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Mixed Poisson Distributions in terms of Special Functions

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Abstract

Mixed Poisson distributions can be expressed in explicit, recursive and expectation forms. It can also be expressed in terms of special functions.

This paper expresses mixed Poisson distributions and their probability generating functions in terms of Confluent Hypergeometric functions and modified Bessel function of the third kind.

Keywords: Mixed Poisson; Confluent Hypergeometric; Bessel funtion; Probability generating function

1 Introduction

The main difficulty with the use of mixed Poisson distributions is that, its probability mass function is difficult to evaluate, except for a few mixing distributions. One way of solving this problem is to express the mixed Poisson distributions in terms of special functions.

In this paper, mixed Poisson distributions and their probability generating functions have been expressed in terms of confluent hypergeometric functions. Beta, gamma and Pareto distributions and their generalizations have been used as mixing distributions. The work has been given in Section 2 where the definition of Confluent hypergeometric has been given and its relations.

Mixed Poisson distributions have also been expressed in terms of modified Bessel function of the third kind. The mixing distributions for this case are: inverse gamma, inverse Gaussian, reciprocal inverse Gaussian and generalised Gaussian distributions. The work is in Section 3 where the modified Bessel function of the third kind has been defined and some properties stated. In Section 4 we have some concluding remarks.

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2 Confluent hypergeometric functions

The confluent hypergeometric function, also known as Kummer's series is defined as:

$$
\begin{aligned}
{}_1F_1(a;c;x) &= 1+\frac{a}{c}\frac{x}{1!}+\frac{a(a+1)}{c(c+1)}\frac{x^2}{2!}+\ldots \\
&= \sum_{n=0}^{\infty}\frac{a(a+1)(a+2)\ldots(a+n-1)}{c(c+1)(c+2)\ldots(c+n-1)}\frac{x^n}{n!} \qquad (1)
\end{aligned}
$$

$$where\ c \neq 0,-1,-2,\ldots$$

An integral representation is

$${}_1F_1(a;c;x)=\frac{1}{B(a,c;x)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{xt}dt \qquad (2)$$

Another confluent hypergeometric (Tricomi) function has the integral representation as

$$\psi(a,c;x)=\frac{1}{\Gamma(a)}\int_0^{\infty} t^{a-1}(1+t)^{c-a-1}e^{-xt}dt \qquad (3)$$

The following relations hold:

$$\psi(a;c;x)=x^{1-c}\psi(a-c+1;2-c;x) \qquad (4)$$

$$\psi(a;c;x)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}{}_1F_1(a;c;x)+\frac{\Gamma(c-1)x^{1-c}}{\Gamma(a)}{}_1F_1(a-c+1;2-c;x) \qquad (5)$$

Incomplete Gamma function is defined as

$$
\begin{aligned}
\gamma(a,x) &= \int_0^x t^{a-1}e^{-t}dt \\
&= a^{-1}x^a e^{-x}{}_1F_1(1;a+1;x) \qquad (6)
\end{aligned}
$$

2.1 Mixing with 2-parameter Beta distribution

2.1.1 Beta I distribution

The probability density function of Beta I distribution is given by

$$g(\lambda)=\frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)},\ 0<\lambda<1;\ \alpha,\beta>0 \qquad (7)$$

Therefore the mixed Poisson distribution is given by

$$\begin{aligned} f(x) &= \int_0^1 \frac{e^{-\lambda t}(\lambda t)^x}{x!}\frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}d\lambda \\ &= \frac{t^x}{x!B(\alpha,\beta)}\int_0^1 \lambda^{x+\alpha-1}(1-\lambda)^{(x+\alpha)+\beta-(x+\alpha)-1}e^{-\lambda t}d\lambda \\ &= \frac{t^x}{x!}\frac{B(x+\alpha,\beta)}{B(\alpha,\beta)}\, {}_1F_1(x+\alpha;x+\alpha+\beta;-t) \end{aligned} \tag{8}$$

$$for\ x = 0, 1, 2, \ldots$$

In terms of probability generating function (pgf), we have

$$\begin{aligned} G_X(s,t) &= \frac{1}{B(\alpha,\beta)}\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\alpha+\beta-\alpha-1}e^{-[t(1-s)]\lambda}d\lambda \\ &= {}_1F_1(\alpha;\alpha+\beta;-t(1-s)) \end{aligned} \tag{9}$$

as obtained by Gurland (1958) and Katti (1966).

2.1.2 Rectangular distribution

The probability density function of a Rectangular distribution is given by

$$g(\lambda) = \frac{1}{b-a};\ a \le \lambda \le b \tag{10}$$

Therefore

$$\begin{aligned} f(x) &= \int_a^b e^{-\lambda t}\frac{(\lambda t)^x}{x!}\frac{d\lambda}{b-a} \\ &= \frac{t^x}{x!(b-a)}\left\{\int_0^b e^{-\lambda t}\lambda^x d\lambda - \int_0^a e^{-\lambda t}\lambda^x d\lambda\right\} \end{aligned}$$

Let

$$y = \lambda t \Longrightarrow dy = t d\lambda \Longrightarrow d\lambda = \frac{dy}{t}$$

Therefore

$$\begin{aligned} f(x) &= \frac{t^x}{x!(b-a)}\left\{\int_0^{bt} e^{-y}\frac{y^x}{t^{x+1}}dy - \int_0^{at} e^{-y}\frac{y^x}{t^{x+1}}dy\right\} \\ &= \frac{1}{x!(b-a)t}\left\{\gamma(x+1,bt)-\gamma(x+1,at)\right\} \\ &= \frac{1}{x!(b-a)t}\left\{\frac{1}{x+1}(bt)^{x+1}\,{}_1F_1(x+1;x+2;bt)\right\} \\ &\quad -\frac{1}{x!(b-a)t}\left\{\frac{1}{x+1}(at)^{x+1}\,{}_1F_1(x+1;x+2;at)\right\} \end{aligned}$$

Therefore,

$$f(x) = \frac{t^x(b^x-a^x)}{(x+1)!(b-a)}\left\{{}_1F_1(x+1;x+2;bt) -{}_1F_1(x+1;x+2;at)\right\} \quad (11)$$

In terms of pgf

$$\begin{aligned} G_X(s,t) &= \int_a^b e^{-\lambda t(1-s)}\frac{d\lambda}{b-a} \\ &= \frac{1}{(b-a)(1-s)t}\left\{e^{-bt(1-s)}-e^{-at(1-s)}\right\} \quad (12) \end{aligned}$$

as obtained by Bhattacharya and Holla (1965) for $t = 1$.

2.1.3 Beta II distribution

The second kind of beta distribution also known as inverted beta distribution is given by

$$g(\lambda) = \frac{\lambda^{p-1}}{B(p,q)(1+\lambda)^{p+q}};\ \lambda > 0;\ p,q > 0 \quad (13)$$

Therefore

$$\begin{aligned} f(x) &= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^x}{x!}\frac{\lambda^{p-1}}{B(p,q)(1+\lambda)^{p+q}}d\lambda \\ &= \frac{t^x}{x!B(p,q)}\int_0^\infty \lambda^{x+p-1}(1+\lambda)^{x+1-q-(x+p)-1}e^{-\lambda t}d\lambda \\ &= \frac{t^x}{x!}\frac{\Gamma(x+p)}{B(p,q)}\psi(x+p,x-q+1,t) \quad (14) \end{aligned}$$

In terms of pgf

$$\begin{aligned} G_X(s,t) &= \frac{1}{B(p,q)}\int_0^{\infty}\lambda^{p-1}(1+\lambda)^{1-q-p-1}e^{-\lambda t(1-s)}d\lambda \\ &= \frac{\Gamma(p)}{B(p,q)}\psi(p,1-q,t(1-s)),\ 0<q<1 \end{aligned} \tag{15}$$

2.2 Mixing with 3-parameter Beta distribution

2.2.1 Scaled Beta distribution

Let the classical Beta (Beta I) be written as

$$w(t)=\frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha,\beta)},\ 0<t<1;\ \alpha,\beta>0$$

By the tranformation

$$t=\frac{\lambda}{\mu}\Longrightarrow\lambda=\mu t\ and\ \frac{dt}{d\lambda}=\frac{1}{\mu}$$

we have the scaled Beta distribution

$$g(\lambda)=\frac{\lambda^{\alpha-1}(\mu-\lambda)^{\beta-1}}{\mu^{\alpha+\beta-1}B(\alpha,\beta)},\ 0<\lambda<\mu;\ \alpha,\beta>0 \tag{16}$$

so that the mixed Poisson distribution becomes

$$f(x)=\frac{t^x}{x!\mu^{\alpha+\beta-1}B(\alpha,\beta)}\int_0^{\mu}e^{-\lambda t}\lambda^{\alpha+x-1}(\mu-\lambda)^{\beta-1}d\lambda$$

