b ϵ S and a,b ϵ T.

Since m,n ϵ M, m≠0, n≠0, a, b ϵ S, and S is a SubArtex space of A, by Proposition 3.2.2,

ma Λ nb ϵ S and ma V nb ϵ S

Similarly, since m,nϵM, m≠0, n≠0, a, b ϵT, and T is a SubArtex space of A, by Proposition 3.2.2,

ma Λ nb ϵ T and ma V nb ϵ T

Therefore, ma Λ nb ϵ S∩T = B and ma V nb ϵ S∩T = B

Therefore, by Proposition 3.2.2, B = S ∩ T is a SubArtex space of A.

Hence, a nonempty intersection of two SubArtex spaces is a subartex space.

**Proposition 3.2.4 :** A nonempty intersection of a family of SubArtex spaces of an Artex space A over a bi-monoid M is a SubArtex space of A.

 

**Proof :** Let (A, Λ ,V) be an Artex space over a bi-monoid ( M,+, . )

Let { $S_\alpha$ / α∊J } be a family of SubArtex spaces of A such that ∩ $S_\alpha$ is nonempty.

Let S = ∩ $S_\alpha$

Let m,n ϵ M, m≠0, n≠0, and a, b ϵ S.

a,b ϵ S implies a,b ϵ $S_\alpha$ , for each α∊J.

Since a,b ϵ $S_\alpha$ , for each α∊J, and $S_\alpha$ is a SubArtex space of A, by Proposition 3.2.2,

ma Λ nb ϵ $S_\alpha$ and ma V nb ϵ $S_\alpha$ , for each α∊J

Therefore, ma Λ nb ϵ ∩ $S_\alpha$ and ma V nb ϵ ∩ $S_\alpha$ , for each α∊J

That is, ma Λ nb ϵ S and ma V nb ϵ ∩ S

By Proposition 3.2.2, S = ∩ $S_\alpha$ is a SubArtex space of A

Hence, the nonempty intersection of a family of SubArtex spaces of an Artex space A over a bi-monoid M is a SubArtex space of A.

## 4 Problem

**4.1.1 Problem :** The union of two SubArtex spaces of an Artex space over a bi-monoid need not be a SubArtex space.

**Solution :** Let us prove this result by giving an example

Let R' = $R^+$ ∪ {0}, where $R^+$ is the set of all positive real numbers

and let W = {0,1,2,3,…..}

(R', ≤) is a lattice in which Λ and V are defined by a Λ b = mini {a,b} and a V b = maxi {a,b} , for all a,b ϵ R'.

Clearly for each m ϵ W,m≠0, and for each a∊ R', there exists ma∊ R'.

Also, (i) m(a Λ b) = ma Λ mb

(ii) m(a V b) = ma V mb

(iii) ma Λ na ≤ (m +n)a and ma V na ≤ (m + n)a

(iv) (mn)a = m(na) , for all m,n ϵ W, m≠0, n≠0, and a,b ϵ R'

(v) 1.a = a , for all a ϵ R'

Therefore, R' is an Artex Space Over the bi-monoid ( W , + , . )

Generally, if $\Lambda_1$, $\Lambda_2$, and $\Lambda_3$ are the cap operations of A , B and C respectively, then the cap of AXBXC is denoted by Λ and if $V_1$, $V_2$, and $V_3$ are the cup operations of A , B and C respectively, then the cup of AXBXC is denoted by V

Here, $\Lambda_1$, $\Lambda_2$, and $\Lambda_3$ denote the same meaning minimum of two elements in R' and $V_1$, $V_2$, and $V_3$ denote the same meaning maximum of two elements in R'

 

Now by Proposition 2.5.1 and Corollary 2.5.2 , $R'^3 = R'XR'XR'$ is an Artex over W, where cap and cup operations are denoted by Λ and V respectively

Let S = { (a,0,0) / a∈R' } and Let T = { (0,b,0) / b∈R' }

Claim 1: S is a SubArtex space of $R'^3$

Let m,n∈W, and m≠0,n≠0, and x, y ∈ S, where x = $(a_1,0,0)$ and y = $(a_2,0,0)$, $a_1$, $a_2$∈R'

Now mx Λ ny = $m(a_1,0,0)$ Λ $n(a_2,0,0)$

= $(ma_1,m0,m0)$ Λ $(na_2,n0,n0)$

= $(ma_1,0,0)$ Λ $(na_2,0,0)$

= $(ma_1 \Lambda_1 na_2, 0 \Lambda_2 0, 0 \Lambda_3 0)$

= $( ma_1 \Lambda_1 na_2, 0 ,0 )$

Since m,n∈W, m≠0,n≠0, and $a_1$, $a_2$∈R', and R' is an Artex space over W, $ma_1 \Lambda_1 na_2$ ∈ R'

Therefore, $( ma_1 \Lambda_1 na_2, 0 ,0 )$ ∈ S

That is, mx Λ ny ∈ S

Now mx V ny = $m(a_1,0,0)$ V $n(a_2,0,0)$

= $(ma_1,m0,m0)$ V $(na_2,n0,n0)$

= $(ma_1,0,0)$ V $(na_2,0,0)$

= $(ma_1 V_1 na_2, 0 V_2 0, 0 V_3 0)$

= $( ma_1 V_1 na_2, 0 ,0 )$

Since m,n∈W, m≠0,n≠0, and $a_1$, $a_2$∈R', and R' is an Artex space over W, $ma_1 V_1 na_2$ ∈ R'

Therefore, $( ma_1 V_1 na_2, 0 ,0 )$ ∈ S

That is, mx V ny ∈ S

Therefore by Proposition 3.2.2, S is a SubArtex space of $R'^3$.

Hence Claim 1

Similarly T is a SubArtex space of $R'^3$

Claim : S ∪ T is not a SubArtex space of $R'^3$

Let us take x= (2,0,0) and y = (0,3,0)

Clearly x∈S and y∈T

Therefore, x,y ∈ S ∪ T

But x V y = (2,0,0)V (0,3,0)

= $(2V_1 0 , 0V_2 3 , 0V_3 0)$

$= (2,3,0)$ which does not belong to $S \cup T$

Therefore, $S \cup T$ is not a SubArtex space of $R'^3$

Hence the union of two SubArtex spaces need not be a SubArtex space.

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