Putting

$$\lambda=\mu z\Longrightarrow z=\frac{\lambda}{\mu}\ and\ d\lambda=\mu dz$$

then

$$\begin{aligned} f(x) &= \frac{t^x\mu^x}{x!B(\alpha,\beta)}\int_0^1 z^{\alpha+x-1}(1-z)^{\alpha+x+\beta-(\alpha+x)-1}e^{-\mu z t}dz \\ &= \frac{(\mu t)^x}{x!}\frac{B(\alpha+x,\beta)}{B(\alpha,\beta)}\,{}_1F_1(\alpha+x;\alpha+x+\beta;-\mu t),\ x=0,1,2, \end{aligned} \tag{17}$$

In terms of pgf,

$$G_X(s,t)=\frac{1}{\mu^{\alpha+\beta-1}B(\alpha,\beta)}\int_0^{\mu}\lambda^{\alpha-1}(\mu-\lambda)^{\beta-1}e^{-\lambda t(1-s)}d\lambda$$

putting

$$\lambda = \mu z \Longrightarrow z = \frac{\lambda}{\mu} \; and \; d\lambda = \mu dz$$

gives

$$\begin{aligned} G_X(s,t) &= \frac{\mu^{\alpha-1+\beta-1+1}}{\mu^{\alpha+\beta-1}B(\alpha,\beta)} \int_0^1 z^{\alpha-1}(1-z)^{\alpha+\beta-\alpha-1} e^{-\mu t(1-s)z} dz \\ &= \frac{B(\alpha,\beta)}{B(\alpha,\beta)} \, {}_1F_1(\alpha;\alpha+\beta;-\mu t(1-s)) \qquad (18) \end{aligned}$$

as obtained by Willmot (1986) when $t = 1$.
A more general situation is given by letting

$$y = \frac{\lambda - \sigma}{\mu} \Longrightarrow \lambda = \mu y + \sigma \; and \; \frac{dy}{d\lambda} = \frac{1}{\mu}$$

If

$$g(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}, \; 0 < y < 1; \; \alpha,\beta > 0$$

then

$$g(y) = \frac{(\lambda-\sigma)^{\alpha-1}((\mu+\sigma)-\lambda)^{\beta-1}}{\mu^{\alpha+\beta-1}B(\alpha,\beta)}$$

$$for \; \sigma < \lambda < \sigma + \mu; \; \alpha,\beta,\mu,\sigma > 0$$

therefore

$$f(x) = \frac{t^x}{x!\mu^{\alpha+\beta-1}B(\alpha,\beta)} \int_\sigma^{\sigma+\mu} \lambda^x (\lambda-\sigma)^{\alpha-1} [(\sigma+\mu)-\lambda]^{\beta-1} e^{-\lambda t} d\lambda$$

Putting

$$z = \frac{\lambda-\sigma}{\mu} \Longrightarrow \lambda = \mu z + \sigma \; and \; d\lambda = \mu dz$$

gives

$$\begin{aligned} f(x) &= \frac{t^x}{x!\mu^{\alpha+\beta-1}B(\alpha,\beta)} \int_0^1 (\mu z+\sigma)^x (\mu z)^{\alpha-1} [(\sigma+\mu)-\mu z-\sigma]^{\beta-1} e^{-(\mu z+\sigma)t} \mu dz \\ &= \frac{t^x e^{-\sigma t}}{x!B(\alpha,\beta)} \int_0^1 (\mu z+\sigma)^x z^{\alpha-1}(1-z)^{\beta-1} e^{-\mu z t} dz \\ &= \frac{t^x e^{-\sigma t}}{x!B(\alpha,\beta)} \int_0^1 \left\{ \sum_{k=0}^{x} \binom{x}{k} \sigma^{x-k} (\mu z)^k \right\} z^{\alpha-1}(1-z)^{\beta-1} e^{-\mu z t} dz \\ &= \frac{t^x e^{-\sigma t}}{x!B(\alpha,\beta)} \sum_{k=0}^{x} \left\{ \binom{x}{k} \sigma^{x-k} \mu^k \, {}_1F_1(\alpha+k;\alpha+k+\beta;-\mu t) \right\} \qquad (19) \end{aligned}$$

In terms of pgf,

$$G_X(s,t) = \frac{1}{\mu^{\alpha+\beta-1}B(\alpha,\beta)} \int_{\sigma}^{\sigma+\mu} e^{-\lambda t(1-s)} (\lambda-\sigma)^{\alpha-1} [(\mu+\sigma)-\lambda]^{\beta-1} d\lambda$$

Putting

$$z = \frac{\lambda-\sigma}{\mu} \Longrightarrow dz = \mu d\lambda \; and \; \lambda = \mu z + \sigma$$

we have

$$\begin{aligned} G_X(s,t) &= \frac{e^{-\sigma t(1-s)}}{B(\alpha,\beta)} \int_0^1 z^{\alpha-1} (1-z)^{\alpha+\beta-\alpha-1} e^{-\mu t(1-s)z} dz \\ &= e^{-\sigma t(1-s)} {}_1F_1(\alpha; \alpha+\beta; -\mu t(1-s)) \end{aligned} \quad (20)$$

When $\alpha = t = 1$, we have

$$G_X(s) = e^{\sigma(s-1)} {}_1F_1(1; 1+\beta; \mu(s-1)) \quad (21)$$

as obtained by Willmot (1986).

2.2.2 The Full Beta Model

Kempton (1975) mixed two Gamma distributions to obtain what he called Full Beta model given by

$$\begin{aligned} g(\lambda) &= \int_0^\infty \frac{a^p}{\Gamma(p)} e^{-a\lambda} \lambda^{p-1} \cdot \frac{1}{b^q \Gamma(q)} e^{-\frac{a}{b}} a^{q-1} da \\ &= \frac{b^p}{B(p,q)} \frac{\lambda^{p-1}}{(1+b\lambda)^{p+q}}, \; \lambda > 0; b, p, q > 0 \end{aligned} \quad (22)$$

This distribution can also be obtained by the following transformation: From the Beta II *pdf* given by

$$g(y) = \frac{y^{p-1}}{B(p,q)(1+y)^{p+q}}; \; y > 0; \; p, q > 0$$

Let

$$y = b\lambda \Longrightarrow \frac{dy}{d\lambda} = b$$

therefore

$$g(\lambda) = \frac{b^p}{B(p,q)} \frac{\lambda^{p-1}}{(1+b\lambda)^{p+q}}; \; y > 0; \; b, p, q > 0$$

Therefore the mixed Poisson distribution is given by,

$$f(x) = \frac{b^p}{B(p,q)} \frac{t^x}{x!} \int_0^\infty e^{-\lambda t} \lambda^{x+p-1} (1+b\lambda)^{-p-q} \, d\lambda$$

letting

$$z = b\lambda \Longrightarrow \lambda = \frac{z}{b} \; and \; d\lambda = \frac{dz}{b}$$

we get

$$\begin{aligned} f(x) &= \frac{b^p t^x}{B(p,q)\, x!} \frac{1}{b^{x+p}} \int_0^\infty z^{x+p-1} (1+z)^{x+1-q-(x+p)-1} e^{-\frac{t}{b}z} dz \\ &= \frac{b^p t^x}{B(p,q)\, x!} \frac{\Gamma(x+p)}{b^{x+p}} \psi\left(x+p, x+1-q; \frac{t}{b}\right) \end{aligned}$$

Using the relation (4); i.e.,

$$\psi(a,c;x) = x^{1-c} \psi(1+a-c, 2-c; x)$$

we obtain

$$f(x) = \left(\frac{t}{b}\right)^q \frac{\Gamma(x+p)}{B(p,q)\, x!} \psi\left(p+q, 1-x+q; \frac{t}{b}\right) \; for \; x = 0,1,2,\ldots \quad (23)$$

as given by Gupta and Ong (2005) when $t = 1$.
In terms of pgf

$$G_X(s,t) = \frac{b^p}{B(p,q)} \int_0^\infty \lambda^{p-1} (1+b\lambda)^{-p-q} e^{-\lambda t(1-s)} d\lambda$$

Putting

$$z = b\lambda \Longrightarrow \lambda = \frac{z}{b} \; and \; d\lambda = \frac{dz}{b}$$

therefore

$$\begin{aligned} G_X(s,t) &= \frac{1}{B(p,q)} \int_0^\infty z^{p-1} (1+z)^{1-p-q-1} e^{-\frac{t}{b}(1-s)z} dz \\ &= \frac{\Gamma(p)}{B(p,q)} \psi\left(p, 1-q; \frac{t}{b}(1-s)\right) \end{aligned} \quad (24)$$

2.3 Mixing with 4-parameter Beta distributions

Pearson Type I and VI distributions are cases of 4-parameter Beta distributions.

2.3.1 Pearson Type I distribution

The pdf of Pearson Type I mixing distribution is given by:

$$g(\lambda) = \frac{1}{B(p,q)} \frac{(\lambda - a)^{p-1}}{(b-a)^{p-1}} \frac{(b-\lambda)^{q-1}}{(b-a)^{q-1}} \frac{1}{b-a} \tag{25}$$

Therefore the mixed Poisson distribution is as follows

$$\begin{aligned} f_x(t) &= \int_a^b e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{1}{B(p,q)} \frac{(\lambda - a)^{p-1}}{(b-a)^{p-1}} \frac{(b-\lambda)^{q-1}}{(b-a)^{q-1}} \frac{d\lambda}{b-a} \\ &= \frac{t^x}{x! B(p,q)} \int_a^b e^{-\lambda t} \lambda^x \left(\frac{\lambda - a}{b-a} \right)^{p-1} \left[1 - \frac{\lambda - a}{b-a} \right]^{q-1} \frac{d\lambda}{b-a} \end{aligned}$$

Let

$$z = \frac{\lambda - a}{b-a} \Longrightarrow \lambda = a + (b-a)z \ and \ d\lambda = (b-a)\,dz$$

Therefore

$$\begin{aligned} f_x(t) &= \frac{t^x e^{-at}}{x! B(p,q)} \int_0^1 e^{-(b-a)tz} [a + (b-a)z]^x z^{p-1} (1-z)^{q-1} dz \\ &= \frac{t^x e^{-at}}{x! B(p,q)} \int_0^1 e^{-(b-a)tz} \left[\sum_{k=0}^{x} \binom{x}{k}^k a^{x-k} (b-a)^k z^k \right] z^{p-1} (1-z)^{q-1} dz \\ &= \frac{t^x e^{-at}}{x!} \sum_{k=0}^{x} \left\{ \binom{x}{k}^k a^{x-k} (b-a)^k \frac{B(k+p,q)}{B(p,q)} \, {}_1F_1(k+p; k+p+q; -(b-a)t) \right\} \\ &= \frac{(at)^x e^{-at}}{x!} \frac{\Gamma(p+q)}{\Gamma(p)} \\ & \quad \sum_{k=0}^{x} \binom{x}{k} \left(\frac{b-a}{a} \right)^k \frac{\Gamma(k+q)}{\Gamma(k+p+q)} {}_1F_1(k+p; k+p+q; -(b-a)t) \end{aligned} \tag{26}$$

When $a = 0$,

$$f_x(t) = \frac{(bt)^x}{x! B(p,q)} B(x+p,q)_1 F_1(x+p, x+p+q; -bt)$$

Both $f_x(t)$ and $f_0(t)$ are as given by Albrecht (1984).
In terms of pgf

$$\begin{aligned} G_X(s,t) &= \int_0^1 e^{-[a+(b-a)z]t(1-s)} z^{p-1} (1-z)^{q-1} dz \\ &= e^{-at(1-s)} B(p,q)_1 F_1(p; p+q; -(b-a)t(1-s)) \end{aligned}$$

2.3.2 Pearson Type VI distribution

The pdf of Pearson Type VI mixing distribution is given by

$$g(\lambda) = \frac{\left(\frac{\lambda-d}{d-c}\right)^{b-a-1} \frac{1}{d-c}}{B(a, b-a)\left(1+\frac{\lambda-d}{d-c}\right)^{b}}, \ \lambda > d \tag{27}$$

Therefore the mixed Poisson distribution is given by

$$f_x(t) = \int_d^\infty e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{\left(\frac{\lambda-d}{d-c}\right)^{b-a-1} \frac{1}{d-c}}{B(a, b-a)\left(1+\frac{\lambda-d}{d-c}\right)^{b}} d\lambda$$

putting

$$z = \frac{\lambda - d}{d-c} \Longrightarrow \lambda = d + (d-c)\, z \ and\ d\lambda = (d-c)\, dz$$

we get

$$\begin{aligned} f_x(t) &= \frac{t^x e^{-dt}}{x! B(a, b-a)} \int_0^\infty (d + (d-c)\, z)^x z^{b-a-1} (1+z)^{-b} e^{-(d-c)tz} dz \\ &= \frac{t^x e^{-dt}}{x! B(a, b-a)} \int_0^\infty \left\{ \sum_{k=0}^{x} \binom{x}{k} d^{x-k} (d-c)^k z^{k+b-a-1} \right\} (1+z)^{-b} e^{-(d-c)tz} dz \\ &= \frac{(dt)^x e^{-dt}}{x! B(a, b-a)} \\ & \sum_{k=0}^{x} \left\{ \binom{x}{k} \left(\frac{d-c}{d}\right)^k \Gamma(k+b-a)\, \psi(k+b-a, k-a+1; (d-c)\, t) \right\} \end{aligned} \tag{28}$$

Therefore when $x = 0$ we have:

$$\begin{aligned} f_0(t) &= \frac{e^{-dt}}{B(a, b-a)} \Gamma(b-a)\, \psi(b-a, 1+a; (d-c)\, t) \\ &= \frac{e^{-dt}\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \Gamma(b-a)\, \psi(b-a, 1+a; (d-c)\, t) \\ &= e^{-dt} \frac{\Gamma(b)}{\Gamma(a)} \psi(b-a, 1+a; (d-c)\, t) \end{aligned}$$

Both $f_x(t)$ and $f_0(t)$ are given by Albretch (1984) who states that the result are neater than those of Philipson $(1960,\ p\ 152)$.

In terms of pgf,

$$\begin{aligned}
G_X(s,t) &= \int_d^\infty \frac{e^{-\lambda t(1-s)}\left(\frac{\lambda-d}{d-c}\right)^{b-a-1}}{B(a,b-a)\left(1+\frac{\lambda-d}{d-c}\right)^b}\frac{d\lambda}{d-c} \\
&= \frac{e^{-dt(1-s)}}{B(a,b-a)}\int_0^\infty z^{b-a-1}(1+z)^{-a+1-(b-a)-1}e^{-(d-c)t(1-s)z}dz \\
&= \frac{e^{-dt(1-s)}}{B(a,b-a)}\Gamma(b-a)\,\psi(b-a,1-a;(d-c)\,t\,(1-s)) \qquad (29)
\end{aligned}$$

2.4 Mixing with Gamma distributions

2.4.1 Shifted Gamma (Pearson Type III) distribution

The pdf of a Shifted Gamma distribution is given by:

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta(\lambda-\mu)}(\lambda-\mu)^{\alpha-1};\ \lambda>\mu>0;\ \alpha,\beta>0 \qquad (30)$$

The Poisson - Shifted Gamma distribution is therfore given by

$$\begin{aligned}
f_x(t) &= \frac{t^x}{x!}\frac{\beta^\alpha}{\Gamma(\alpha)}\int_\mu^\infty e^{-\lambda t}\lambda^x e^{-\beta(\lambda-\mu)}(\lambda-\mu)^{\alpha-1}d\lambda \\
&= \frac{t^x\beta^\alpha e^{-\mu t}}{x!\Gamma(\alpha)}\int_\mu^\infty \lambda^x(\lambda-\mu)^{\alpha-1}e^{-(t+\beta)(\lambda-\mu)}d\lambda
\end{aligned}$$

Putting

$$z=\lambda-\mu \Longrightarrow \lambda = z+\mu \ and \ d\lambda = dz$$

we obtain

$$f_x(t) = \frac{t^x\beta^\alpha e^{-\mu t}}{x!\Gamma(\alpha)}\int_0^\infty (z+\mu)^x z^{\alpha-1}e^{-(t+\beta)z}dz$$

Next, letting

$$z=\mu y \Longrightarrow dz = \mu dy$$

we get

$$
\begin{aligned}
f_x(t) &= \frac{t^x \beta^\alpha e^{-\mu t}}{x!\Gamma(\alpha)} \int_0^\infty \mu^x (1+y)^x \mu^{\alpha-1} y^{\alpha-1} e^{-(t+\beta)\mu y} \mu dy \\
&= \frac{t^x \beta^\alpha e^{-\mu t} \mu^{x+\alpha}}{x!\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} (1+y)^{\alpha+x+1-\alpha-1} e^{-(t+\beta)\mu y} dy \\
&= \frac{(\mu t)^x (\mu\beta)^\alpha e^{-\mu t}}{x!\Gamma(\alpha)} \psi(\alpha, \alpha+x+1; (t+\beta)\mu) \qquad (31)
\end{aligned}
$$

as obtained by Rolski *et al* $(1999, p\ 372)$. Using the relation (4),Albretcht (1984) obtained

$$
f_x(t) = \frac{t^x \beta^\alpha e^{-\mu t}}{x!} [(t+\beta)]^{-(\alpha+x)} \psi(-x, 1-\alpha-x, (t+\beta)\mu) \qquad (32)
$$

In terms of pgf

$$
\begin{aligned}
G_X(s,t) &= \int_\mu^\infty e^{-\lambda t(1-s)} \frac{\beta^a}{\Gamma(\alpha)} e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1} d\lambda \\
&= \frac{\beta^a}{\Gamma(\alpha)} e^{-\mu t(1-s)} \int_\mu^\infty (\lambda-\mu)^{\alpha-1} e^{-(\lambda-\mu)[t(1-s)+\beta]} d\lambda \\
&= \frac{\beta^a}{\Gamma(\alpha)} e^{-\mu t(1-s)} \int_0^\infty z^{\alpha-1} (1+z)^{\alpha+1-\alpha-1} e^{-z[t(1-s)+\beta]} dz \\
&= \frac{\beta^a}{\Gamma(\alpha)} e^{-\mu t(1-s)} \Gamma(\alpha) \psi(\alpha, \alpha+1; t(1-s)+\beta) \qquad (33)
\end{aligned}
$$

by letting

$$
z = \lambda - \mu.
$$

2.4.2 Gamma truncated from above

A two-parameter Gamma *pdf* is given by

$$
h(y) = \frac{a^b}{\Gamma(b)} e^{-ay} y^{b-1};\ y > 0;\ a, b > 0 \qquad (34)
$$

Let us consider the integral

$$
I = \int_0^p e^{-ay} y^{b-1} dy
$$

put

$$x = ay \Longrightarrow y = \frac{x}{a} \; and \; dy = \frac{dx}{a}$$

so that

$$\begin{aligned} I &= \int_0^{ap} e^{-x}\left(\frac{x}{a}\right)^{b-1}\frac{dx}{a} \\ &= \frac{1}{a^b}\int_0^{ap} e^{-x}x^{b-1}dx \\ &= \frac{1}{a^b}\gamma(b, ap) \end{aligned}$$

Therefore

$$\frac{a^b}{\Gamma(b)}\int_0^p e^{-ay}y^{b-1}dy = \frac{a^b}{\Gamma(b)}\frac{1}{a^b}\gamma(b, ap) = \frac{\gamma(b, ap)}{\Gamma(b)}$$

where $\gamma(b, ap)$ is a truncated Gamma. Therefore,

$$\begin{aligned} \frac{a^b}{\Gamma(b)}\frac{\Gamma(b)}{\gamma(b, ap)}\int_0^p e^{-ay}y^{b-1}dy &= 1 \\ \int_0^p \frac{a^b}{\gamma(b, ap)}e^{-ay}y^{b-1}dy &= 1 \end{aligned}$$

which also implies that

$$\int_0^p e^{-ay}y^{b-1}dy = \frac{\gamma(b, ap)}{a^b}$$

Thus the mixing distribution to be considered is the Gamma truncated from above which is given by:

$$g(\lambda) = \frac{a^b}{\gamma(b, ap)}e^{-a\lambda}\lambda^{b-1}, \; 0 < \lambda < p; \; a, b > 0 \tag{35}$$

The corresponding mixed Poisson distribution is

$$\begin{aligned} f_x(t) &= \frac{t^x a^b}{x!\gamma(b, ap)}\int_0^p e^{-\lambda(t+a)}\lambda^{x+b-1}d\lambda \\ &= \frac{t^x a^b}{x!\gamma(b, ap)}\frac{\gamma(x+b, (t+a)p)}{(t+a)^{x+b}} \\ &= \frac{(pt)^x (ap)^b}{x!(pt+ap)^{x+b}}\frac{\gamma(x+b, (pt+ap))}{\gamma(b, ap)} \end{aligned}$$

But

$$\gamma(a,x) = a^{-1}x^a{}_1F_1(a;a+1;-x)$$

therefore

$$\begin{aligned} f_x(t) &= \frac{(pt)^x(ap)^b}{x!(pt+ap)^{x+b}}\frac{(x+b)^{-1}(pt+ap)^{x+b}{}_1F_1(x+b;x+b+1;-pt-ap)}{b^{-1}(ap)^b{}_1F_1(b;b+1;-ap)} \\ &= \frac{(pt)^x}{x!}\frac{b}{b+x}\frac{{}_1F_1(x+b;x+b+1;-pt-ap)}{{}_1F_1(b;b+1;-ap)} \end{aligned} \tag{36}$$

If we use the relation (6), then

$$f_x(t) = \frac{(pt)^x(ap)^b}{x!(pt+ap)^{x+b}}\frac{(x+b)^{-1}(pt+ap)^{x+b}}{b^{-1}(ap)^b}\frac{e^{-pt-ap}{}_1F_1(1;x+b+1;pt+ap)}{e^{-ap}{}_1F_1(1;b+1;ap)} \tag{37}$$

as given by Johnson, Kemp and Kotz (2005).
In terms of pgf

$$\begin{aligned} G_X(s,t) &= \int_0^p e^{-\lambda t(1-s)}\frac{a^be^{-a\lambda}\lambda^{b-1}}{\gamma(b,ap)}d\lambda \\ &= \frac{a^b}{[t(1-s)+a]^b}\frac{\gamma(b,(t(1-s)+a)p)}{\gamma(b,ap)} \\ &= \frac{{}_1F_1(b;b+1;-pt(1-s)-ap)}{b^{-1}(ap)^b{}_1F_1(-ap)} \end{aligned} \tag{38}$$

also given by Johnson, Kemp and Kotz (2005).

2.4.3 Gamma truncated from below

A Gamma distribution with two parameters α and β has pdf

$$h(y) = \frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta y}y^{\alpha-1}, \; y>0; \; \alpha,\beta>0$$

Therefore

$$\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-\beta y}y^{\alpha-1}dy = 1$$

Therefore

$$\begin{aligned} \frac{\beta^\alpha}{\Gamma(\alpha)}\left\{\int_0^{\lambda_0} e^{-\beta y}y^{\alpha-1}dy + \int_{\lambda_0}^\infty e^{-\beta y}y^{\alpha-1}dy\right\} &= 1 \\ \frac{\gamma(\alpha,\beta\lambda_0)}{\Gamma(\alpha)} + \frac{\beta^\alpha}{\Gamma(\alpha)}\int_{\lambda_0}^\infty e^{-\beta y}y^{\alpha-1}dy &= 1 \end{aligned}$$

therefore

$$\begin{aligned}\int_{\lambda_0}^{\infty} e^{-\beta y} y^{\alpha-1} dy &= \frac{\Gamma(\alpha)}{\beta^\alpha} - \frac{\gamma(\alpha, \beta\lambda_0)}{\beta^\alpha} \\ &= \frac{1}{\beta^\alpha}\{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)\}\end{aligned}$$

which implies that

$$\int_{\lambda_0}^{\infty} \frac{\beta^\alpha e^{-\beta y} y^{\alpha-1} dy}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} = 1$$

Therefore the pdf of a Gamma distribution truncated from below is given by:

$$g(\lambda) = \frac{\beta^\alpha e^{-\beta\lambda} \lambda^{\alpha-1}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)}, \ \lambda > \lambda_0 \tag{39}$$

so that the mixed distribution becomes

$$\begin{aligned}f_x(t) &= \int_{\lambda_0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{\beta^\alpha e^{-\beta\lambda} \lambda^{\alpha-1}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} d\lambda \\ &= \frac{1}{x!}\left(\frac{\beta}{t+\beta}\right)^\alpha \left(\frac{t}{t+\beta}\right)^x \frac{\Gamma(\alpha+x) - \gamma(\alpha+x, (t+\beta)\lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \quad (40)\end{aligned}$$

In terms of pgf, we have;

$$\begin{aligned}G_X(s,t) &= \int_{\lambda_0}^{\infty} e^{-\lambda t(1-s)} \frac{\beta^\alpha e^{-\beta\lambda} \lambda^{\alpha-1}}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} d\lambda \\ &= \left(\frac{\beta}{\beta + t(1-s)}\right)^\alpha \frac{\Gamma(\alpha) - \gamma(\alpha, [t(1-s)+\beta]\lambda_0)}{\Gamma(\alpha) - \gamma(\alpha, \beta\lambda_0)} \quad (41)\end{aligned}$$

2.4.4 Gamma truncated from both sides

Consider the integral

$$\begin{aligned}\int_a^b e^{-\beta y} y^{\alpha-1} dy &= \int_0^b e^{-\beta y} y^{\alpha-1} dy - \int_0^a e^{-\beta y} y^{\alpha-1} dy \\ &= \frac{\gamma(\alpha, \beta b)}{\beta^\alpha} - \frac{\gamma(\alpha, \beta a)}{\beta^\alpha}\end{aligned}$$

Therefore

$$\int_a^b \beta^\alpha e^{-\beta y} y^{\alpha-1} dy = \gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)$$

Therefore

$$\int_a^b \frac{\beta^\alpha e^{-\beta y} y^{\alpha-1}}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} dy = 1$$

Hence the pdf of Gamma distribution truncated from both sides is given by;

$$g(\lambda) = \frac{\beta^\alpha e^{-\beta\lambda} \lambda^{\alpha-1}}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)}, \quad 0 < a < \lambda < b < \infty; \ \alpha, \beta > 0 \tag{42}$$

The Poisson - Gamma truncated from both sides is thus given by;

$$\begin{aligned} f_x(t) &= \int_a^b e^{-\lambda t} \frac{(\lambda t)^x}{x!} \frac{\beta^\alpha e^{-\beta\lambda} \lambda^{\alpha-1}}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} d\lambda \\ &= \frac{t^x \beta^\alpha}{x! [\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)]} \int_a^b \lambda^{x+\alpha-1} e^{-\lambda(t+\beta)} d\lambda \\ &= \frac{1}{x!} \left(\frac{t}{t+\beta}\right)^x \left(\frac{\beta}{t+\beta}\right)^\alpha \\ &\quad \frac{\gamma(x+\alpha, (t+\beta) b) - \gamma(x+\alpha, (t+\beta) a)}{[\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)]} \end{aligned} \tag{43}$$

In terms of pgf

$$\begin{aligned} G_X(s,t) &= \int_a^b e^{-\lambda t(1-s)} \frac{\beta^\alpha e^{-\beta\lambda} \lambda^{\alpha-1}}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)} d\lambda \\ &= \left[\frac{\beta}{\beta + t(1-s)}\right]^\alpha \\ &\quad \left\{\frac{\gamma(\alpha, [\beta + t(1-s)] b) - \gamma(\alpha, [\beta + t(1-s)] a)}{\gamma(\alpha, \beta b) - \gamma(\alpha, \beta a)}\right\} \end{aligned} \tag{44}$$

2.4.5 Truncated Pearson Type III distribution

The Pearson differential equation is given by

$$\frac{1}{y} \frac{dy}{dx} = -\frac{a+x}{c_0 + c_1 x + c_2 x^2}$$

where

$$y = f(x)$$

is a probability distribution function.

Pearson Type III corresponds to the case of $c_2 = 0$ and $c_1 \neq 0$.

Therefore

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= -\frac{x+a}{c_1x+c_0} \\ &= -\frac{1}{c_1}\left[\frac{x+a}{x+\frac{c_0}{c_1}}\right] \\ &= -\frac{1}{c_1}+\frac{\frac{c_0}{c_1}-a}{c_1x+c_0}\end{aligned}$$

Therefore

$$\begin{aligned}\int\frac{dy}{y} &= \int\left[-\frac{1}{c_1}+\frac{\frac{c_0}{c_1}-a}{c_1x+c_0}\right]dx \\ \log y &= -\frac{x}{c_1}+\left(c_0c_1^{-1}-a\right)c_1^{-1}\log\left(c_1x+c_0\right)+\log K \\ &= -\frac{x}{c_1}+\log\left(c_1x+c_0\right)^m+\log K\end{aligned}$$

where

$$m=c_1^{-1}\left(c_0c_1^{-1}-a\right)$$

Therefore

$$\begin{aligned}\log y &= \log e^{-\frac{x}{c_1}}+\log\left(c_1x+c_0\right)^m+\log K \\ y &= Ke^{-\frac{x}{c_1}}\left(c_1x+c_0\right)^m,\ c_1\neq 0\end{aligned}$$

If $c_1>0$,then $c_1x+c_0>0$ implies $x>-\frac{c_0}{c_1}$.
If $c_1<0$,let $c_1=-\delta$ where $\delta>0$ then $c_1x+c_0>0$ which implies that

$$-\delta x+c_0>0\Longrightarrow-\delta x>-c_0\Longrightarrow\delta x<c_0\Longrightarrow x<\frac{c_0}{\delta}$$

so that

$$c_1<0\Longrightarrow x<-\frac{c_0}{c_1}$$

a case we want to consider.
Therefore

$$\begin{aligned}y &= Ke^{-\frac{x}{c_1}}\left(c_1x+c_0\right)^m \\ &= Ke^{\frac{x}{\delta}}\left(c_0-\delta x\right)^m \\ &= K\delta^m e^{\frac{x}{\delta}}\left(\frac{c_0}{\delta}-x\right)^m\end{aligned}$$

Putting

$$\frac{c_0}{\delta}=1\ and\ \alpha=\frac{1}{\delta}$$

we obtain

$$y = \frac{K}{\alpha^m} e^{\alpha x} (1-x)^m, \; x < 1$$

But

$$y = f(x)$$

Therefore

$$\begin{aligned} \int_0^1 f(x)\, dx &= \frac{K}{\alpha^m} \int_0^1 e^{\alpha x} (1-x)^m \, dx \\ 1 &= \frac{K}{\alpha^m} \int_0^1 e^{\alpha x} (1-x)^m \, dx \end{aligned}$$

Consider the integral

$$\begin{aligned} \int_0^1 e^{\alpha x} (1-x)^m \, dx &= \int_0^1 x^{1-1} (1-x)^{2+m-1-1} e^{\alpha x} dx \\ &= B(1, m+1)_1 F_1 (1; m+2; \alpha) \end{aligned}$$

Putting

$$\beta = m+2 \Longrightarrow \beta - 1 = m+1 \; and \; m = \beta - 2$$

we get

$$\int_0^1 e^{\alpha x} (1-x)^{\beta-2} dx = B(1, \beta-1)_1 F_1 (1; \beta; \alpha)$$

therefore

$$\int_0^1 \frac{(1-x)^{\beta-2} e^{\alpha x}}{B(1, \beta-1)_1 F_1 (1; \beta; \alpha)} dx = 1$$

Thus the mixing distribution (Truncated Pearson Type III) under consideration is

$$g(\lambda) = \frac{(1-\lambda)^{\beta-2} e^{\alpha\lambda}}{B(1, \beta-1)_1 F_1 (1; \beta; \alpha)}, \; 0 < \lambda < 1 \tag{45}$$

Therefore the mixed Poisson distribution is given by;

$$
\begin{aligned}
f_x(t) &= \int_0^1 e^{-\lambda t}\frac{(\lambda t)^x}{x!}\frac{(1-\lambda)^{\beta-2}e^{\alpha\lambda}}{B(1,\beta-1)\,{}_1F_1(1;\beta;\alpha)}d\lambda \\
&= \frac{t^x}{x!B(1,\beta-1)\,{}_1F_1(1;\beta;\alpha)}\int_0^1 \lambda^{(x+1)-1}(1-\lambda)^{x+\beta-(x+1)-1}e^{(\alpha-t)\lambda}d\lambda \\
&= \frac{t^x}{x!}\frac{B(x+1;\beta-1)}{B(1,\beta-1)}\frac{{}_1F_1(x+1;x+\beta;\alpha-t)}{{}_1F_1(1;\beta;\alpha)} \\
&= t^x\frac{\Gamma(\beta)}{\Gamma(x+\beta)}\frac{{}_1F_1(x+1;x+\beta;\alpha-t)}{{}_1F_1(1;\beta;\alpha)} \qquad (46)
\end{aligned}
$$

In terms of pgf

$$
\begin{aligned}
G_X(s,t) &= \int_0^1 e^{-\lambda t(1-s)}\frac{(1-\lambda)^{\beta-2}e^{\alpha\lambda}}{B(1,\beta-1)\,{}_1F_1(1;\beta;\alpha)}d\lambda \\
&= \frac{1}{B(1,\beta-1)\,{}_1F_1(1;\beta;\alpha)}\int_0^1 \lambda^{1-1}(1-\lambda)^{\beta-1-1}e^{[\alpha-t(1-s)]\lambda}d\lambda \\
&= \frac{{}_1F_1(1;\beta;\alpha-t+ts)}{{}_1F_1(1;\beta;\alpha)} \qquad (47)
\end{aligned}
$$

When $\alpha = t$, we have

$$G_X(s,t) = \frac{{}_1F_1(1;\beta;ts)}{{}_1F_1(1;\beta;\alpha)}$$

as given in Johnson, Kemp and Kotz (2005, p 370, eqn (8.82)).

2.5 Mixing with Pareto distribution

2.5.1 Pareto I distribution

Let

$$g(\lambda) = \frac{\alpha\beta^\alpha}{\lambda^{\alpha+1}},\ \lambda > \beta > 0;\ \alpha > 0 \qquad (48)$$

This is the Pareto distribution, sometimes called Pareto of the first kind. Willmot (1993) calls it Shifted Pareto. The mixed Poisson distribution becomes

$$f_x(t) = \frac{\alpha t^x\beta^\alpha}{x!}\int_\beta^\infty e^{-\lambda t}\lambda^{x-\alpha-1}d\lambda$$

Let

$$\lambda = z + \beta \Longrightarrow z = \lambda - \beta \ and \ d\lambda = dz$$

Therefore

$$f_x(t) = \frac{\alpha t^x \beta^\alpha e^{-\beta t}}{x!} \int_0^\infty (z+\beta)^{x-\alpha-1} e^{-zt} dz$$

Putting

$$z = \beta y \Longrightarrow dz = \beta dy$$

we get

$$\begin{aligned} f_x(t) &= \frac{\alpha t^x \beta^\alpha e^{-\beta t}}{x!} \int_0^\infty \beta^{x-\alpha-1} (y+1)^{x-\alpha-1} e^{-\beta y t} \beta dy \\ &= \frac{\alpha (t\beta)^x e^{-\beta t}}{x!} \int_0^\infty y^{1-1} (y+1)^{x-\alpha+1-1-1} e^{-\beta y t} dy \\ &= \frac{\alpha (t\beta)^x e^{-\beta t}}{x!} \psi(1, x-\alpha+1; \beta t) \qquad (49) \end{aligned}$$

In terms of pgf

$$G_X(s,t) = \alpha\beta^\alpha \int_\beta^\infty \lambda^{-\alpha-1} e^{-\lambda t(1-s)} d\lambda$$

Putting

$$\lambda = z + \beta \Longrightarrow z = \lambda - \beta \ and \ dz = d\lambda$$

gives

$$G_X(s,t) = \alpha\beta^\alpha e^{-\beta t(1-s)} \int_0^\infty (z+\beta)^{-\alpha-1} e^{-t(1-s)z} dz$$

Putting

$$z = \beta y \Longrightarrow dz = \beta dy$$

$$\begin{aligned} G_X(s,t) &= \alpha\beta^\alpha e^{-\beta t(1-s)} \int_0^\infty \beta^{-\alpha-1} (1+y)^{-\alpha-1} e^{-t(1-s)\beta y} \beta dy \qquad (50) \\ &= \alpha e^{-\beta t(1-s)} \int_0^\infty y^{1-1} (1+y)^{1-\alpha-1-1} e^{-t(1-s)\beta y} dy \qquad (51) \\ &= \alpha e^{-\beta t(1-s)} \psi(1, 1-\alpha; \beta t(1-s)) \qquad (52) \end{aligned}$$

2.5.2 Pareto II (Lomax) distribution

The Pareto II (Lomax) distribution *pdf* is given by

$$g(\lambda) = \frac{\alpha\beta^{\alpha}}{(\lambda+\beta)^{\alpha+1}};\ \lambda > 0;\ \alpha, \beta > 0 \tag{53}$$

The mixed Poisson distribution becomes

$$f_x(t) = \frac{t^x}{x!}\alpha\beta^{\alpha}\int_0^{\infty}\lambda^x(\lambda+\beta)^{-\alpha-1}e^{-\lambda t}d\lambda$$

Let

$$\lambda = \beta u \Longrightarrow d\lambda = \beta du$$

Therefore

$$\begin{aligned} f_x(t) &= \frac{t^x}{x!}\alpha\beta^{\alpha}\int_0^{\infty}\beta^x u^x \beta^{-\alpha}(1+u)^{-\alpha-1}e^{-\beta tu}du \\ &= \frac{t^x}{x!}\alpha\beta^{x}\int_0^{\infty}u^{x+1-1}(1+u)^{1+x-\alpha-(x+1)-1}e^{-\beta tu}du \\ &= \alpha(\beta t)^x\psi(x+1, x-\alpha+1; \beta t) \end{aligned} \tag{54}$$

In terms of pgf,

$$\begin{aligned} G_X(s,t) &= \alpha\beta^{\alpha}\int_0^{\infty}(\lambda+\beta)^{-\alpha-1}e^{-\lambda t(1-s)}d\lambda \\ &= \alpha\beta^{\alpha}\int_0^{\infty}(\beta u+\beta)^{-\alpha-1}e^{-\beta ut(1-s)}\beta du \\ &= \alpha\int_0^{\infty}u^{1-1}(1+u)^{1-\alpha-1-1}e^{-\beta t(1-s)u}du \\ &= \alpha\psi(1, 1-\alpha; \beta t(1-s)) \end{aligned} \tag{55}$$

2.5.3 Generalized Pareto distribution

The pdf of a Generalized Pareto distribution also known as Gamma- Gamma is given by

$$\begin{aligned} g(\lambda) &= \int_0^{\infty}\frac{k^{\beta}}{\Gamma(\beta)}e^{-k\lambda}\lambda^{\beta-1}\frac{\mu^{\alpha}}{\Gamma(\alpha)}e^{-\mu k}k^{\alpha-1}dk \\ &= \frac{\mu^{\alpha}\lambda^{\beta-1}}{B(\alpha,\beta)(\lambda+\mu)^{\alpha+\beta}};\ \lambda > 0;\ \alpha, \beta, \mu > 0 \end{aligned} \tag{56}$$

Now, the mixed Poisson distribution becomes

$$f_x(t) = \frac{t^x \mu^\alpha}{x! B(\alpha,\beta)} \int_0^\infty \lambda^{x+\beta-1} (\lambda+\mu)^{-\alpha-\beta} e^{-\lambda t} d\lambda$$

Putting

$$\lambda = \mu z \Longrightarrow d\lambda = \mu dz$$

we get

$$\begin{aligned} f_x(t) &= \frac{t^x \mu^\alpha}{x! B(\alpha,\beta)} \int_0^\infty \mu^{x+\beta-1-\alpha-\beta+1} z^{x+\beta-1} (1+z)^{-\alpha-\beta} e^{-\mu z t} dz \\ &= \frac{(\mu t)^x}{x! B(\alpha,\beta)} \int_0^\infty z^{x+\beta-1} (1+z)^{x+1-\alpha-(x+\beta)-1} e^{-\mu z t} dz \end{aligned}$$

Therefore

$$f_x(t) = \frac{(\mu t)^x}{x! B(\alpha,\beta)} \Gamma(x+\beta)\, \psi(x+\beta, x-\alpha+1; \mu t) \tag{57}$$

as given by Willmot $(1993, p\ 119)$.
In terms of pgf,

$$\begin{aligned} G_x(s,t) &= \frac{\mu^\alpha}{B(\alpha,\beta)} \int_0^\infty \lambda^{\beta-1} (\lambda+\mu)^{-\alpha-\beta} e^{-\lambda t(1-s)} d\lambda \\ &= \frac{\mu^\alpha}{B(\alpha,\beta)} \int_0^\infty (\mu z)^{\beta-1} (\mu z+\mu)^{-\alpha-\beta} e^{-\mu z t(1-s)} \mu dz \\ &= \frac{1}{B(\alpha,\beta)} \int_0^\infty z^{\beta-1} (1+z)^{1-\alpha-\beta-1} e^{-\mu t(1-s) z} dz \\ &= \frac{1}{B(\alpha,\beta)} \Gamma(\beta)\, \psi(\beta, 1-\alpha; \mu t(1-s)) \end{aligned}$$

Therefore

$$G_X(s,t) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \psi(\beta, 1-\alpha; \mu t(1-s)) \tag{58}$$

Using the relation (5),

$$\begin{aligned} f_x(t) &= \frac{(\mu t)^x}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(x+\beta)}{\Gamma(x+1)} \Gamma(\alpha-x)\, {}_1F_1(x+\beta; 1+x-\alpha; \mu t) \\ &+ \frac{(\mu t)^x}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(x+\beta)}{\Gamma(x+1)}\, {}_1F_1(\alpha+\beta; 1-x+\alpha; \mu t) \end{aligned}$$

This result is similar to that of Bruno et al (2006) who used different notations for $t = 1$.

The pgf becomes

$$G_X(s,t) = {}_1F_1(\beta; 1-\alpha; \mu t(1-s)) + \frac{\Gamma(-\alpha)}{B(\alpha,\beta)}[\mu t(1-s)]^{\alpha}{}_1F_1(\alpha+\beta; 1+\alpha; \mu t(1-s)) \tag{59}$$

3 Mixed Poisson Distributions in terms of Modified Bessel function of the third kind

In this section, mixed Poisson distribution are expressed in terms of modified Bessel function of the third kind denoted by

$$K_v(\omega) = \frac{1}{2}\int_0^{\infty} x^{v-1} e^{-\frac{\omega}{2}\left(x+\frac{1}{x}\right)} dx \tag{60}$$

which is a function of ω with index v. Some properties of the Bessel function are:

$$\begin{aligned}
K_v(\omega) &= K_{-v}(\omega) \\
K_{v+1}(\omega) &= \frac{2v}{\omega}K_v(\omega) + K_{v-1}(\omega) \\
K'_v(\omega) &= \frac{d}{d\omega}K_v(\omega) = -\frac{1}{2}\left[K_{v-1}(\omega) + K_{v+1}(\omega)\right] \\
K_{v+\frac{1}{2}}(\omega) &= \sqrt{\frac{\pi}{2\omega}}e^{-\omega}\left\{1 + \sum_{i=1}^{v}\frac{(v+i)!\,(2\omega)^{-i}}{(v-i)!i!}\right\}
\end{aligned}$$

3.1 Inverse Gamma distribution

The pdf of Inverse Gamma distribution is given by

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}e^{-\frac{\beta}{\lambda}}\lambda^{-\alpha-1};\ \lambda > 0;\ \alpha,\beta > 0 \tag{61}$$

which is obtained as by replacing y with $\frac{1}{\lambda}$ in the Gamma pdf with two parameters α and β given by:

$$h(y) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}e^{-\beta y}y^{\alpha-1};\ y > 0;\ \alpha,\beta > 0 \tag{62}$$

Therefore the mixed Poisson distribution is given by

$$f_x(t) = \frac{t^x}{x!}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\lambda^{x-\alpha-1}e^{-t\left(\lambda+\frac{\beta}{t}\frac{1}{\lambda}\right)}d\lambda$$

Let

$$\lambda = \sqrt{\frac{\beta}{t}} z \Longrightarrow d\lambda = \sqrt{\frac{\beta}{t}} dz$$

Therefore

$$\begin{aligned} f_x(t) &= \frac{t^x}{x!} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\sqrt{\frac{\beta}{t}} z \right)^{x-\alpha-1} e^{-t\sqrt{\frac{\beta}{t}}\left(z+\frac{1}{z}\right)} \sqrt{\frac{\beta}{t}} dz \\ &= \frac{t^x}{x!} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t}} \right)^{x-\alpha} \int_0^\infty z^{x-\alpha-1} e^{-\frac{2\sqrt{\beta t}}{2}\left(z+\frac{1}{z}\right)} dz \\ &= \frac{2}{x!} \frac{(\beta t)^{\frac{x+\alpha}{2}}}{\Gamma(\alpha)} K_{x-\alpha}\left(2\sqrt{\beta t}\right);\ x = 0, 1, 2, \ldots \end{aligned} \quad (63)$$

In terms of pgf,

$$\begin{aligned} G_X(s,t) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{-\alpha-1} e^{-\lambda t(1-s) - \frac{\beta}{\lambda}} d\lambda \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{-\alpha-1} e^{-t(1-s)\left[\lambda + \frac{\beta}{t(1-s)}\frac{1}{\lambda}\right]} d\lambda \end{aligned}$$

Let

$$\lambda = \sqrt{\frac{\beta}{t(1-s)}} z \Longrightarrow d\lambda = \sqrt{\frac{\beta}{t(1-s)}} dz$$

$$\begin{aligned} G_X(s,t) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t(1-s)}} \right)^{-\alpha} \int_0^\infty z^{-\alpha-1} e^{-\frac{2\sqrt{\beta t(1-s)}}{2}\left(z+\frac{1}{z}\right)} dz \\ &= \frac{2\beta^\alpha}{\Gamma(\alpha)} \left(\sqrt{\frac{\beta}{t(1-s)}} \right)^{-\alpha} K_{-\alpha}\left(2\sqrt{\beta t(1-s)}\right) \end{aligned} \quad (64)$$

3.2 Pearson Type V distribution

The pdf of Pearson Type V distribution is given by

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\frac{\beta}{\lambda - c}} (\lambda - c)^{-(\alpha+1)},\ \lambda > c;\ \alpha, \beta > 0 \quad (65)$$

The Poisson-Pearson Type V distribution is therefore given by

$$f_x(t) = \frac{t^x}{x!} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_c^\infty \lambda^x (\lambda - c)^{-(\alpha+1)} e^{-\lambda t - \frac{\beta}{\lambda - c}} d\lambda$$

Putting $z = \lambda - c$ we have

$$\begin{aligned} f_x(t) &= \frac{t^x}{x!}\frac{\beta^\alpha}{\Gamma(\alpha)}e^{-ct}\int_0^\infty (z+c)^x z^{-\alpha-1}e^{-tz-\frac{\beta}{z}}dz \\ &= \frac{t^x\beta^\alpha e^{-ct}}{x!\Gamma(\alpha)}\left\{\sum_{k=0}^{x}\binom{x}{k}c^{x-k}\int_0^\infty z^{k-\alpha-1}e^{-t\left(z+\frac{\beta}{t}\frac{1}{z}\right)}dz\right\} \end{aligned}$$

Let

$$z = \sqrt{\frac{\beta}{t}}u \Longrightarrow dz = \sqrt{\frac{\beta}{t}}du$$

then,

$$\begin{aligned} f_x(t) &= \frac{t^x\beta^\alpha e^{-ct}}{x!\Gamma(\alpha)}\left\{\sum_{k=0}^{x}\binom{x}{k}c^{x-k}\int_0^\infty\left(\sqrt{\frac{\beta}{t}}\right)^{k-\alpha}u^{k-\alpha-1}e^{-\frac{2\sqrt{\beta t}}{2}\left(u+\frac{1}{u}\right)}dz\right\} \\ &= \frac{t^x\beta^\alpha e^{-ct}}{x!\Gamma(\alpha)}\sum_{k=0}^{x}\binom{x}{k}c^{x-k}\left(\frac{\beta}{t}\right)^{\frac{k-\alpha}{2}}2K_{k-\alpha}\left(2\sqrt{\beta t}\right) \\ &= \frac{2\beta^\alpha(ct)^x e^{-ct}}{x!\Gamma(\alpha)}\sum_{k=0}^{x}\binom{x}{k}\frac{1}{c^k}\left(\frac{\beta}{t}\right)^{\frac{k-\alpha}{2}}K_{k-\alpha}\left(2\sqrt{\beta t}\right) \qquad (66) \end{aligned}$$

When $c = 0$, we obtain the result for Poisson-Inverse Gamma distribution.

In terms of pgf

$$G_X(s,t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_c^\infty(\lambda-c)^{-(\alpha+1)}e^{-\lambda t(1-s)-\frac{\beta}{\lambda-c}}d\lambda$$

Put

$$z = \lambda - c \Longrightarrow dz = d\lambda$$

Therefore,

$$\begin{aligned}
G_X(s,t) &= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty z^{-\alpha-1}e^{-t(1-s)(z+c)-\frac{\beta}{z}}dz \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}e^{-t(1-s)c}\int_0^\infty z^{-\alpha-1}e^{-t(1-s)\left[z+\frac{\beta}{t(1-s)}\frac{1}{z}\right]}dz \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}e^{-t(1-s)c}\int_0^\infty \left(\sqrt{\frac{\beta}{t(1-s)}}\right)^{-\alpha} y^{-\alpha-1}e^{-\frac{2\sqrt{\beta t(1-s)}}{2}\left(y+\frac{1}{y}\right)}dy \\
&= 2\frac{\beta^\alpha}{\Gamma(\alpha)}\left(\sqrt{\frac{\beta}{t(1-s)}}\right)^{-\alpha} e^{-t(1-s)c}K_{-\alpha}\left(2\sqrt{\beta t(1-s)}\right) \qquad (67)
\end{aligned}$$

3.3 Inverse Gaussian distribution

The pdf of Inverse Gaussian distribution is given by:

$$g(\lambda) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}}\lambda^{-\frac{3}{2}}\exp\left\{-\frac{\phi\lambda}{2\mu^2} - \frac{\phi}{2\lambda}\right\} \qquad (68)$$

The Poisson-Inverse Gaussian ditribution is therefore given by:

$$\begin{aligned}
f_x(t) &= \frac{t^x}{x!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}}\int_0^\infty \lambda^{x-\frac{1}{2}-1}\exp\left\{-\lambda t - \frac{\phi}{2\mu^2}\lambda - \frac{\phi}{2\mu^2}\frac{\mu^2}{\lambda}\right\}d\lambda \\
&= \frac{t^x}{x!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}}\int_0^\infty \lambda^{x-\frac{1}{2}-1}\exp\left\{-\left(\frac{2\mu^2 t+\phi}{2\mu^2}\right)\left[\lambda + \frac{\mu^2}{2\mu^2 t+\phi}\frac{\phi}{\lambda}\right]\right\}d\lambda
\end{aligned}$$

Put

$$\lambda = \sqrt{\frac{\mu^2}{2\mu^2 t+\phi}}z \Longrightarrow d\lambda = \sqrt{\frac{\mu^2}{2\mu^2 t+\phi}}dz$$

Therefore

$$\begin{aligned}
f_x(t) &= \frac{t^x}{x!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}}\left(\sqrt{\frac{\mu^2\phi}{2\mu^2 t+\phi}}\right)^{x-\frac{1}{2}}\int_0^\infty z^{x-\frac{1}{2}-1}\exp\left\{-\frac{1}{2}\sqrt{\frac{(2\mu^2 t+\phi)\phi}{\mu^2}}\left(z+\frac{1}{z}\right)\right\}dz \\
&= \frac{t^x}{x!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}}\left(\sqrt{\frac{\mu^2\phi}{2\mu^2 t+\phi}}\right)^{x-\frac{1}{2}} K_{x-\frac{1}{2}}\left(\sqrt{\frac{(2\mu^2 t+\phi)\phi}{\mu^2}}\right) \qquad (69)
\end{aligned}$$

In terms of pgf

$$\begin{aligned}
G_X(s,t) &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{1}{2}-1} e^{-\left[\frac{2\mu^2 t(1-s)+\phi}{2\mu^2}\right]\lambda-\frac{\phi}{2}\frac{1}{\lambda}} d\lambda \\
&= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{1}{2}-1} e^{-\left(\frac{2\mu^2 t(1-s)+\phi}{2\mu^2}\right)\left[\lambda+\frac{\phi\mu^2}{2\mu^2 t(1-s)+\phi}\frac{1}{\lambda}\right]} d\lambda
\end{aligned}$$

Put

$$d\lambda = \sqrt{\frac{\phi\mu^2}{2\mu^2 t(1-s)+\phi}}dz$$

Therefore

$$\begin{aligned}
G_X(s,t) &= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi\mu^2}{2\mu^2 t(1-s)+\phi}}\right)^{-\frac{1}{2}} \int_0^\infty z^{-\frac{1}{2}-1} e^{-\frac{1}{2}\sqrt{\frac{\phi[2\mu^2 t(1-s)+\phi]}{\mu^2}}\left(z+\frac{1}{z}\right)} dz \\
&= \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi\mu^2}{2\mu^2 t(1-s)+\phi}}\right)^{-\frac{1}{2}} \\
&\quad 2K_{-\frac{1}{2}}\left(\sqrt{\frac{\phi\left[2\mu^2 t(1-s)+\phi\right]}{\mu^2}}\right) \qquad (70)
\end{aligned}$$

Using Willmot's notation, $\phi = \mu^2/\beta$

$$\begin{aligned}
G_X(s,t) &= \left(\frac{\mu^2}{2\beta\pi}\right)^{\frac{1}{2}} e^{\frac{\mu}{\beta}} \left(\sqrt{\frac{\mu^4}{2\beta\mu^2 t(1-s)+\mu^2}}\right)^{-\frac{1}{2}} 2K_{-\frac{1}{2}}\left(\sqrt{\frac{2\mu^2 t(1-s)+\frac{\mu^2}{\beta}}{\beta}}\right) \\
&= 2\left(\frac{\mu^2}{2\beta\pi}\right)^{\frac{1}{2}} e^{\frac{\mu}{\beta}} \mu^{-\frac{1}{2}} \left(\sqrt{2\beta t(1-s)+1}\right)^{\frac{1}{2}} K_{-\frac{1}{2}}\left(\frac{\mu}{\beta}\sqrt{2\beta t(1-s)+1}\right) \\
&= \left(\frac{2\mu}{\beta\pi}\right)^{\frac{1}{2}} e^{\frac{\mu}{\beta}} \left(\sqrt{2\beta t(1-s)+1}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\frac{\mu}{\beta}\sqrt{2\beta t(1-s)+1}\right)
\end{aligned}$$

But

$$K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$$

Therefore

$$\begin{aligned}
G_X(s,t) &= \left(\frac{2\mu}{\beta\pi}\right)^{\frac{1}{2}} e^{\frac{\mu}{\beta}} \frac{1}{\left(\frac{2\mu}{\beta\pi}\right)^{\frac{1}{2}}} e^{-\frac{\mu}{\beta}\sqrt{2\beta t(1-s)+1}} \\
&= e^{\frac{\mu}{\beta}} e^{-\frac{\mu}{\beta}\sqrt{2\beta t(1-s)+1}} \\
&= \exp\left\{-\frac{\mu}{\beta}\left[\sqrt{1-2\beta t(s-1)}-1\right]\right\} \qquad (71)
\end{aligned}$$

3.4 Reciprocal Inverse Gaussian distribution

The pdf of Reciprocal Inverse Gaussian distribution is given by;

$$g(\lambda) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\phi/\mu} \lambda^{-\frac{1}{2}} \exp\left\{-\frac{\phi}{2}\lambda - \frac{\phi}{2\mu^2\lambda}\right\};\ \lambda > 0 \tag{72}$$

The Poisson-Reciprocal Inverse Gaussian distribution becomes

$$\begin{aligned} f_x(t) &= \frac{t^x}{x!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{x-\frac{1}{2}} \exp\left\{-\lambda t - \frac{\phi}{2}\lambda - \frac{\phi}{2\mu^2\lambda}\right\} d\lambda \\ &= \frac{t^x}{x!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{x+\frac{1}{2}-1} \exp\left\{-\frac{2t+\phi}{2}\lambda - \frac{\phi}{2\mu^2}\frac{1}{\lambda}\right\} d\lambda \\ &= \frac{t^x}{x!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{x+\frac{1}{2}-1} \exp\left\{-\frac{2t+\phi}{2}\left(\lambda - \frac{\phi}{\mu^2(2t+\phi)}\frac{1}{\lambda}\right)\right\} d\lambda \end{aligned}$$

Put

$$\lambda = \sqrt{\frac{\phi}{\mu^2(2t+\phi)}} z \Longrightarrow d\lambda = \sqrt{\frac{\phi}{\mu^2(2t+\phi)}} dz$$

Therefore

$$\begin{aligned} f_x(t) &= \frac{t^x}{x!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}\right)^{x+\frac{1}{2}} \int_0^\infty z^{x+\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\sqrt{\frac{\phi(2t+\phi)}{\mu^2}}\left(z+\frac{1}{z}\right)\right\} dz \\ &= \frac{t^x}{x!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}\right)^{x+\frac{1}{2}} 2K_{x+\frac{1}{2}}\left(\sqrt{\frac{\phi(2t+\phi)}{\mu^2}}\right) \\ &= \frac{t^x}{x!}\left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}\right)^{x} \left(\sqrt{\frac{\phi}{\mu^2(2t+\phi)}}\right)^{\frac{1}{2}} K_{x+\frac{1}{2}}\left(\sqrt{\frac{\phi(2t+\phi)}{\mu^2}}\right) \\ &= \frac{e^{\frac{\phi}{\mu}}}{x!}\left(\frac{t}{\mu}\sqrt{\frac{\phi}{(2t+\phi)}}\right)^{x} \left(\frac{2\phi}{\mu\pi}\sqrt{\frac{\phi}{(2t+\phi)}}\right)^{\frac{1}{2}} K_{x+\frac{1}{2}}\left(\sqrt{\frac{\phi(2t+\phi)}{\mu^2}}\right) \end{aligned} \tag{73}$$

3.5 Generalized Inverse-Gaussian Distribution

The pdf of Generalized Inverse-Gaussian distribution is given by

$$g(\lambda) = \frac{(\psi/\phi)^{\frac{v}{2}}}{2K_v\left(\sqrt{\psi\phi}\right)} \lambda^{v-1} \exp\left\{-\frac{1}{2}\left(\psi\lambda + \frac{\phi}{\lambda}\right)\right\};\ \lambda > 0 \tag{74}$$

with the parameters taking values in one of the ranges:

(i) $\phi > 0,\ \psi \geq 0\ if\ v < 0$

(ii) $\phi > 0,\ \psi > 0\ if\ v = 0$

(iii) $\phi \geq 0,\ \psi = 0\ if\ v > 0$

Therefore the mixed Poisson distribution is

$$f_x(t) = \frac{t^x}{x!}\frac{(\psi/\phi)^{\frac{v}{2}}}{2K_v\left(\sqrt{\psi\phi}\right)}\int_0^\infty \lambda^{x+v-1}\exp\left\{-\frac{1}{2}(2t+\psi)\left[\lambda+\frac{\phi}{(2t+\psi)}\frac{1}{\lambda}\right]\right\}d\lambda$$

Put

$$\lambda = \sqrt{\frac{\phi}{(2t+\psi)}}z \Longrightarrow d\lambda = \sqrt{\frac{\phi}{(2t+\psi)}}dz$$

Therefore

$$\begin{aligned} f_x(t) &= \frac{t^x}{x!}\left(\frac{\psi}{\phi}\right)^{\frac{v}{2}}\left(\frac{\phi}{2t+\psi}\right)^{\frac{x+v}{2}}\frac{1}{2K_v\left(\sqrt{\psi\phi}\right)}\int_0^\infty z^{x+v-1}\exp\left\{-\frac{1}{2}\sqrt{\phi(2t+\psi)}\left(z+\frac{1}{z}\right)\right\}dz \\ &= \frac{t^x}{x!}\left(\frac{\psi}{\phi}\right)^{\frac{v}{2}}\left(\frac{\phi}{2t+\psi}\right)^{\frac{x+v}{2}}\frac{2K_{x+v}\left(\sqrt{\phi(2t+\psi)}\right)}{2K_v\left(\sqrt{\psi\phi}\right)},\ x = 0,1,2,\ldots \end{aligned}$$

In terms of pgf,

$$\begin{aligned} G_X(s,t) &= \frac{(\psi/\phi)^{\frac{v}{2}}}{2K_v\left(\sqrt{\psi\phi}\right)}\int_0^\infty \lambda^{v-1}e^{-\frac{1}{2}\left[(2t(1-s)+\psi)\lambda+\frac{\phi}{\lambda}\right]}d\lambda \\ &= \frac{(\psi/\phi)^{\frac{v}{2}}}{2K_v\left(\sqrt{\psi\phi}\right)}\int_0^\infty \lambda^{v-1}e^{-\frac{1}{2}[2t(1-s)+\psi]\left\{\lambda+\frac{\phi}{2t(1-s)+\psi}\frac{1}{\lambda}\right\}}d\lambda \\ &= \left(\frac{\psi}{\phi}\right)^{\frac{v}{2}}\frac{1}{2K_v\left(\sqrt{\psi\phi}\right)}\left(\sqrt{\frac{\phi}{2t(1-s)+\psi}}\right)^v\int_0^\infty z^{v-1}e^{-\frac{1}{2}\sqrt{2t(1-s)+\psi}\left(z+\frac{1}{z}\right)}dz \\ &= \left(\frac{\psi}{2t(1-s)+\psi}\right)^{\frac{v}{2}}\frac{K_v\left[\sqrt{2t(1-s)+\psi}\right]}{K_v\left(\sqrt{\psi\phi}\right)} \qquad (75) \end{aligned}$$

4 Conclusion

A number of mixed Poisson distributions can be expressed in terms of special functions. This paper has derived Poisson mixtures in terms of confluent hypergeometric functions and modified Bessel functions of the third kind for continuous mixing distributions. These expressions seem quite involved; however, most of them can also be expressed recursively as given by Sarguta and Ottieno (2014); rendering the distributions usable. Algorithms have also been developed by Press et al (1999) and have been used to calculate Generalized Pareto mixtures of Poisson distributions by Bruno et al (2006).

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